Conformal Mapping - Find new flows given a known flow.

Suppose $w(z)$ is the complex potential of a flow in some part of the z-plane.

If $z = f(\zeta)$, $\zeta = \xi + i\eta$, f analytic, then w is also an analytic function of ζ.

Thus, $w(f(\zeta)) = \bar{w}(\zeta)$ is the complex potential of some region of the ζ plane.

The flow in the z-plane is said to be <u>tf'ed</u> into flow in the ζ-plane.

Equi-potential lines & streamlines in z-plane $\rightarrow$ same in ζ, $\perp$ except at singular points of the tf.

Velocity components:

$$u_\xi - i u_\eta = \frac{d\bar{w}}{d\zeta} = \frac{dw}{dz}\frac{dz}{d\zeta}$$

If $w(z) \sim \frac{m - i\Gamma}{2\pi} \log(z - z_0)$ near $z = z_0$

then $\bar{w}(\zeta) \sim \frac{m - i\Gamma}{2\pi} \log(\zeta - \zeta_0)$ near $\zeta = \zeta_0$

(similarly with modification for other flow singularities).

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An elliptic cylinder in translational motion.
(Batchelor, p. 427)

Conformal tf. of the exterior of an ellipse, in the z-plane, to the exterior of a circle in the ζ-plane:

(1) $$z = \zeta + \frac{\lambda^2}{\zeta}, \quad \lambda \text{ real}, \quad z \sim \zeta \text{ for large } \zeta.$$

$$= \cancel{\zeta\left(1 + \frac{\lambda^2}{|\zeta|}\right)}$$

$$= \zeta + \frac{\lambda^2}{|\zeta|^2}\bar{\zeta}, \quad \zeta = \xi + i\eta$$

$$\Rightarrow \quad x = \xi\left(1 + \frac{\lambda^2}{|\zeta|^2}\right), \quad y = \eta\left(1 - \frac{\lambda^2}{|\zeta|^2}\right)$$

If $|\zeta| = c$ (circle of radius c centered at origin)

$$\frac{x^2}{c^2\left(1+\frac{\lambda^2}{c^2}\right)^2} + \frac{y^2}{c^2\left(1-\frac{\lambda^2}{c^2}\right)^2} = 1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$c = \tfrac{1}{2}(a+b), \quad \lambda = \tfrac{1}{2}(a^2-b^2)^{1/2}, \quad \text{choice is } a > b$$

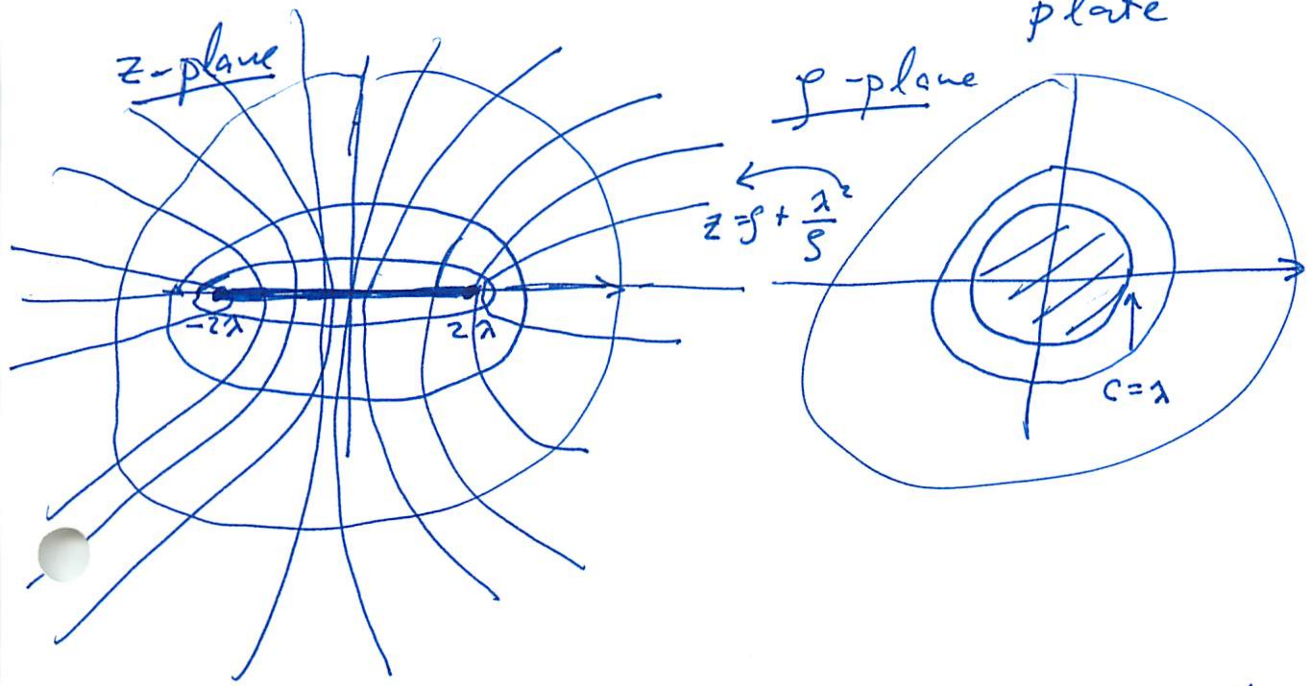
$$\frac{b}{a} = \frac{1-(\lambda/c)^2}{1+(\lambda/c)^2},$$ choice of λ/c determines aspect ratio.

