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Rigid-Body Motion

Consider a collection of mass points connected by springs for which the equations of motion are

$$M_k \frac{d\underline{U}_k}{dt} = \underline{F}_k + \sum_{j \in N(k)} T_{jk} \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|} \tag{1}$$

$$\frac{d\underline{X}_k}{dt} = \underline{U}_k \tag{2}$$

$k = 1 \dots n$

where

$$T_{jk} = T_{kj} \tag{3}$$

The total mass and center of mass of the system, as well as the velocity of the center of mass are defined as follows

(4) $$M = \sum_{k=1}^{n} M_k$$

(5) $$M \underline{X}_{CM} = \sum_{k=1}^{n} M_k \underline{X}_k$$

(6) $$M \underline{U}_{CM} = \sum_{k=1}^{n} M_k \underline{U}_k$$

and we immediately have

(7) $$\frac{d\underline{X}_{CM}}{dt} = \underline{U}_{CM}$$

If we sum equation (1) over $k = 1 \cdots n$, the terms involving T_{jk} become

(8) $$\sum_{k=1}^{n} \sum_{j \in N(k)} T_{jk} \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|} = 0$$

since each pair (j,k) appears twice with the same magnitude and opposite sign.

Thus, we are left with

(9) $$M \frac{d\underline{U}_{CM}}{dt} = \underline{F} = \sum_{k=1}^{n} \underline{F}_k$$

Together with (7), this is the equation of motion for the center of mass.
Note that the spring forces are not involved.

Let $\underline{L}$ be the angular momentum of the system about its center of mass
This is defined by

(10) $$\underline{L}(t) = \sum_{k=1}^{n} M_k (\underline{X}_k - \underline{X}_{CM}) \times \underline{U}_k$$

Note that

(11) $$0 = \sum_{k=1}^{n} M_k (\underline{X}_k - \underline{X}_{CM}) \times \underline{U}_{CM}$$

since U_{CM} factors out of the sum, and then the result is zero by equations (4-5).

Subtracting (11) from (10), we get

$$(12) \qquad \underline{L}(t) = \sum_{k=1}^{n} M_k \left(\underline{X}_k - \underline{X}_{CM} \right) \times \left(\underline{U}_k - \underline{U}_{CM} \right)$$

If we differentiate both sides of this equation with respect to t, the term on the right-hand side that comes from differentiating $\underline{X}_k - \underline{X}_{CM}$ is zero, since

$$(13) \qquad \left(\underline{U}_k - \underline{U}_{CM} \right) \times \left(\underline{U}_k - \underline{U}_{CM} \right) = 0$$

Therefore

$$(14) \qquad \frac{d\underline{L}}{dt} = \sum_{k=1}^{n} \left(\underline{X}_k - \underline{X}_{CM} \right) \times \left(M_k \frac{d\underline{U}_k}{dt} - M_k \frac{d\underline{U}_{CM}}{dt} \right)$$

$$= \sum_{k=1}^{n} \left(\underline{X}_k - \underline{X}_{CM} \right) \times \underline{F}_k$$

$$+ \sum_{k=1}^{n} \sum_{j \in N(k)} T_{jk} \frac{\left(\underline{X}_k - \underline{X}_{CM} \right) \times \left(\underline{X}_j - \underline{X}_k \right)}{\| \underline{X}_j - \underline{X}_k \|}$$

Since $\underline{X}_{CM}$ does not depend on j or k, and since $\underline{X}_k \times \underline{X}_k = 0$, the last term on the right-hand side of (14) can be written as

$$\sum_{k=1}^{n} \sum_{j \in N(k)} T_{jk} \frac{\underline{X}_k \times \underline{X}_j}{\|\underline{X}_j - \underline{X}_k\|} \tag{15}$$

$$- \underline{X}_{CM} \times \sum_{k=1}^{n} \sum_{j \in N(k)} T_{jk} \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|}$$

and each of these terms is separately equal to zero because each pair j, k occurs twice with the same magnitude but opposite sign. Thus, we conclude that

$$\frac{d\underline{L}}{dt} = \sum_{k=1}^{n} \left(\underline{X}_k - \underline{X}_{CM}\right) \times \underline{F}_k \tag{16}$$

$$= \underline{\tau}$$

where $\underline{\tau}$ is the total torque on the system.

