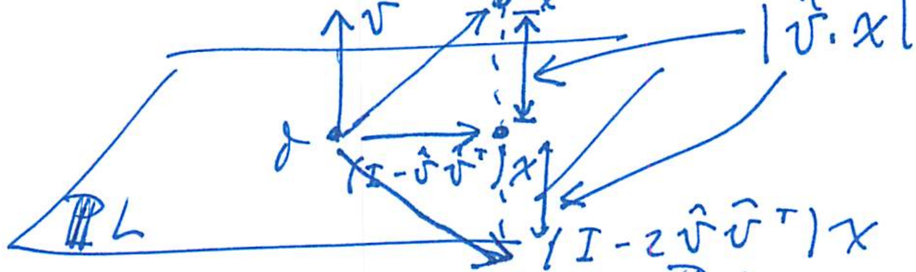
Householder transformations or reflections.

Consider a ~~point~~ vector $x \in \mathbb{R}^n$ and a plane $\mathbb{L}$ containing the origin & have unit normal $\hat{v} = v/\|v\|$



The projection of x onto $\mathbb{L}$ is

$$x - (\hat{v} \cdot x)\hat{v} = \underbrace{(I - \hat{v}\hat{v}^T)}x = P^{(v)} x$$

rank-one perturbation of the identity. Note that $P(v)$ is singular & symmetric.

P is a projection matrix onto a subspace of $\mathbb{R}^n$.

To reflect x across L, we need to pull off a little more

$$H x = x - 2(\hat{v} \cdot x)\hat{v} = (I - 2\hat{v}\hat{v}^T)x$$

H is again symmetric.

$$H^T H = (I - 2\hat{v}\hat{v}^T)(I - 2\hat{v}\hat{v}^T)$$

$$= I - 2\hat{v}\hat{v}^T - 2\hat{v}\hat{v}^T + 4\hat{v}\underbrace{(\hat{v}^T\hat{v})}_{=1}\hat{v}^T$$

$= I$. $H^{-1} = H^T$, & H is an orthogonal matrix.

$\mathcal{H}(v) = I - \frac{2}{v^T v} v v^T$ is called a H.H. tf.

<u>Note</u> $\frac{v v^T}{v^T v} = \hat{v} \hat{v}^T$

Successive left & right appl^n of HhTfs. can be used to make A tridiagonal

<u>Note:</u> Given $x \in \mathbb{R}^n$, $\mathcal{H}x = \underline{x} - \frac{2}{\|v\|_2^2}(v^T \cdot x)\, \underline{v}$

& x, v, & $\mathcal{H}x$ are linearly dependent.

<u>Important lemma</u> Given $x \in \mathbb{R}^n$, $\exists\ \mathcal{H}(v)$

s.t. $\mathcal{H}x = \alpha e_1$ w. $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\alpha \in \mathbb{R}$.

Let's construct directly such an $\mathcal{H}$ (i.e., v):

Assume $\underbrace{v = x + c e_1}_{\text{statement of lin dependence.}}$ & determine c s.t. $\mathcal{H}x = \alpha e_1$ for some α.

$$\cancel{x}\, v = x + c e_1$$

$$\Rightarrow x^T v = x^T x + c\, x^T e_1 = \overset{\text{known.}}{x^T x} + c \overset{\text{known}}{\beta} \qquad \beta = x^T e_1$$

$$v^T v = v^T x + c\, v^T \cdot e_1$$

$$= (x^T x + c\beta) + c\,(x + c e_1)^T e_1$$

$$= x^T x + 2c\beta + c^2$$

$\cancel{\mathcal{H}x = \frac{x}{v^T v}}$

$$Hx = x - 2\frac{v^Tx}{v^Tv}v$$

$$= x - 2\frac{x^Tx + c\beta}{x^Tx + 2c\beta + c^2}(x + ce_1)$$

$$= \frac{(x^Tx + \cancel{2}c\beta + c^2) - 2(x^Tx + \cancel{c}\beta)}{(\quad '' \quad)}x$$

$$- 2c\frac{x^Tx + c\beta}{(\quad '' \quad)}e_1$$

$$= \frac{c^2 - x^Tx}{(\quad '' \quad)}x - 2c\frac{x^Tx + c\beta}{(\quad '' \quad)}e_1$$

$Hx = \alpha e_1 \Rightarrow$ we take $c^2 = x^Tx$ & require that $x^Tx + 2c\beta + c^2 \neq 0$

$$\| \quad 2(x^Tx \pm (\pm\sqrt{x^Tx})\beta)$$

This will be always positive, for $\beta \neq 0$ if we choose the sign as

$$c = \begin{cases} \mathrm{sgn}(\beta)\sqrt{x^Tx} & \beta \neq 0 \\ \sqrt{x^Tx} & \beta = 0 \end{cases}$$

$$\Rightarrow Hx = -2c\frac{x^Tx + c\beta}{2(x^Tx + c\beta)}e_1 = -ce_1 \quad \checkmark \ \underline{\text{done}}$$

$$\alpha = -c$$

But why?

