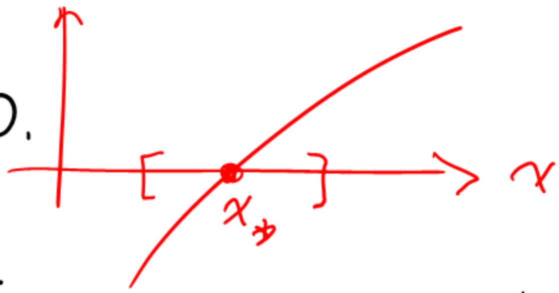
Back to "local analysis"

Consider a function $f(x)$ in a small nghd of a zero x_*:

Assume $f'(x_*) > 0$. & f smooth.



In a suff'ly small nghd of x_*, f will be very nearly a linear function. So, let's consider it's approximation $\tilde{f}(x) = \alpha(x - x_*)$

$$\alpha = \tilde{f}'(x_*)$$

Consider the fixed pt. iteration: (*)

of ✓

$$x_{k+1} = x_k - \lambda \tilde{f}(x_k) \qquad (**)$$

(**) is called a "relaxation" & λ a "relaxation parameter." For our approximation,

$$\tilde{g}(x) = x - \lambda\alpha(x - x_*)$$

$$\tilde{g}'(x) = (1 - \lambda\alpha)$$

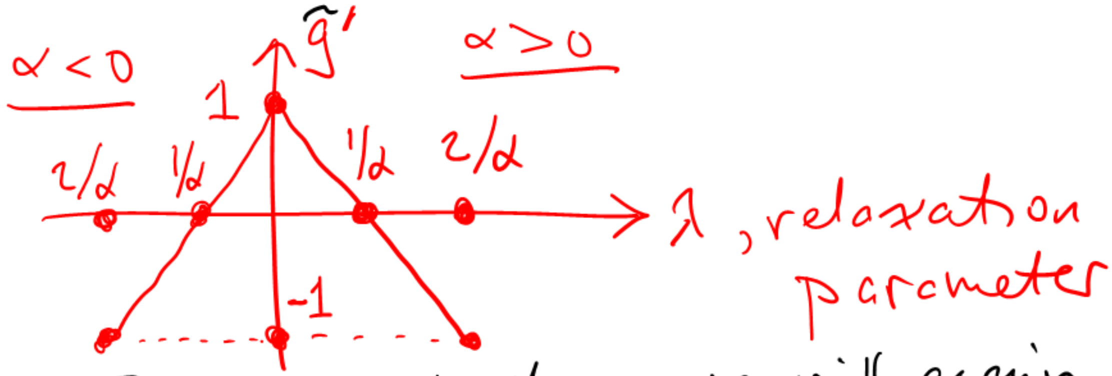
$\tilde{g}$ will be a contraction if $|1 - \lambda\alpha| < 1$

$$\Rightarrow -1 < 1 - \lambda\alpha < 1$$

$$\Rightarrow -2 < -\lambda\alpha < 0$$

$\alpha > 0$, so must take $\lambda > 0$. $\Rightarrow 2 > \lambda\alpha > 0$

or $0 < \lambda < 2/\alpha$



What if $\alpha < 0$? For contraction we will again have
$$-2 < -\lambda\alpha < 0$$
which can be satisfied only by taking $\lambda < 0$ (i.e. to get $-\lambda\alpha < 0$)
$$2 > \lambda\alpha > 0$$
or since $\alpha < 0$: $\frac{2}{\alpha} < \lambda < 0$

Q: what is the "optimal" choice for λ? $\lambda = \frac{1}{\alpha}$
Conv. in 1 itn!

So, here is the theorem about relaxations (slightly incorrect in the text)

Consider a continuously diff'ble fn $f(x)$ with $f(x_*) = 0$ & $f'(x_*) > 0$ $(f'(x_*) < 0)$ [only need smoothness in a nghd of x_*]. and consider the relaxation fixed point iteration $x_{k+1} = x_k - \lambda f(x_k) = g_\lambda(x_k)$

<u>Thm</u> For $f'(x_*) > 0$ $[f'(x_*) < 0]$,
$\exists\ \lambda_* > 0$ $[\lambda_* < 0]$ & $\delta > 0$ s.t.
$x_k \to x_*$ for any $0 < \lambda \le \lambda_*$ $[\lambda_* \le \lambda < 0]$
with $x_0 \in [x_* - \delta, x_* + \delta]$.