Note that $\underline{L}$ and $\underline{\tau}$ are both defined relative to the center of mass.

Thus, in summary

$$(17) \qquad M \frac{d\underline{U}_{CM}}{dt} = \underline{F} = \sum_{k=1}^{n} \underline{F}_k$$

$$(18) \qquad \frac{d\underline{X}_{CM}}{dt} = \underline{U}_{CM}$$

$$(19) \qquad \frac{d\underline{L}}{dt} = \underline{\tau} = \sum_{k=1}^{n} (\underline{X}_k - \underline{X}_{CM}) \times \underline{F}_k$$

where

$$(20) \qquad \underline{L} = \sum_{k=1}^{n} M_k (\underline{X}_k - \underline{X}_{CM}) \times (\underline{U}_k - \underline{U}_{CM})$$

These equations hold <u>whether or not</u> the system is rigid, but when the system is rigid they <u>determine</u> its motion.

The velocities of all of the points of a rigid body are determined at any given time t by the velocity of the center of mass $\underline{U}_{cm}(t)$ and by the angular velocity $\underline{\Omega}(t)$ of the whole body about its center of mass. The individual point velocities are given by

(21) $$\underline{U}_k(t) = \underline{U}_{cm}(t) + \underline{\Omega}(t) \times \left(\underline{X}_k(t) - \underline{X}_{cm}(t)\right)$$

Substituting (21) into (20), we find a relationship between $\underline{L}(t)$ and $\underline{\Omega}(t)$, namely

(22) $$\underline{L}(t) = \sum_{k=1}^{n} M_k \left(\underline{X}_k(t) - \underline{X}_{cm}(t)\right) \times \left(\underline{\Omega}(t) \times \left(\underline{X}_k(t) - \underline{X}_{cm}(t)\right)\right)$$

Now we need the vector identity

(23) $$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b})$$

This gives

(24) $$\underline{L}(t) = \sum_{k=1}^{n} M_k \|\underline{X}_k(t) - \underline{X}_{CM}(t)\|^2 \, \underline{\Omega}(t)$$

$$- \sum_{k=1}^{n} M_k \left(\underline{X}_k(t) - \underline{X}_{CM}(t)\right)\left(\underline{X}_k(t) - \underline{X}_{CM}(t)\right) \cdot \underline{\Omega}(t)$$

In matrix notation, with the vectors as column vectors, this can be written as

(25) $$\underline{L}(t) = I(t)\,\underline{\Omega}(t)$$

where $I(t)$ is the following 3×3 symmetric matrix called the moment of inertia tensor:

(26) $$I(t) = \sum_{k=1}^{n} M_k \left(\|\underline{\tilde{X}}_k(t)\|^2 E - \underline{\tilde{X}}_k(t) \left(\underline{\tilde{X}}_k(t)\right)^T \right)$$

with

(27) $$\underline{\tilde{X}}_k(t) = \underline{X}_k(t) - \underline{X}_{CM}(t)$$

Here E is the 3×3 identity matrix, and the superscript ^{T} denotes the transpose of a matrix.

