Eigenvalues of Symmetric Matrices

Given $A \in \mathbb{R}^{n\times n}$, ~~$x \in \mathbb{R}^n$ is called~~

~~(x,λ)~~ $(x \in \mathbb{R}^n, \lambda \in \mathbb{C})$ is called an eigenvector eigenvalue pair $(x \neq 0)$ if

$$Ax = \lambda x$$

Arise in many instances - stability, analysis of algorithms & iterative schemes.

Their determination is ~~a~~ a nonlinear problem unless λ is known.

Properties of symmetric matrices: Let A be such.

① $\exists$ n linearly independent eigenvectors $x_i \in \mathbb{R}^n$ with eigenvalues $\lambda_i \in \mathbb{R}$

[eigenvalues are real]

② eigenvectors with distinct eigenvalues are orthogonal, i.e. $\lambda_i \neq \lambda_j \Rightarrow x_i^T x_j = 0$

③ If λ_i has multiplicity $m \geq 1$, $\exists$ a basis of orthogonal eigenvectors corresponding to λ_i

④ Given $Q \in \mathbb{R}^{n\times n}$ an orthogonal matrix $B = Q^T A Q$ have the same eigenvalues.

⑤ $\exists$ an orthonormal basis of $\mathbb{R}^n$ of e.Vs of A

Gershgorin Thms

Given any $A \in \mathbb{C}^{n \times n}$. Where, more or less are your eigenvalues?

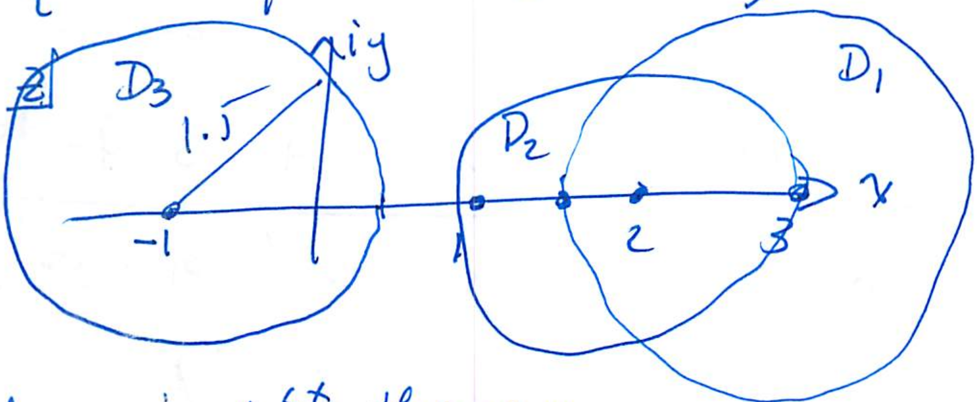
Defn Let $D_i = \left\{ z \in \mathbb{C} \;\middle|\; |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| = R_i \right\}$ absolute row sum.

$i = 1(1)n$

D_i is called a Gershgorin disk

Ex $B = \begin{bmatrix} 3 & +1 & -0.5 \\ 1 & 2 & 0 \\ 1 & 0.5 & -1 \end{bmatrix}$ $\quad R_1 = 1.5$, $R_2 = 1$, $R_3 = 1.5$

$D_3 = \{ z \in \mathbb{C} \mid |z - (-1)| \leq 1.5 \}$



Gerschgorin's 1st theorem

All eigenvalues of A lie in the union of its Gerschgorin disks, i.e. $\lambda_k \in \bigcup_{i=1}^{n} D_i \quad k = 1(1)n$

Pf. Given $(x \neq 0, \lambda)$

$$Ax = \lambda x \iff \sum_{j=1}^{n} a_{ij} x_j = \lambda x_i, \; i = 1(1)n$$

Choose $k \ni |x_k| \geq |x_i| \;\forall i$

Then $|\lambda - a_{kk}||x_k| = |\lambda x_k - a_{kk} x_k|$

$$= \left|\sum_{j=1}^{n} a_{kj} x_j - a_{kk} x_k\right| = \left|\sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} x_j\right|$$