Pf. Assume $f'(x_*) = \alpha > 0$. Continuous f'
$\Rightarrow \exists\, \delta > 0$ s.t. $f'(x) > \frac{1}{2}\alpha \quad \forall\, |x - x_*| < \delta$.
Let M = upper bd on f' in this interval
so that $M > \frac{1}{2}\alpha$. Hence:

$$-M \le -f'(x) \le -\frac{1}{2}\alpha$$

& for any $\lambda > 0$

$$1 - \lambda M \le 1 - \lambda f'(x) \le 1 - \frac{1}{2}\lambda\alpha$$

Choose, because why not, λ_* s.t.

$$-v = 1 - \lambda_* M \le 1 - \lambda_* f'(x) \le 1 - \frac{1}{2}\lambda_*\alpha = v > 0$$

$$\Rightarrow -1 + \lambda_* M = 1 - \frac{1}{2}\lambda_*\alpha$$

$$\Rightarrow \lambda_*\left(M + \frac{1}{2}\alpha\right) = 2 \Rightarrow \boxed{\lambda_* = \frac{4}{\alpha + 2M} > 0}$$

or: $v = \lambda M - 1$

$$= \frac{4M - \alpha - 2M}{\alpha + 2M} = \frac{2M - \alpha}{2M + \alpha} < 1$$

$$\Rightarrow |g'_{\lambda_*}(x)| < 1 \quad \forall\, |x - x_*| \le \delta$$

And moreover, for any $0 < \lambda \le \lambda_*$

$$|g'_\lambda(x)| < 1 \quad \forall\, |x - x_*| < \delta$$

Apply Thm 1.5 of book.

Works the same way for $f'(x_*) < 0$.

Newton's Method

What if instead we consider

$$x_{k+1} = x_k - \lambda(x_k) f(x_k)$$

$$g(x) = x - \lambda(x) f(x)$$

$$g'(x) = 1 - \lambda'(x) f(x) - \lambda(x) f'(x)$$

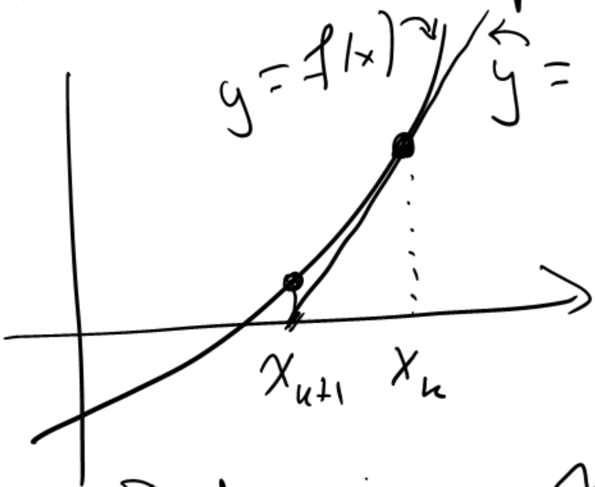
$$g'(x_*) = 1 - \lambda(x_*) f'(x_*)$$

If $\lambda(x_k) = 1/f'(x_k)$, we will have the smallest absolute $g'(x_*) = 0$

$$x_{k+1} = x_k - f(x_k)/f'(x_k)$$

is <u>Newton's</u> <u>Method</u>.

Another interpretation:



$$y = f(x_k) + f'(x_k)(x - x_k) = M(x; x_k)$$

is the tangent line approximation to f at $x = x_k$.

Determine x_{k+1} by condition

$$M(x_{k+1}; x_k) = 0 \qquad \text{N.M.}$$

$$\Longrightarrow f(x_k) + f'(x_k)(x_{k+1} - x_k) = 0$$

Back of the envelope "local analysis of N.M.

Let $f(x_*) = 0$ and $f'(x_*) \neq 0$ assume f has a continuous 2^{nd} derivative in its nghd.

Consider x_k in the nghd of x_*

$$f(x_*) = f(x_k) + f'(x_k)(x_* - x_k) + \frac{1}{2} f''(\eta_k)(x_* - x_k)^2$$

where η_k lies between x_n & x_*, by Taylor's Thm w. Remainder.

So, $f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2} f''(\eta_k)(x_* - x_k)^2 = 0$

$$\Rightarrow \underbrace{(x_* - x_k) + \frac{f(x_k)}{f'(x_k)}}_{\cancel{x_* - x} \, = (x_* - x_{k+1})} = -\frac{f''(\eta_k)}{2 f'(x_k)} (x_* - x_k)^2$$

Assuming $f'(x_k) \neq 0$

$$|x_* - x_{k+1}| = \left| \frac{f''(\eta_k)}{2 f'(x_k)} \right| |x_* - x_k|^2$$

$$= C_k |x_* - x_k|^2$$