In components,

$$I = \sum_{k=1}^{n} M_k \begin{pmatrix} (\tilde{X}_{k2}^2 + \tilde{X}_{k3}^2) & -\tilde{X}_{k1}\tilde{X}_{k2} & -\tilde{X}_{k1}\tilde{X}_{k3} \\ -\tilde{X}_{k2}\tilde{X}_{k1} & (\tilde{X}_{k1}^2 + \tilde{X}_{k3}^2) & -\tilde{X}_{k2}\tilde{X}_{k3} \\ -\tilde{X}_{k3}\tilde{X}_{k1} & -\tilde{X}_{k3}\tilde{X}_{k2} & (\tilde{X}_{k1}^2 + \tilde{X}_{k2}^2) \end{pmatrix} \tag{28}$$

If the vectors $\tilde{\underline{X}}_k$ are not all co-linear, then the matrix I is positive definite. To prove this, let $\underline{V} \neq 0$ be an arbitrary 3-vector and consider $\underline{V}^T I \underline{V}$. From (26),

$$\underline{V}^T I \underline{V} = \sum_{k=1}^{n} M_k \left(\|\tilde{\underline{X}}_k\|^2 \|\underline{V}\|^2 - \left(\underline{V}^T \tilde{\underline{X}}_k\right)^2 \right) \tag{29}$$

By the Schwarz inequality, every term in the above sum is non-negative and is only equal to zero if $\underline{V}$ is a multiple of $\tilde{\underline{X}}_k$. By hypothesis, the $\tilde{\underline{X}}_k$ are not all colinear, so there must be at least one term in the sum that is strictly positive. Thus $\underline{V}^T I \underline{V} > 0$ for any nonzero $\underline{V}$, so I is positive definite.

It is interesting that the vectors $\tilde{\underline{X}}_1 \cdots \tilde{\underline{X}}_n$ do not need to span $\mathbb{R}^3$ for the matrix I to be positive definite; it is enough that they span a plane. Note, however, that this requires at least three mass points. If there are only two mass points, then the center of mass lies between them, and the vectors $\tilde{\underline{X}}_1$ and $\tilde{\underline{X}}_2$ are colinear.

We can now formulate a simple numerical method for tracking the motion of a rigid body under the influence of applied forces. We work with the variables

(30) $$\underline{X}_{cm}(t),\ \underline{U}_{cm}(t),\ \cdots \tilde{\underline{X}}_k(t) \cdots,\ \underline{L}(t)$$

Given $\cdots \tilde{\underline{X}}_k(t) \cdots$, we can evaluate $I(t)$ from (26), and then we can solve the 3x3 system

(37) $$I(t)\,\underline{\Omega}(t) = \underline{L}(t)$$

for $\underline{\Omega}(t)$. With $\underline{\Omega}(t)$ known, we update $\cdots \tilde{\underline{X}}_k(t) \cdots$ according to

$$\tilde{\underline{X}}_k(t+\Delta t) = R(\underline{\Omega}(t), \Delta t)\, \tilde{\underline{X}}_k(t) \tag{38}$$

in which $R(\underline{\Omega}, \Delta t)$ denotes an exact rotation about the origin at angular velocity $\underline{\Omega}$ for a time interval of duration Δt. This will be written out more explicitly below.

The center of mass is updated via

$$\underline{X}_{cm}(t+\Delta t) = \underline{X}_{cm}(t) + (\Delta t)\, \underline{U}_{cm}(t) \tag{39}$$

Finally, we update the velocity of the center of mass and the angular momentum according to

$$M \frac{\underline{U}_{cm}(t+\Delta t) - \underline{U}_{cm}(t)}{\Delta t} = \sum_{k=1}^{n} \underline{F}_k(t+\Delta t) \tag{40}$$

$$\frac{\underline{\underline{L}}(t+\Delta t) - \underline{\underline{L}}(t)}{\Delta t} = \sum_{k=1}^{n} \tilde{\underline{X}}_k(t+\Delta t) \times \underline{F}_k(t+\Delta t) \tag{41}$$

and this completes the time step.

A crucial feature of the foregoing scheme is that we evaluate $\underline{\Omega}$ as needed from $\underline{L}$. This is very easy to do computationally, since it only requires solving three equations in three unknowns with a matrix that is symmetric and positive definite. By doing this, we avoid having to work with the differential equation for $\underline{\Omega}(t)$, which is much more complicated than the differential equation for $\underline{L}(t)$.