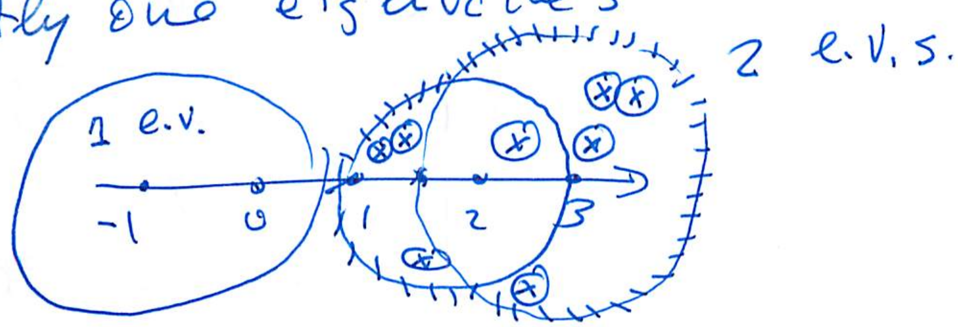
$$\leq \sum_{j=1}^{n} |a_{kj}||x_j| \leq \left(\sum_{j=1}^{n} |a_{kj}|\right)|x_k| = R_k |x_k|$$

$\Rightarrow$ $|\lambda - a_{kk}| \leq R_k$ since $|x_k| \neq 0$. $\Rightarrow \lambda \in D_k$

Gershgorin's 2nd theorem

Divide the D_i's into disjoint sets $D^{(p)}$ & $D^{(q)}$ (one could be the empty set) of p & $q = n-p$ disks. Then $D^{(p)}$ contains p eigenvalues & $D^{(q)}$ contains q.

Cor Disjoint Gershgorin disks contain exactly one eigenvalues



Another example

$$A = \begin{bmatrix} 4 & 0.2 & -0.1 & 0.1 \\ 0.2 & -1 & -0.1 & 0.05 \\ -0.1 & -0.1 & 3 & 0.1 \\ 0.1 & 0.05 & 0.1 & -3 \end{bmatrix}$$

$a_{11} = 4 \quad R_1 = 0.4$

$a_{22} = -1 \quad R_2 = 0.35$

$a_{33} = 3 \quad R_3 = 0.3$

$a_{44} = -3 \quad R_4 = 0.25$

A is symmetric $\Rightarrow$ real eigenvalues

one per interval.

w. disjoint disks, which are really only intervals (Diagonal matrices?)

Ok These theorems gives us estimates what if we want to compute them to some level of accuracy? Any eigenvalues satisfies $P(\lambda) = \det(A - \lambda I) = 0$, i.e. is a root of the n^{th}-degree characteristic polynomial. Finding them is generally tough.

Given a λ we could then solve

$$(A - \lambda I)X = 0$$

i.e. X is in the kernel ~~of~~ or null-space of $A - \lambda I$.

Lets explore other methods.

Power method - Method for finding the largest eigenvalue of $A \in \mathbb{R}^{n \times n}$.

Given $x_0 \in \mathbb{R}^n$, iterate

$$x_{k+1} = A x_k, \quad k = 0, 1, \dots$$

What do you expect to happen?

Simple example $\quad A = \begin{pmatrix} 2 & 0 \\ 0 & 1.5 \end{pmatrix}$

$\lambda_1 = 2, \ \eta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad x_0 = a\eta_1 + b\eta_2$

$\lambda_2 = 1.5, \ \eta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x_1 = 2a\eta_1 + 1.5b\eta_2$

$\vdots$

$x_k = 2^k a\eta_1 + (1.5)^k b\eta_2$

$x_k \to \infty$ as $k \to \infty$, but is dominated by the 1^{st} term.

