Solution to nonlinear equations.

~~Very few equations~~

Find x^* satisfying $f(x^*) = 0$

A common task. Very few equations can be solved in closed form for x^*.

<u>Polynomials</u> No general form for poly. of degree > 4.

How do we find a solution numerically?

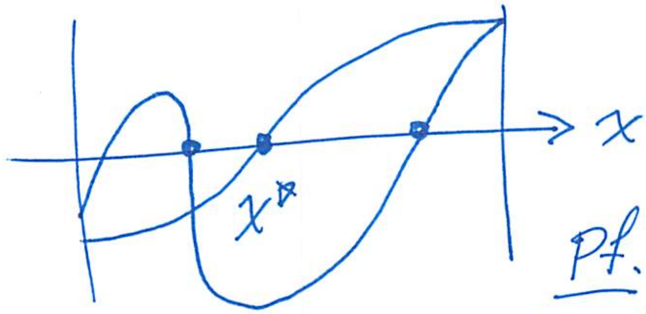
Further, not all equations have (real) solutions, e.g. $x^2+1=0$

Fixed point iteration

Consider a fcn $f: [a,b] \to \mathbb{R}$, $a<b$

<u>Thm</u> (from calculus)

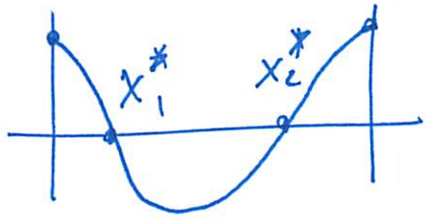
Given a continuous f, with $f(a) \le 0$ & $f(b) \ge 0$ (or vice versa) then $\exists\, x^* \in [a,b]$ satisfying $f(x^*) = 0$.



Can be multiples of course.

<u>Pf.</u> Intermediate Value Theorem. $f(a) \le 0 \le f(b)$

The theorem gives a necessary <u>but</u> not sufficient condition for the existence of a solution. 2



$f(a)$ & $f(b)$ like-signed. IVT argument does not apply, still there are 2 solutions.

Many methods for solving $f(x)=0$ have the form of an iteration

$$x^{k+1} = g(x^k) \qquad (*)$$

where $x^* = g(x^*) \Rightarrow f(x^*) = 0$

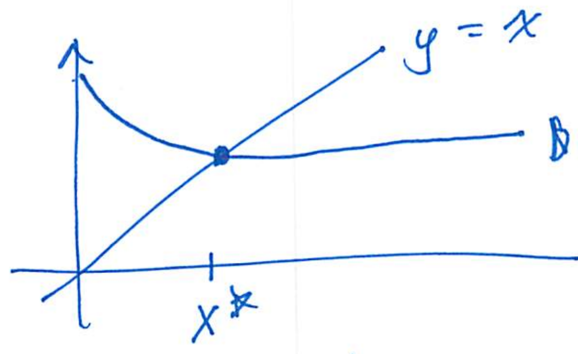
<u>e.g.</u> $f(x) = 0 \Rightarrow f(x) + x = x$

Take $g(x) = f(x) + x$ // or $g(x) = \alpha f(x) + x$, $\alpha \neq 0$.

etc.

$g(x) = x$ is called a <u>fixed point eqn.</u>

Solution x^* is called a <u>fixed point</u>.



$(*)$ is called a fixed point iteration.

Book example (a nice one)

$f(x) = e^x - 2x - 1$, $x \in [1,2]$, is continuous

$f(1) < 0$, $f(2) > 0$

so, $\exists\, x^* \in [1,2]$ s.t. $f(x^*) = 0$

(i) $g(x) = f(x) + x = e^x - x - 1$

~~f(x)~~ $e^x - 2x - 1 \Rightarrow$

(ii.) $x = \ln(1+2x) = g(x)$

(iii) $x = \frac{1}{2}(e^x - 1) = g(x)$ ... an ∞ of possible g's.

Let's say we have g satisfying

$g : [a,b] \to [a,b]$ & g continuous

Given $x_0 \in [a,b]$, consider the fixed point iteration.