We still need to construct the 3×3 matrix $R(\underline{\Omega}, \Delta t)$. Let

(42) $$P(\underline{\Omega}) = \frac{\underline{\Omega}}{\|\underline{\Omega}\|}\left(\frac{\underline{\Omega}}{\|\underline{\Omega}\|}\right)^T$$

so that $P(\underline{\Omega})$ is the orthogonal projection onto the direction of $\underline{\Omega}$, and $E - P(\underline{\Omega})$ is the orthogonal projection onto the plane through the origin normal to $\underline{\Omega}$. (Recall that E is the 3×3 identity matrix.)

Also let $(\underline{\Omega}\times)$ be the 3×3 matrix such that

(43) $$(\underline{\Omega}\times)\underline{X} = \underline{\Omega}\times\underline{X}$$

Writing out the components of $\underline{\Omega}\times\underline{X}$, we have

(44) $$(\underline{\Omega}\times\underline{X})_1 = \Omega_2 X_3 - \Omega_3 X_2$$

(45) $$(\underline{\Omega}\times\underline{X})_2 = \Omega_3 X_1 - \Omega_1 X_3$$

(46) $$(\underline{\Omega}\times\underline{X})_3 = \Omega_1 X_2 - \Omega_2 X_1$$

and therefore

(47) $$(\underline{\Omega}\times) = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

Note that $(\underline{\Omega}\times)$ is an antisymmetric matrix.

Now given any $\underline{X}$ to which $R(\underline{\Omega}, \Delta t)$ should be applied, we first write

$$(48) \qquad \underline{X} = P(\underline{\Omega})\underline{X} + (E - P(\underline{\Omega}))\underline{X}$$

The term $P(\underline{\Omega})\underline{X}$ is unaffected by $R(\underline{\Omega}, \Delta t)$, but the term $(E - P(\underline{\Omega}))\underline{X}$ should be rotated through an angle $\|\underline{\Omega}\|\Delta t$ within the plane through the origin that is normal to $\underline{\Omega}$. To express this, it is useful to have a vector within that plane orthogonal to $(E - P(\underline{\Omega}))\underline{X}$ and having the same length as $(E - P(\underline{\Omega}))\underline{X}$. Such a vector is

$$(49) \qquad \frac{\underline{\Omega}}{\|\underline{\Omega}\|} \times (E - P(\underline{\Omega}))\underline{X}$$

but note that this can be written more simply as

$$(50) \qquad \frac{\underline{\Omega}}{\|\underline{\Omega}\|} \times \underline{X}$$

~~since~~ since $\underline{\Omega} \times \underline{\Omega} = 0$

Putting everything together, we therefore have

(51) $$R(\underline{\Omega}, \Delta t)\, \underline{X}$$
$$= P(\underline{\Omega})\underline{X} + \cos(\|\underline{\Omega}\|\Delta t)\left(E - P(\underline{\Omega})\right)\underline{X} + \sin(\|\underline{\Omega}\|\Delta t)\frac{\underline{\Omega}}{\|\underline{\Omega}\|} \times \underline{X}$$

or, since $\underline{X}$ is arbitrary

(52) $$R(\underline{\Omega}, \Delta t) = P(\underline{\Omega}) + \cos(\|\underline{\Omega}\|\Delta t)\left(E - P(\underline{\Omega})\right) + \sin(\|\underline{\Omega}\|\Delta t)\left(\frac{\underline{\Omega}}{\|\underline{\Omega}\|} \times\right)$$

As a check on the foregoing, we set

(53) $$\underline{X}(t) = R(\underline{\Omega}, t)\,\underline{X}(0)$$

with $\underline{\Omega}$ constant. We need to show that

(54) $$\frac{d\underline{X}}{dt}(t) = \underline{\Omega} \times \underline{X}(t)$$

