**Contraction Mapping Theorem (2)**

Let \( g \) be real-valued, continuous on \([a, b]\) (bddd)

\[
g(x) = \left\{ \begin{array}{ll}
|a| & x < a \\
0 & a \leq x \leq b \\
|b| & x > b
\end{array} \right.
\]

Let \( g \) be a contraction

(i.e. \( |g(x) - g(y)| \leq L|x-y| \forall x \in [a, b] \) w. 0 < \( L < 1 \))

Then \( \exists ! \) \( x^* \in [a, b] \) s.t. \( x^* = g(x^*) \) (fixed pt.)

Further (*) converges to \( x^* \) for any \( x_0 \in [a, b] \)

(i.e. \( x_{k+1} = g(x_k) \) for any \( x_0 \in [a, b] \))

**Pf.** Existence: Thm 2 (Brouwer)

Uniqueness: Let \( x^{**} \) be another fixed point

\[
|g(x^{**}) - g(x^*)| = |x^{**} - x^*| \leq L |x^{**} - x^*|, \quad L < 1
\]

\[
= \Rightarrow x^{**} = x^*
\]

Convergence:

\[
|x_k - x^*| = |g(x_{k-1}) - g(x^*)| \leq L |x_{k-1} - x^*| 
\]

\[
\leq L^k |x_0 - x^*| \to 0 \quad \text{as} \quad k \to \infty \quad \text{since} \quad L < 1
\]

Bisection method: successive halving.
Geometrically, what is going on?

$g(x) = x$, $g'(x) < 1$

$x_0$, $x_1$, $x_2$

Divergence & no uniqueness

$x = x_*$
Back to example \( f(x) = e^x - 2x - 1 \) \( x \in [1, 2] \)

Generate a g from f by inverting

\[ e^x = 1 + 2x \Rightarrow x = \ln (1 + 2x) = g(x) \]

Iteration:

\[ x_{k+1} = \ln (1 + 2x_k) \]

Recall we know \( \exists X_0 \in [1, 2] \) by checking signs of end-points and applying IVT.

\( g \) is differentiable on \([1, 2]\); \( \forall \ g' = \frac{2}{1 + 2x} > 0 \)

\( w. \ g'' = \frac{-4}{(1 + 2x)^2} < 0 \) on \([1, 2]\)

Hence \( g' \) is monotonically decreasing.

\[ |g(x) - g(y)| = |g'(y)||x - y| \]

Since largest \( |g'\) must occur at the left end-point,

So, \( g = \ln (1 + 2x) \) is a contraction on \([1, 2]\). \( L = \frac{2}{3} \).

Book: MMAAAM \( (x_0 = 1.2564312086262) \) 3.4.7.

\( x_0 = 1 \) 10 runs give \( x_1 = 1.255260 \) (correct from 1.25)

Is this fast? Is it slow?

It is guaranteed!
Question: Let's say we want a certain number of correct digits (i.e., we must satisfy a tolerance) and we (somehow) know $L$.

Consider the iteration (*) where $g$ is a contraction as in Thm 3, and ask what is the smallest $k$ s.t. $|x_k - x_0| < \varepsilon$.

Call this $k_0(\varepsilon)$.

Then $k_0(\varepsilon) \leq \left[ \ln \left( \frac{\ln |x_1 - x_0| - \ln(\varepsilon (1-L))}{\ln (1/L)} \right) \right] + 1$

where $[x] = \text{largest integer} = \lfloor x \rfloor$ (floor($x$) in MATLAB).

Proof: See the book.

Note fixed $L$, let $\varepsilon \in (0, L)$, $K(\varepsilon, L) \sim \frac{-\ln \varepsilon}{\ln 1/2} \to \infty$.

Fixed $\varepsilon$, let $L \in (0, L)$ again. $K(\varepsilon, L) \sim \frac{\varepsilon}{\ln \frac{1}{L}} \to \infty$.