Fixed λ

$\frac{b}{a} \to 1$ as $c \to \infty$

circles

$\sigma \to 0$ as $c \to \lambda$, $a \to 2\lambda$, $b \to 0$, the "flat" plate



Inverse tf:

$$\zeta = \frac{1}{2} z + \frac{1}{2}\left(z^2 - 4\lambda^2\right)^{1/2} \qquad (2)$$

branch cut at $y = 0$. Take branch which gives positive value of $(z^2 - 4\lambda^2)^{1/2}$ for $x > 2\lambda$, $y = 0$

(~~[illegible]~~ Other branch maps z-plane outside of ellipse into interior of $|\zeta| = c$).

Check this.

Tf's (1) & (2) has been used ~~in more complicated~~ ~~[illegible]~~ to transform circles to more complicated shapes.

Flow in the ζ-plane:

A circular cylinder of radius c held in a stream of uniform velocity $(-U,-V)$ at infty, w. circulation Γ

(3) $$W(\zeta) = \underbrace{-(U-iV)\zeta - (U+iV)\frac{c^2}{\zeta}}_{\text{the circle thm}} + \frac{\Gamma}{2\pi i}\ln\frac{\zeta}{c}$$

Eqs (3) + (1) given (parametrically) the flow around the elliptic cylinder with velocity $(-U,-V)$ at ∞, & circulation Γ.

Let $\zeta = \sigma e^{i\nu}$, $U+iV = |\underline{U}|e^{-i\alpha}$

$$x = \sigma\left(1+\frac{\lambda^2}{\sigma^2}\right)\cos\nu, \quad y = \sigma\left(1-\frac{\lambda^2}{\sigma^2}\right)\sin\nu$$

Eq (3) becomes

$$w(\zeta) = -(U^2+V^2)^{1/2}\left\{\sigma e^{i(\nu+\alpha)} + \frac{c^2}{\sigma}e^{-i(\nu+\alpha)}\right\} + \frac{\Gamma}{2\pi}\left(\ln\frac{\sigma}{c} + i\nu\right)$$

$$\Rightarrow \quad \phi = -(U^2+V^2)^{1/2}\left(\sigma + \frac{c^2}{\sigma}\right)\cos(\nu+\alpha) + \frac{\Gamma}{2\pi}\nu$$

$$\psi = -(U^2+V^2)^{1/2}\left(\sigma - \frac{c^2}{\sigma}\right)\sin(\nu+\alpha) - \frac{\Gamma}{2\pi}\log\frac{\sigma}{c}$$

If $\psi = K$ along a streamline in the ζ-plane

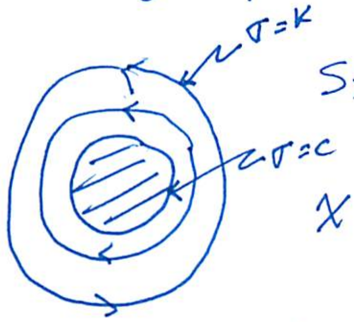
$K = R(\nu, \sigma)$, locally $\sigma = \sigma(\nu; K)$

In the z-plane

$(x(\sigma,\nu), y(\sigma,\nu))$ traces the stream-line.