Consider $A = \left(\begin{array}{c|c} \alpha_0 & y_0^T \\ \hline \underbrace{y_0}_{\in \mathbb{R}^{n-1}} & C_0 \end{array}\right) \in \mathbb{R}^{n \times n}$

and $\mathcal{H}^{(n)}(w_1)$ where $w_1 = \begin{pmatrix} 0 \\ \hline v_1 \in \mathbb{R}^{n-1} \end{pmatrix}$

$$w_1 w_1^T = \left(\begin{array}{c|c} 0 & 0^T \\ \hline 0 & v_1 v_1^T \end{array}\right), \quad w_1^T w_1 = v_1^T v_1$$

$$\Rightarrow \mathcal{H}^{(n)}(w_1) = I - 2\frac{w_1 w_1^T}{v_1^T v_1} = \left(\begin{array}{c|c} 1 & 0^T \\ \hline 0 & \underbrace{I - 2\frac{v_1 v_1^T}{v_1^T v_1}}_{= \mathcal{H}^{(n-1)}(v_1)} \end{array}\right)$$

$$\mathcal{H}^{(n)}(w_1) A = \left(\begin{array}{c|c} 1 & 0^T \\ \hline 0 & \mathcal{H}^{(n-1)}(v_1) \end{array}\right) \left(\begin{array}{c|c} \alpha_0 & y_0^T \\ \hline y_0 & C_0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} \alpha_0 & y_0^T \\ \hline \mathcal{H}^{n-1} y_0 & \mathcal{H}^{n-1} C_0 \end{array}\right)$$

<u>Personal notation</u> here $\mathcal{H}^{(n)}$ only refers to the dimension, $n \times n$, of $\mathcal{H}^{(n)}$, i.e $\mathcal{H}^{(n-1)} \in \mathbb{R}^{(n-1)\times(n-1)}$

$$\mathcal{H}^{(n)} A \mathcal{H}^{(n)} = \left(\begin{array}{c|c} \alpha_0 & y_0^T \\ \hline \mathcal{H}^{(n-1)} y_0 & \mathcal{H}^{n-1} C_0 \end{array}\right) \left(\begin{array}{c|c} 1 & 0^T \\ \hline 0 & \mathcal{H}^{(n-1)} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \alpha_0 & (\mathcal{H}^{(n-1)} y_0)^T \\ \hline \mathcal{H}^{(n-1)} y_0 & \mathcal{H}^{n-1} C_0 \mathcal{H}^{n-1} \end{array}\right) = \left(\begin{array}{c|c} \alpha_0 & (\mathcal{H}^{(n-1)} y_0)^T \\ \hline \mathcal{H}^{(n-1)} y_0 & C_1 \end{array}\right)$$

Given $y \in \mathbb{R}^{n-1}$, pick v_1 s.t.

$$\mathcal{H}^{(n-1)} y_0 = \alpha_1 e_1 \in \mathbb{R}^{n-1}$$

$$\Rightarrow \mathcal{H}^{(n)} A \mathcal{H}^{(n)} = \left(\begin{array}{c|c|c} \alpha_0 & \alpha_1 & 0^T \\ \alpha_1 & \gamma_1 & y_1^T \\ \hline 0 & y_1 & C_1 \end{array}\right)$$

$$\text{If } w_2 = \begin{pmatrix} 0 \\ 0 \\ v_2 \in \mathbb{R}^{n-2} \end{pmatrix}; \quad \mathcal{H}^{(n)}(w_2) = \left(\begin{array}{cc|c} 1 & 0 & 0^T \\ 0 & 1 & 0^T \\ \hline 0 & 0 & \mathcal{H}^{(n-2)}(v_2) \end{array}\right)$$

$$\mathcal{H}^{(n)}(w_2) A = \left(\begin{array}{c|c} I_2 & 0^T \\ & 0^T \\ \hline 0 \; 0 & \mathcal{H}^{(n-2)}(v_2) \end{array}\right) \left(\begin{array}{cc|c} \alpha_0 & \alpha_1 & 0 \\ \alpha_1 & \gamma_1 & y_1^T \\ \hline 0 & y_1 & C_1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} \alpha_0 & \alpha_1 & 0^T \\ \alpha_1 & \gamma_1 & y_1^T \\ \hline 0 & \mathcal{H}^{n-2} y_1 & \mathcal{H}^{(n-2)} C_1 \end{array}\right)$$

$$\mathcal{H}^{(n)}(w_2)\, A\, \mathcal{H}^{(n)}(w_2)$$

$$= \left(\begin{array}{cc|c} \alpha_0 & \alpha_1 & 0^T \\ \alpha_1 & \gamma_1 & (\mathcal{H}^{(n-2)} y_1)^T \\ \hline 0 & \mathcal{H}^{(n-2)} y_1 & \mathcal{H}^{(n-2)} C_1 \mathcal{H}^{(n-2)} \end{array}\right)$$

Choose v_2 s.t. $\mathcal{H}^{(n-2)} v_2 = \alpha_2 e_1 \in \mathbb{R}^{n-2}$

$$= \left(\begin{array}{cc|cccc} \alpha_0 & \alpha_1 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \gamma_1 & \alpha_2 & 0 & \cdots & 0 \\ \hline 0 & \alpha_2 & & & & \\ 0 & 0 & & & & \\ \vdots & \vdots & & C_2 & & \\ 0 & 0 & & & & \end{array}\right)$$

Note $\begin{pmatrix} \alpha_0 & \alpha_1 & 0 \\ \alpha_1 & \gamma_1 & \alpha_2 \\ 0 & \alpha_2 & \gamma_2 \end{pmatrix}$ is tridiagonal.

Stop when $C_k \in \mathbb{R}^{2\times 2}$ i.e. $k = n-2$

Put another way and follow the book's notation:

Let $\mathcal{H}^{(n)}(w_1) = H_{(n,n-1)}$

$\mathcal{H}^{(n)}(w_k) = H_{(n,n-k)}$

$$T = H_{(n,2)} \cdots H_{(n,n-2)} H_{(n,n-1)}\, A\, H_{(n,n-1)} H_{(n,n-2)} \cdots H_{(n,2)}$$

$= Q_n^T A Q_n$ has the same e.v.s as A.

Opn Ct: $\frac{1}{3} n^3$