Contraction Mapping Thm (3)

Let g be real-valued, continuous on $[a,b]$ (bdd)

w. $g(x) \to [a,b]$ $\forall x \in [a,b]$.

Let g be a contraction

(i.e. $|g(x)-g(y)| \le L|x-y|$ $\forall x \in [a,b]$ w. $0<L<1$)

Then $\exists!\, x_* \in [a,b]$ s.t. $x_* = g(x_*)$ (i.e. a fixed pt.)

Further $(*)$ converges to x_* for any $x_0 \in [a,b]$

(i.e. $x_{k+1} = g(x_k)$ for any $x_0 \in [a,b]$)

<u>Pf.</u> <u>Existence</u>: Thm 2 (Brouwer)

<u>Uniqueness</u>: Let x_{**} be another fixed point

$\cancel{|x_{**} - x_*|}$

$|g(x_{**}) - g(x_*)| = |x_{**} - x_*| \le L|x_{**} - x_*|$, $L<1$

$\Rightarrow x_{**} = x_*$

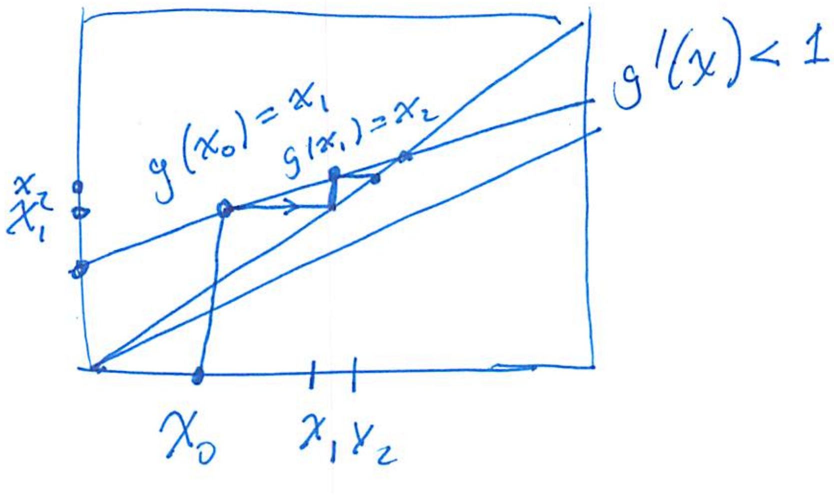
<u>Convergence</u>:

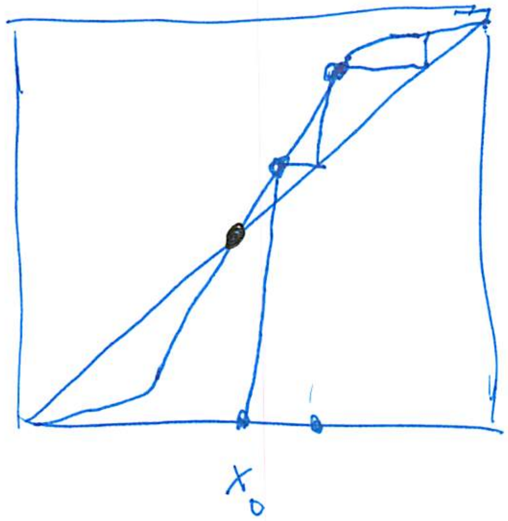
$|x_k - x_*| = |g(x_{k-1}) - g(x_*)| \le L|x_{k-1} - x_*|$

$\le L^k |x_0 - x_*| \to 0$ as $k \to \infty$ since $L<1$

Bisection method: successive halving.

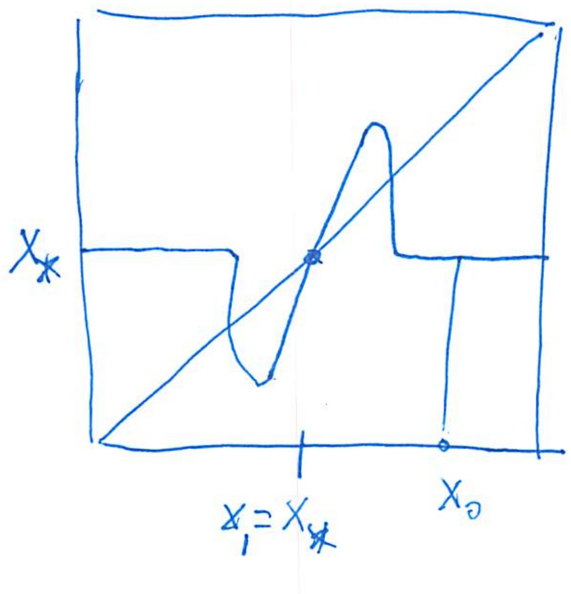
Geometrically, what is going on?





divergence &

no-uniqueness



<u>Back to example</u> $f(x) = e^x - 2x - 1$ $x \in [1,2]$

Generate a g from f by inverting

$$e^x = 1 + 2x \Rightarrow x = \ln(1+2x) \doteq g(x)$$

<u>Iteration:</u> $x_{k+1} = \ln(1 + 2x_k)$

Recall we know $\exists\, x_* \in [1,2]$ by checking signs of end-points and applying IVT.

g is cont. diff'ble on $[1,2]$; $g' = \frac{2}{1+2x} > 0$

w. $g'' = \frac{-4}{(1+2x)^2} < 0$ on $[1,2]$

Hence g' is monotonically ~~increasing~~ decreasing

$|g(x) - g(y)| = |g'(\eta)||x-y|$ By mean-value Thm, $\exists\, \eta$ between x & y

~~Biggest $g'(x)$ must be at $x=2$~~

$$\le \max_{z \in [a,b]} |g'(z)||x-y| \le \frac{2}{3}|x-y|$$

since largest $|g'|$ must occur at the left end-point.

