Consider a function \( f(x) \) in a small nhd of \( x_\ast \).

Assume \( f'(x_\ast) > 0 \).

If \( f \) smooth.

In a sufficiently small nhd of \( x_\ast \), \( f \) will be very nearly a linear function. So, let's consider its approximation \( \hat{f}(x) = \lambda(x - x_\ast) \).

\[ \lambda = \frac{f'}{f(x_\ast)} \]

Consider the fixed pt. iteration:

\[ x_{k+1} = x_k - \lambda \hat{f}(x_k) \quad (\# \#) \]

(\# \#) is called a "relaxation" & \( \lambda \) a "relaxation parameter." For our approximation, \( \hat{g}(x) = x - \lambda x / (x - x_\ast) \)

\[ \hat{g}'(x) = (1 - \lambda x) \]

\( \hat{g} \) will be a contraction if \( |1 - \lambda x| < 1 \)

\[ \Rightarrow -1 < 1 - \lambda x < 1 \quad \text{or} \quad 0 < x \frac{2}{\lambda} \]

\( x > 0 \), so must \( -2 < -\lambda x < 0 \)

Take \( x > 0 \). \( \Rightarrow 2 > -\lambda x > 0 \)
What if \( \lambda < 0 \)? For contraction we will again have
\[
-2 < -\lambda \delta < 0
\]
which can be satisfied only by taking \( \lambda < 0 \) (i.e. to get \( -\lambda \delta < 0 \))

\[
2 > \lambda \delta > 0
\]

or

\[
\lambda < 0: \frac{2}{\lambda} < \lambda < 0
\]

So, here is the theorem about relaxations (slightly incorrect in the text)

Consider a continuously differentiable function \( f(x) \) with \( f'(x_\ast) = 0 \) and \( f''(x_\ast) > 0 \) \( (f'(x_\ast) < 0) \)

[only need smoothness in a neighborhood of \( x_\ast \)].

and consider the relaxation fixed point iteration

\[
x_{k+1} = x_k - \lambda f(x_k) = g_\lambda(x_k)
\]

Then for \( f'(x_\ast) > 0 \) \( (f'(x_\ast) < 0) \),

\[
\exists \lambda > 0 \quad \exists s > 0 \quad \text{s.t.}
\]

\[
x_k \to x_\ast \quad \text{for any } 0 < \lambda \leq \lambda_\ast \] \( (\lambda_\ast < 0) \)

with \( x_0 \in [x_\ast - s, x_\ast + s] \)
Proof. Assume $F'(x_\delta) = \alpha > 0$. Continuous $F' \Rightarrow \exists \delta > 0 \text{ s.t. } F'(x) > \frac{1}{2} \alpha \text{ for } |x-x_\delta| < \delta$.

Let $M = \text{upper bd on } F' \text{ in this interval}$ so that $M > \frac{1}{2} \alpha$. Hence:

$$-M < -F'(x) \leq -\frac{1}{2} \alpha$$

& for any $\lambda > 0$

$$1 - \lambda M \leq 1 - \lambda F'(x) \leq 1 - \frac{1}{2} \lambda \alpha$$

Choose $\lambda = \sqrt{\frac{2M - \alpha}{\alpha + 2M}} > 0$

$$\Rightarrow \lambda(\lambda + \frac{1}{2} \alpha) = 1 \Rightarrow \lambda = \frac{y}{\alpha + 2M} > 0$$

or:

$$\delta = \lambda M - 1$$

$$= \frac{yM - \alpha - 2M}{\alpha + 2M} = \frac{2M - \alpha}{2M + \alpha} < 1$$

$$\Rightarrow |F'(x)| < 1 \text{ for } |x - x_\delta| < \delta$$

And moreover, for any $\delta < \lambda \leq \lambda_*$

$$|F'(x)| < 1 \text{ for } |x - x_\delta| < \delta$$

Apply Thm 1.5 of the book.

Works the same way for $F'(x_\delta) < 0$. 
Newton's Method

What if instead we consider:

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

\[ g(x) = x - x_n f(x) f(x_n) \]

\[ g'(x) = 1 - x_n f(x) f'(x) - x_n f'(x) f'(x_n) \]

If \( \frac{1}{f'(x_n)} = \frac{1}{f'(x_n)} \), we will have the smallest absolute \( g'(x_n) = 0 \)

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

is Newton's Method.

Another interpretation:

\[ g = f(x) \]

\[ g' = f(x_n) + f'(x_n)(x - x_n) \]

is the tangent line approximation to \( f \) at \( x = x_n \).

Determine \( x_{n+1} \) by condition

\[ M(x_{n+1}, x_n) = 0 \quad \text{N.M.} \]

\[ \Rightarrow f(x_n) + f'(x_n)/(x_{n+1} - x_n) = 0 \]
Each of the envelope "local analysis of N.M."
Let $f(x_*) = 0$ and assume $f$ has a
continuous 2nd derivative in its nght.
Consider $x_k$ in the nght of $x_*$

$$f(x_*) = f(x_k) + f'(x_k)(x_* - x_k)$$
$$\quad + \frac{1}{2} f''(\eta_k)(x_* - x_k)^2$$

where $\eta_k$ lies between $x_k \in x_*$. by
Taylor's Thm w. Remainder.

So,

$$f(x_k) + f'(x_k)(x_* - x_k) + \frac{1}{2} f''(\eta_k)(x_* - x_k)^2 = 0$$

$$\Rightarrow (x_* - x_k) + \frac{f(x_k)}{f'(x_k)} = - \frac{f''(\eta_k)}{2 f'(x_k)} (x_* - x_k)^2$$

Assuming $f'(x_k) \neq 0$

$$x_* - x_k = (x_* - x_k)$$

$$|x_* - x_k| = \left| \frac{f''(\eta_k)}{2 f'(x_k)} \right| (x_* - x_k)^2$$

$$= c_k |x_* - x_k|^2$$