Normalized x_k: $\quad y_k = \dfrac{x_k}{\|x_k\|}$

$\|x_k\| = \left(2^{2k}a^2 + (1.5)^{2k}b^2\right)^{1/2}$

$= 2^k \underbrace{\left[a^2 + \left(\frac{1.5}{2}\right)^{2k} b^2\right]^{1/2}}_{\to |a|}$

& $\quad y_k \to \eta_1/\|\eta_1\|$, i.e. picks out the eigenvector with the largest eigenvalue. (in absolute value)

Thm Let λ_1 be a simple eigenvalue of the symmetric matrix $A \in \mathbb{R}^{n\times n}$ satisfying $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$

"well-separated"

Let $x_0 \in \mathbb{R}^n$ be not $\perp$ to the (1-dim'l) eigenspace of λ_1. Then

$$y_k = \frac{x_k}{\|x_k\|} \; ; \text{ where } x_{k+1} = A x_k ,$$

converges to a normalized eigenvector of λ_1.

Pf. Let $\{\eta_i\}_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$, of eigenvectors of A.

[That is $A\eta_i = \lambda_i \eta_i$, ~~$i \neq j$~~ $\eta_i^T \eta_j = \delta_{jk}$]

We can write $x_0 = \sum_{i=1}^n \alpha_i \eta_i$ [Q: what are the α_i's?]

where $\alpha_i = \eta_i^T x_0$ & $\alpha_1 \neq 0$.

$$x_k = A x_{k-1} = A^k x_0 = \sum_{i=1}^n \alpha_i A^k \eta_i = \sum_{i=1}^n \alpha_i \lambda_i^k \eta_i$$

$$= \alpha_1 \lambda_1^k \underbrace{\left[\eta_1 + \sum_{i=2}^n \frac{\alpha_i}{\alpha_1}\left(\frac{\lambda_i}{\lambda_1}\right)^k \eta_i\right]}_{z_k}$$

$z_k \to \eta_1$ since $|\lambda_i| < |\lambda_1|$ $i \neq 1$.

$\Rightarrow y_k = \frac{x_k}{\|x_k\|} = \pm \frac{z_k}{\|z_k\|} \to \pm \eta_1$. ✓

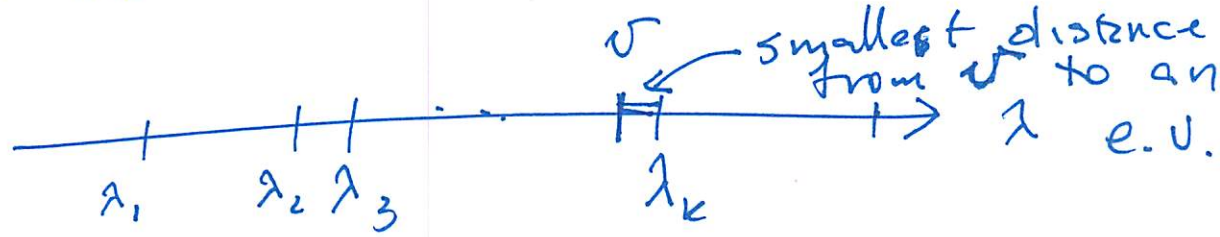
Inverse power method

What if you want to find the smallest absolute eigenvalue of a n.s. matrix A?

$x_{k+1} = A^{-1} x_k$; i.e. the ~~largest~~ smallest abs. e.v.

or $A x_{k+1} = x_k$ of A is the largest of A^{-1}.

Let's consider $A - vI$ which has eigenvalues $\lambda_i(A) - v$ $i = 1(1)n$. The largest e.v. of $(A - vI)^{-1}$ is ~~that closest to~~ $(\lambda_k - v)$ where λ_k is the closest to λ.



$$x_{k+1} = (A - vI)^{-1} x_k$$

or $(A - vI)\, x_{k+1} = x_k$

Methods to compute all eigenvalues (the "spectrum") of a symmetric matrix A.

2 steps (1) Find an orthogonal matrix Q so that $A' = Q^T A Q$ is tridiagonal.

Note: eigenvalues are invariant

(2) Use QR algorithm to iteratively determine the e.v. of A'.