(*) $$x_{k+1} = g(x_k) \quad k = 0, 1, 2, \ldots$$

If $x_k \to \xi$ $\left(\xi = \lim_{k\to\infty} x_k\right)$

then $\xi = g(\xi)$

Brouwer's Fixed Pt. Thm

Let $g : [a, b] \to \mathbb{R}$, $-\infty < a < b < -\infty$.

be continuous on $[a, b]$.

Let $g(x) \in [a, b]$ $\forall$ $x \in [a, b]$

(i.e. g takes the interval into itself)

Then, $\exists$ $\xi \in [a, b]$ s.t. $\xi = g(\xi)$,

(i.e. $\exists$ a fixed point of $g(x) = x$).

<u>Pf.</u> Let $f(x) = x - g(x)$

$g(a) \in [a, b] \Rightarrow f(a) = a - g(a) \le 0$

$g(b) \in [a, b] \Rightarrow f(b) = b - g(b) \ge 0$

$\Rightarrow$ ~~$\exists$ x^* s.t.~~ So, $f(a) \le 0 \le f(b)$

w. f continuous $\Rightarrow$ $\exists$ x^* s.t.

$f(x^*) = 0 \Leftrightarrow x^* = g(x^*)$

A fn f can produce an infinity of different fixed point eqns.

When does (*) converge?

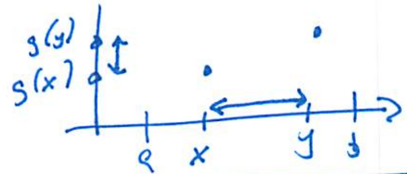
<u>Defn</u> (Contraction mapping)

A fn $g : [a,b] \to \mathbb{R}$ is called a <u>contraction</u>

if $\exists\, 0<L<1$ s.t.

$$\underbrace{|g(x)-g(y)| \leq L|x-y|}_{\text{Called a Lipschitz condition (for any } L>0)} \quad \forall\, x,y \in [a,b]$$

What does this mean?



It contracts!

<u>Contraction Mapping Thm</u>

Let $g : [a,b] \to \mathbb{R}$, $g(x) \in [a,b]$ $\forall x \in [a,b]$

be continuous & a contraction

$\exists!$ fixed point x^* ($x^* = g(x^*)$)

and ~~the~~ the iteration (*) converges

to it from any $x_0 \in [a,b]$.

<u>Pf.</u> (1) Existence is done (Brouwer F.P. Thm)

(2) Uniqueness - Assume x^{**} is also a fixed pt.

$$|x^* - x^{**}| = |g(x^*) - g(x^{**})| \leq L|x^* - x^{**}|$$

$$L < 1 \Rightarrow x^* = x^{**}$$

(3) $|x_{k+1} - x^*| = |g(x_k) - g(x^*)| \leq L|x_k - x^*|$

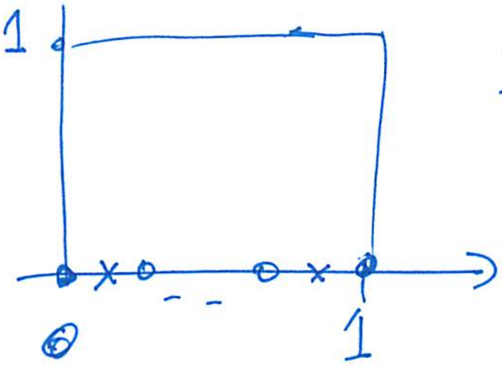
$\dots \leq L^k|x_0 - x^*| \to 0$ as $k \to \infty$ since $L<1$.

When does (*) converse?

<u>Defn</u> (contraction mapping)

A fn $g : [a, b] \to \mathbb{R}$ is called a contraction

if $\cancel{|g(x) - g(y)| \le L|x - y| \quad \forall x, y \in [}$



$$I_0 = [0, 1]$$

$$I_1 = g(I_0) \subseteq I_0$$

$$\operatorname{diam}(I_1) \leq L \operatorname{diam}(I_0)$$

Generally

$$I_{k+1} = g(I_k) \subseteq I_k$$

$$\operatorname{diam}(I_k) \leq L \operatorname{diam}(I_{k-1})$$
$$\vdots$$
$$\leq L^k \operatorname{diam}(I_0)$$
$$\to 0.$$

Maps sets into smaller subsets.

x^* is an attracting point for <u>every</u> sequence.