<u>Ex:</u> If $(U, V) \equiv \underset{\sim}{0}$, i.e. only circulatory flows, $\Gamma \neq 0$.

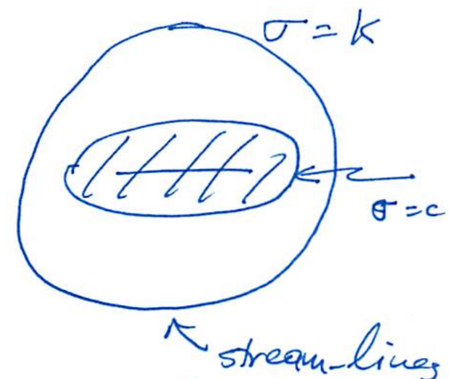
In ζ-plane, stream-lines are circles



Stream-lines: $\sigma = K$

$$x = K\left(1 + \frac{\lambda^2}{K^2}\right)\cos\nu$$

$$y = K\left(1 - \frac{\lambda^2}{K^2}\right)\sin\nu$$



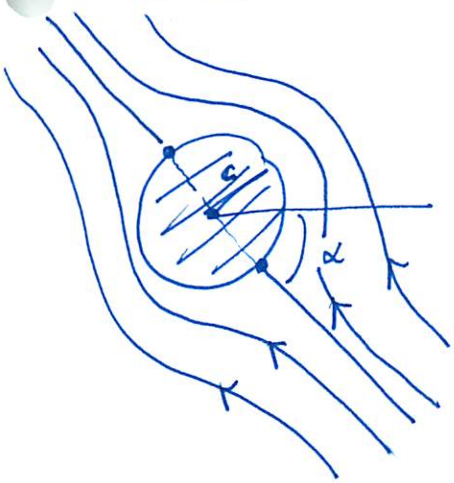
streamlines are "con-focal" ellipses.

Circulation of the ellipse

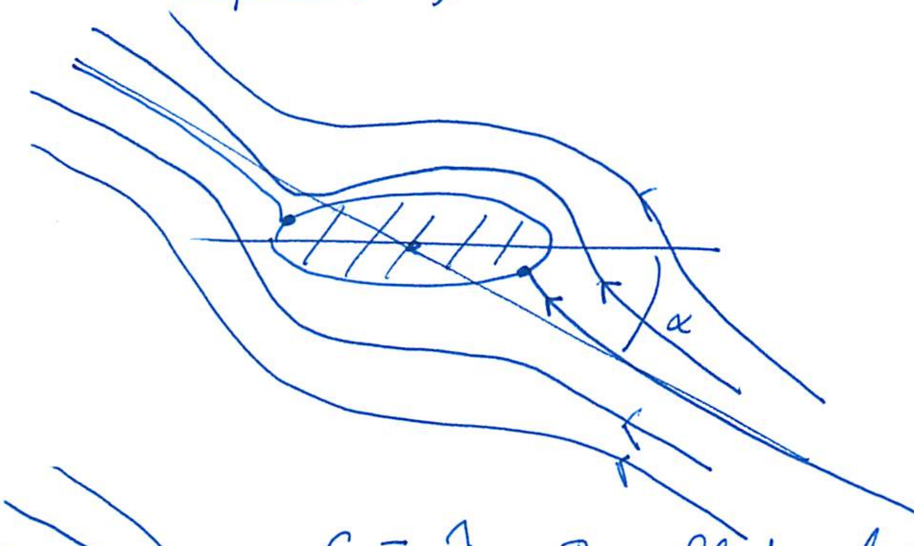
$$C = \oint \frac{dw}{dz}dz = \oint_{\sigma=c} \frac{dw}{d\zeta}d\zeta = \Gamma.$$