So, ~~g is a~~ $g = \ln(1+2x)$ is a contraction on $[1,2]$ w. $L = \frac{2}{3}$.

<u>Book.</u> ($x_* = 1.2564312086262$) 3 itns of Newt. from 1.25 or 5 itns from 1.0

$x_0 = 1$, 1~~0~~ itns give $x_{11} = 1.255800$

Is this fast? Is it slow?

It is guaranteed!

<u>Question</u> Let's say we want a certain # of correct digits (i.e. we must satisfy a tolerance) & we (somehow) know L.

~~Theorem~~ Consider the iteration (*) where g is a contraction as in Thm 3, and ask what is the smallest k s.t. $|x_k - x_*| < \varepsilon$. Call this <u>$k_0(\varepsilon)$</u>.

<u>Thm 4</u> $$k_0(\varepsilon) \le \left[\frac{\ln|x_1 - x_0| - \ln(\varepsilon(1-L))}{\ln(1/L)} \right]^< + 1 = \bar{k}(\varepsilon, L)$$

where $[x]^<$ = largest integer (floor(x) in matlab).

<u>Pf.</u> See the book.

<u>Note</u> fixed L, let $\varepsilon \downarrow 0$, $\bar{k}(\varepsilon, L) \sim \frac{-\ln\varepsilon}{\ln 1/L} \to \infty$

fixed ε, let $L \downarrow 0$

again $\bar{k}(\varepsilon, L) \sim \cancel{\frac{\ln\varepsilon}{\ln 1/L}} \to 1^+$

<u>Comment</u> tolerances for errors should be expressed in relative terms, i.e., $\frac{|x_k - x_*|}{|x_*|} < \delta$ ← w.o. units.

$$f(x) = e^{x} - 2x - 1$$

$$f'(x) = e^{x} - 2$$

$$\boxed{x_{k+1} = x_k - \cancel{\frac{e^{x_k} - 2}{e^{x_k} - 2x_k - 1}}}$$

L2.5

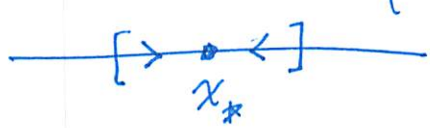
Then $-\log_{10}\left(\frac{|x_k - x_*|}{|x_*|}\right)$ estimates the number of correct digits.

~~Most of the time we will be dealing w.~~

Note: The conditions of the CMT guarantee that x_* is an <u>attracting</u> (or stable) fixed point.

(i.e. $\exists \delta > 0$ s.t. for all $|x_0 - x_*| < \delta$

$x_k \to x_*$)



If $\exists \delta$ s.t. $x_k \not\to x_*$ w. $\forall |x_0 - x_*| < \delta$

we called it unstable

We've already estimated the Lipschitz constant L for one diff'ble fn.

Let g be diff'ble on $[a,b]$ w. $g(x) \in [a,b]$ $\forall x \in [a,b]$

~~MVT Given $g:[a,b] \to \mathbb{R}$ w.~~

Assume wlog $x < y$:

MVT $\frac{|g(x) - g(y)|}{|x-y|} = |g'(\eta)|$ w. $\eta \in (x,y) \subset [a,b]$

Take $L = \max_{\eta \in [a,b]} |g'(\eta)|$ and we have a contraction if $L < 1$.

Thus $|g'(x_*)|<1$ will control the speed of convergence of $(*)$ in the nghd of x_* (i.e. once you get close).

Now, take

~~Thm~~ Assumptions as before w. g continuously diff'ble, with x_* a fixed point. Assume $|g'(x_*)| < 1$. Continuity of g' means $\exists \delta > 0$ s.t.

$$|g'(x)| < 1 \text{ for all } |x - x_*| < \delta$$

~~in a nghd of x_* s.t.~~

Hence, g is a contraction in this nghd and $x_k \to x_*$ for any $|x_0 - x_*| < \delta$.

<u>Thm</u> Under these assumptions, $(*)$ converges to x_* for any x_0 suff'ly close to x_*.

<u>Note:</u> The CMT guarantees you "global convergence" from any $x_0 \in [a,b]$. If you can ~~only~~ control the derivative of g at the fixed point (not nec. everywhere), then you can get "local convergence".

Defn Let $x_* = \lim_{k\to\infty} x_k$

We say $x_k \to x_*$ at least linearly, if

$\exists$ a sequence $\{\varepsilon\}_k$ w. $\varepsilon_k > 0$, $\varepsilon_k \to 0$

~~s.t.~~ and a $\mu \in (0,1)$ s.t.

$$|x_k - x_*| \le \varepsilon_k \quad k = 0, 1, \dots$$

and $$\lim_{k\to\infty} \frac{\varepsilon_{k+1}}{\varepsilon_k} = \mu$$

Ex. For CMT, we have

$$|x_k - x_*| \le L^k |x_0 - x_*| \le L^k |b-a|$$

$$0 < L < 1$$

Take $\varepsilon_k = L^k (b-a)$

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = L = \mu$$

Hence contraction mapping converge at least linearly

$\rho = -\log_{10}\mu$ is called the rate.

If $\mu = 0$, then convergence is called superlinear