Eigenvalues of Symmetric Matrices

Given \( A \in \mathbb{R}^{n \times n} \), \( x \in \mathbb{R}^n \) is called an eigenvalue eigenvector pair \( (x \neq 0) \) if

\[ A x = \lambda x \]

Arise in many instances - stability, analysis of algorithms & iterative schemes.

Their determination is a non-linear problem unless \( \lambda \) is known.

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Properties of symmetric matrices: Let \( \lambda \) be such.

1. \( \exists \) \( n \) linearly independent eigenvectors \( x_i \in \mathbb{R}^n \) with \( \lambda_i \) eigenvalues \( x_i \in \mathbb{R} \)
   [eigenvalues are real]

2. Eigenvectors with distinct eigenvalues are orthogonal, i.e. \( x_i \neq x_j \Rightarrow x_i^T x_j = 0 \)

3. If \( \lambda \) has multiplicity \( m \geq 1 \), \( \exists \) a basis of orthogonal eigenvectors corresponding to \( \lambda \)

4. Given \( Q \in \mathbb{R}^{n \times n} \) an orthogonal matrix \( B = Q^T A Q \) have the same eigenvalues.

5. \( \exists \) an orthonormal basis of \( \mathbb{R}^n \) \( \lambda \) e.v.s of \( A \)
Gershgorin Theorem

Given any $A \in \mathbb{C}^{n \times n}$. Where, more or less are your eigenvalues?

**Definition** Let $D_i = \left\{ z \in \mathbb{C} \mid |z - A_{ii}| \leq \sum_{j=1, j \neq i}^{n} |A_{ij}| = R_i \right\}$

$D_i$ is called a Gershgorin disk

**Example**

$$B = \begin{bmatrix} 3 & 1 & -0.5 \\ 1 & 2 & 0 \\ 1 & 0.5 & -1 \end{bmatrix} \quad R_1 = 1.5 \\
R_2 = 1 \\
R_3 = 1.5$$

$D_3 = \left\{ z \in \mathbb{C} \mid |z - (-1)| \leq 1.5 \right\}$

**Picture**

Gershgorin's 1st theorem

All eigenvalues of $A$ lie in the union of its Gershgorin disks, i.e. $A_k \in \bigcup_{i=1}^{n} D_i, \; k = 1(1)n$

**Proof**

Given $(X \neq 0, x)$

$A x = \lambda x \iff \sum_{i=1}^{n} A_{ij} x_j = \lambda x_j, \; i = 1(1)n$

Choose $k \in \mathbb{C} \mid x_k \geq |x_i| \; \forall i$
Then

\[ |a - a_{xx}| |x_n| = |a x_n - a_{xx} x_n| \]

\[ = |\sum_{i=1}^{n} a_{xi} x_i - a_{xx} x_n| = |\sum_{i \neq n} a_{xi} x_i| \]

\[ \leq \sum_{i=1}^{n} |a_{xi}| |x_i| \leq \left( \sum_{i=1}^{n} |a_{ii}| \right) |x_n| = R_n |x_n| \]

\[ \Rightarrow |a - a_{xx}| \leq R_n \quad \text{since} \quad |x_n| \neq 0. \Rightarrow x \in D_n \]

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Gershgorin's 2nd Theorem

Divide the $D_i$'s into disjoint sets $D^{(p)}$ and $D^{(q)}$ (one could be the empty set) of $p \neq q = n - p$ disks. Then $D^{(p)}$ contains $p$ eigenvalues and $D^{(q)}$ contains $q$.

Cor. Disjoint Gershgorin disks contain exactly one eigenvalue.
A is symmetric $\Rightarrow$ real eigenvalues

\begin{bmatrix}
-3 & 1 \\
-1 & 3
\end{bmatrix}

\Rightarrow \text{one per interval.}

w. disjoint disks, which are really only intervals (Diagonal matrices?)

Oh, these theorems gives us estimates what if we want to compute them to some level of accuracy? Any eigenvalues satisfies $P(\lambda) = \det(A - \lambda I) = 0$, i.e. $\lambda$ is a root of the $n$th-degree characteristic polynomial. Finding them is generally tough. Given $A$ we could then solve

$(A - \lambda I)X = 0$

i.e. $X$ is in the kernel or null-space of $A - \lambda I$.