Comment: tolerances for errors should be expressed in relative terms, i.e., $\frac{|x_k - x^n|}{|x^n|} < \delta \in \mathbb{R}$.
\[
\begin{align*}
\frac{e^{-x^2}}{1 - e^{-x^2}} - x^2 &= 1 - x^2 - e^{-x^2} \\
\Rightarrow \quad (x)_{1+} &= e^{-x^2} \\
1 - x^2 - e^{-x^2} &= (x)_{f}
\end{align*}
\]
Then \(-\log_{10} \left( \frac{|x_n - x_*|}{|x_*|} \right)\) estimates the number of correct digits.

Most of the time we will be dealing with

Note: The conditions of the CMT guarantee that \(x_*\) is an attracting fixed point.

(i.e., \(\exists \delta > 0\) s.t. for all \(|x_0 - x_*| < \delta\)

\[ x_n \to x_* \quad \text{as} \quad n \to \infty \]

If \(\exists \delta > 0\) s.t. \(x_n \to x_*\) w. \(\forall |x_0 - x_*| < \delta\)
we called it unstable

\[ \left[ \begin{array}{c} \text{\textbullet} \end{array} \right] \]

We've already estimated the Lipschitz constant \(L\) for one differentiable function.

Let \(g\) be differentiable on \([a, b]\) w. \(g'(x) \in \mathbb{R}\) \(\forall x \in [a, b]\).

\[ \sup_{x \in [a, b]} \left| g'(x) \right| = \left| g'(\eta) \right| \text{ w. } \eta \in (x, y) \subset \mathbb{R} \]

Assume w.l.o.g. \(x < y\):

\[ \frac{|g(x) - g(y)|}{|x - y|} = \left| g'(\eta) \right| \text{ w. } \eta \in (x, y) \subset \mathbb{R} \]

Take \(L = \max_{\eta \in [a, b]} \left| g'(\eta) \right|\) and we have a contradiction if \(L < 1\).
Thus $|g'(x_*)| < 1$ will control the speed of convergence of $(*)$ in the neighborhood of $x_*$ (i.e., once you get close).

Now, take the assumptions as before w.r.t. $g$ continuously adjustable, with $x_*$ a fixed point. Assume $|g'(x_*)| < 1$. Continuity of $g'$ means there exists $\epsilon > 0$ such that $|g'(x)| < 1$ for all $|x - x_*| < \epsilon$.

Hence, $g$ is a contraction in this neighborhood and $x_k \to x_*$ for any $|x_0 - x_*| < \epsilon$.

Thus, under these assumptions, $(*)$ converges to $x_*$ for any $x_0$ sufficiently close to $x_*$. 

Note: The CMT guarantees you "global convergence" from any $x_0 \in [a, b]$. If you can control the derivative of $g$ at the fixed point (not necessarily everywhere), then you can get "local convergence".
**Defn.** Let \( X_k = \lim_{k \to \infty} X_k \).

We say \( X_n \to X_\ast \) at least linearly, if for a sequence \( \varepsilon_k \) w. \( \varepsilon_k > 0, \varepsilon_k \to 0 \) s.t. and a \( \mu \in (0, 1) \) s.t.

\[
|X_n - X_\ast| \leq \varepsilon_k \quad k = 0, 1, \ldots
\]

and \( \lim_{k \to \infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \mu \).

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**Ex.** For CMT, we have

\[
|X_n - X_\ast| \leq L^n |X_0 - X_\ast| \leq L^n (b-a)
\]

\( 0 < L < 1 \) \( \Rightarrow \) mapping converge at least linearly.

Take \( \varepsilon_k = L^k (b-a) \)

\[
\frac{\varepsilon_{k+1}}{\varepsilon_k} = L = \mu
\]

\( \Rightarrow \) \( \varepsilon = -\log \mu \) is called the rate.

If \( \mu = 0 \), then convergence is called superlinear.