$\Gamma = 0$

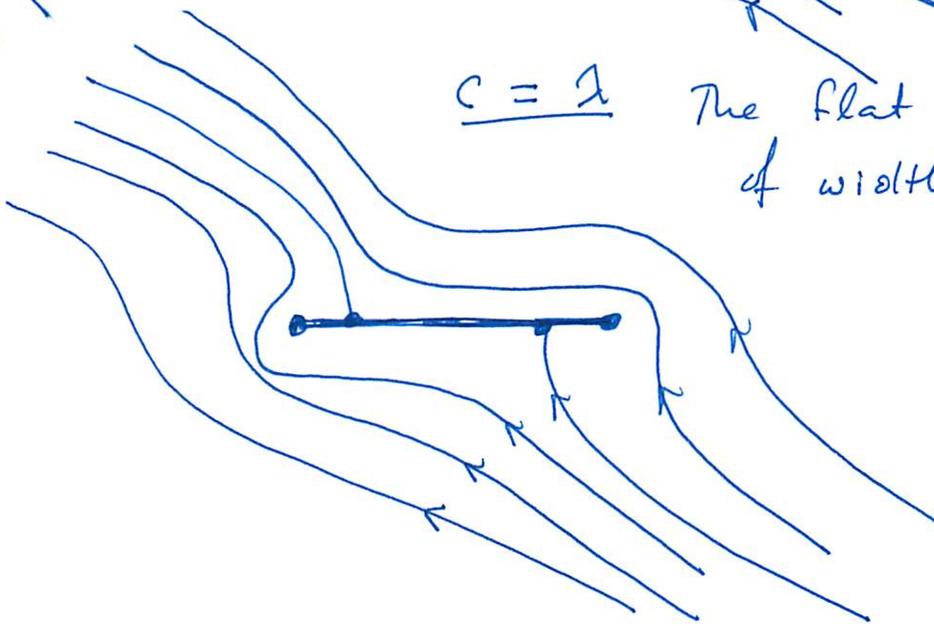
ζ-plane



z-plane , $c > \lambda$



$c = \lambda$ The flat plate of width 4λ.



Velocity on the cylinder surface

$$u - iv = \left(\frac{dw}{d\xi} \Big/ \frac{dz}{d\xi} \right) \Bigg|_{\mu = c}$$

$$= \frac{-i(U^2+V^2)^{1/2}(a+b)\sin(\nu+\alpha) - i\Gamma/2\pi}{i a \sin\nu + b\cos\nu}$$

$\underline{\Gamma = 0}$: $\quad \nu = -\alpha, \ \nu = \pi - \alpha$ stagnation points

$\underline{\Gamma \neq 0}$ As before, the stagnation points shift, with positions on the elliptic cylineder calculable.

$\underline{b = 0,\ c = \lambda}$ The flat plate
$a = 2\lambda$

$$u - iv\Big|_{\text{plate}} = \frac{-i(U^2+V^2)^{1/2}\, 2\lambda \sin(\nu+\alpha) - i\Gamma/2\pi}{2i\lambda \sin\nu}$$

<u>Note:</u> $\mu = 0$

Generally an infinite ~~value~~ velocity for $\nu = 0, \pi$

Flow around a tip or sharp corner generally produces a divergence.

However, note that Γ can be chosen so as to move the stagnation points to the ends, ~~and make~~ to make the flow smooth at the "trailing ~~ede~~ edge" $\nu = \pi$.