Let's explore other methods.

Power method - Method for finding the largest eigenvalue of $A \in \mathbb{R}^{n \times n}$.

Given $X_0 \in \mathbb{R}^n$, iterate

$X_{k+1} = A X_k$, $k = 0, 1, \ldots$
What do you expect to happen?

Simple example

\[ A = \begin{pmatrix} 2 & 0 \\ 0 & 1.5 \end{pmatrix} \]

\[ \lambda_1 = 2, \quad \phi_{\lambda_1} = (0, 1) \]

\[ \lambda_2 = 1.5, \quad \phi_{\lambda_2} = (1) \]

\[ x_0 = a \phi_{\lambda_1} + b \phi_{\lambda_2} \]

\[ x_1 = 2 a \phi_{\lambda_1} + 1.5 b \phi_{\lambda_2} \]

\[ x_n = 2^n a \phi_{\lambda_1} + (1.5)^n b \phi_{\lambda_2} \]

\[ x_n \to \infty \text{ as } n \to \infty, \] but is dominated by the 1st term.

Normalized \( x_n \):

\[ y_n = \frac{x_n}{\| x_n \|} \]

\[ \| x_n \| = \left( 2^{2n} a^2 + (1.5)^{2n} b^2 \right)^{1/2} \]

\[ = 2^n \left[ a^2 + \left( \frac{1.5}{2} \right)^{2n} b^2 \right]^{1/2} \]

\[ \to |a|^{2n} \]

\[ y_n \to \frac{x_1}{\| x_1 \|}, \] i.e. picks out the eigenvector with the largest eigenvalue. (In absolute value)
Thm. Let $\lambda_1$ be a simple eigenvalue of the symmetric matrix $A \in \mathbb{R}^{n \times n}$ satisfying $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots \geq |\lambda_n|$ ("well-separated"). Let $x_0 \in \mathbb{R}^n$ be not 1 to the (1-dim'1) eigenspace of $\lambda_1$. Then

$$y_k = \frac{x_k}{\|x_k\|}; \text{ where } x_{k+1} = Ax_k$$

converges to a normalized eigenvector of $\lambda_1$.

Pf. Let $\{\eta_i\}_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$ of eigenvectors of $A$.

That is $A\eta_i = \lambda_i \eta_i$, $\eta_i^T \eta_i = 1$.

We can write $x_0 = \sum_{i=1}^n \alpha_i \eta_i$.

where $\alpha_i = \eta_i^T x_0 \neq 0$.

$x_k = A x_{k-1} = A^k x_0 = \sum_{i=1}^n \alpha_i A^k \eta_i = \sum_{i=1}^n \alpha_i \lambda_i^k \eta_i$

$= \alpha_1 \lambda_1^k \left[ \eta_1 + \sum_{i=2}^n \frac{\alpha_i}{\alpha_1} \left( \frac{\lambda_i}{\lambda_1} \right)^k \eta_i \right]$

$\Rightarrow y_k = \frac{x_k}{\|x_k\|} = \pm \frac{Z_k}{\|Z_k\|} \Rightarrow \pm y_1 - \sqrt{2}$
Inverse power method

What if you want to find the smallest absolute eigenvalue $\lambda$ of a n.s. matrix $A$?

$X_{k+1} = A^{-1} X_k$; i.e. the largest abs. e.v. of $A$ is the largest $\lambda A^{-1}$.

Let's consider $A - \mu I$ which has eigenvalues $\lambda_i(A) - \mu$; i.e. $n$. The largest e.v. of $(A - \mu I)^{-1}$ is the closest to $1$.

where $\lambda_k$ is the closest to $1$.

$$
\lambda_1 \quad \lambda_2 \quad \lambda_3
$$

$$
\vdots
\lambda_k
$$

$$X_{k+1} = (A - \mu I)^{-1} X_k
$$

or $(A - \mu I) X_{k+1} = X_k$

Methods to compute all eigenvalues (the "spectrum")

- A symmetric matrix $A$.

2 steps:

1. Find an orthogonal matrix $Q$ so that $A' = Q^T A Q$ is tridiagonal.
2. Use QR algorithm to iteratively determine the e.v. of $A'$.