Req. that u is finite:

$$u = \frac{-(U^2+V^2)^{1/2}\, a \,\sin(\nu+\alpha) - \Gamma/2\pi}{a \sin \nu}, \quad v = 0.$$

~~$u\big|_{\nu=\pi} = (U^2+V^2)^{1/2}$~~

If we choose

$$\Gamma = 2\pi a (U^2+V^2)^{1/2} \sin\alpha \quad > 0$$

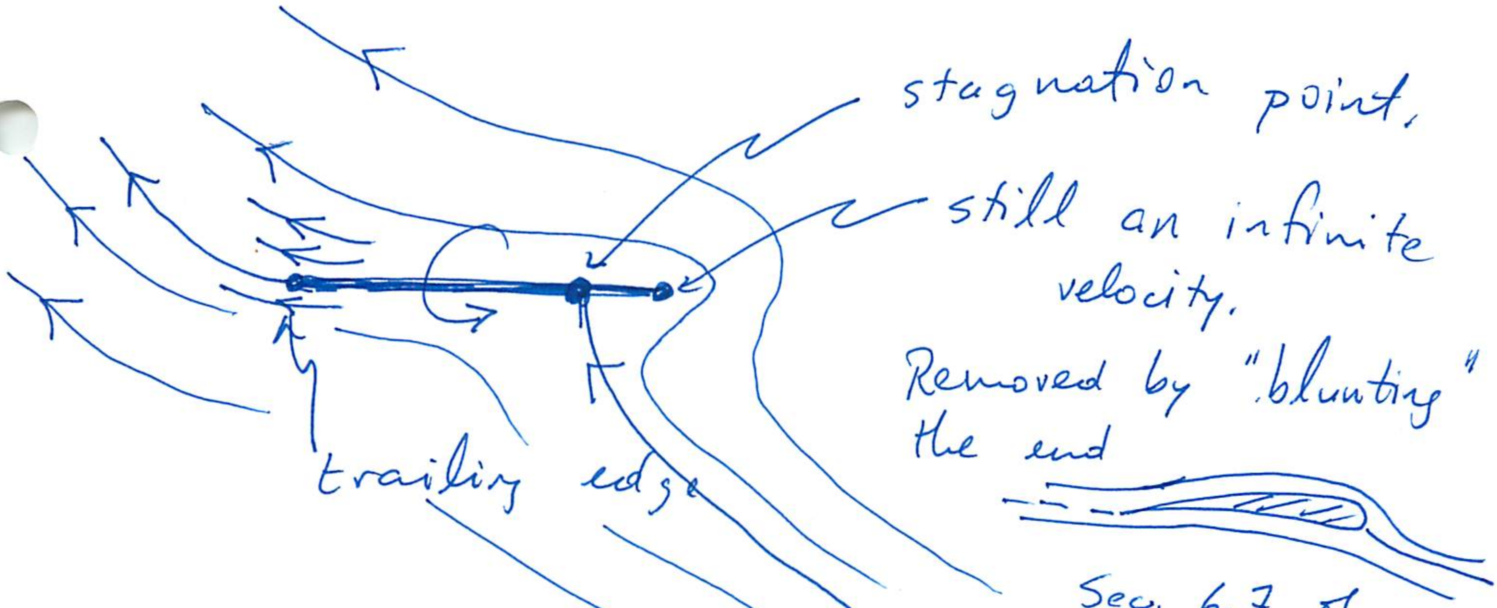
we have cancelling zeroes

$$u = -\frac{a(U^2+V^2)^{1/2}}{a}\left[\frac{\sin(\nu+\alpha)+\sin\alpha}{\sin\nu}\right]$$

$$\lim_{\nu\to\pi} \frac{\sin(\nu+\alpha)+\sin\alpha}{\sin\nu} = \lim_{\nu\to\pi}\frac{\cos(\nu+\alpha)}{\cos\nu}$$

$$= \frac{\cos(\alpha+\pi)}{\cos\pi} = \cos\alpha$$

$$u \to -(U^2+V^2)^{1/2}\cos\alpha$$



Sec. 6.7 of Batchelor.

<u>The point:</u> Asking that the flow depart smoothly at the trailing edge generates a circulation, and hence lift.

Called the <u>Joukowski</u> <u>condition.</u>

~~The~~

In practice: In steady motion; viscosity acting in the <u>bdy</u> <u>layer</u> generates the circulation around the "air-foil" necessary to remove the velocity singularity at the trailing edge, and to thereafter ignore the effects of viscosity.

Called <u>Joukowski's</u> <u>Hypothesis</u>

Justified (at least qualitatively) by boundary layer thry.