We have

(55) $$\underline{X}(t) = P(\underline{\Omega})\underline{X}(0) + \cos(\|\underline{\Omega}\|t)(E - P(\underline{\Omega}))\underline{X}(0) + \frac{\sin(\|\underline{\Omega}\|t)}{\|\underline{\Omega}\|}\,\underline{\Omega} \times \underline{X}(0)$$

and therefore

(56) $$\underline{\Omega} \times \underline{X}(t) = \cos(\|\underline{\Omega}\|t)(\underline{\Omega} \times \underline{X}(0)) + \frac{\sin(\|\underline{\Omega}\|t)}{\|\underline{\Omega}\|}\,\underline{\Omega} \times (\underline{\Omega} \times \underline{X}(0))$$

By making use of (23), we can rewrite this as

(57) $$\underline{\Omega} \times \underline{X}(t) = \cos(\|\underline{\Omega}\|t)(\underline{\Omega} \times \underline{X}(0))$$

$$+ \frac{\sin(\|\underline{\Omega}\|t)}{\|\underline{\Omega}\|}\left(\underline{\Omega}(\underline{\Omega}\cdot\underline{X}(0)) - \|\underline{\Omega}\|^2 \underline{X}(0)\right)$$

$$= \cos(\|\underline{\Omega}\|t)(\underline{\Omega} \times \underline{X}(0))$$

$$- \|\underline{\Omega}\| \sin(\|\underline{\Omega}\|t)(E - P(\underline{\Omega}))\underline{X}(0)$$

On the other hand, by differentiating with respect to t in (55), we get

(58) $$\frac{d\underline{X}}{dt}(t) = -\|\underline{\Omega}\| \sin(\|\underline{\Omega}\|t)(E - P(\underline{\Omega}))\underline{X}(0)$$

$$+ \cos(\|\underline{\Omega}\|t)(\underline{\Omega} \times \underline{X}(0))$$

Comparison of (57) & (58) shows that (54) is indeed satisfied. Note also that $R(\underline{\Omega}, 0) = E$. This confirms the interpretation claimed above for $R(\underline{\Omega}, \Delta t)$.

An important remark about the foregoing scheme is that we do not really need to know all of the details about the distribution of mass within the body, since all that matters is the total mass M and the moment of inertia tensor $I(t)$.

The matrix $I(t)$ can be updated in the following way. From (26) evaluated at time $t+\Delta t$, we have

(59) $$I(t+\Delta t) = \sum_{k=1}^{n} M_k \Big(\|\underline{\tilde{X}}_k(t+\Delta t)\|^2 E - \underline{\tilde{X}}_k(t+\Delta t)\big(\underline{\tilde{X}}_k(t+\Delta t)\big)^T \Big)$$

From (38), we can express $\underline{\tilde{X}}_k(t+\Delta t)$ in terms of $\underline{\tilde{X}}_k(t)$. Since $R(\Omega(t), \Delta t)$ is a rotation (and hence is given by an orthogonal matrix), we have

(60) $$\|\underline{\tilde{X}}_k(t+\Delta t)\| = \|\underline{\tilde{X}}_k(t)\|$$

and also

(61) $$R(\underline{\Omega}(t), \Delta t)\, R^T(\underline{\Omega}(t), \Delta t) = E$$

It therefore follows from (59) & (38) that

(62) $$I(t+\Delta t) = R(\underline{\Omega}(t), \Delta t)\, I(t)\, R^T(\underline{\Omega}(t), \Delta t)$$

If n is large, this is a much less expensive way to update $I(t)$ than to move all of the individual points and then to evaluate $I(t+\Delta t)$ directly from (59).

Note, however, that we still need to update at least the positions of those points ~~where~~ at which forces are applied, both because the forces themselves may depend on the positions of those points, and also because the torque is position dependent even if the force is not.

A further simplification can be achieved by using coordinates that are attached to the body. In such coordinates $I(t)$ is actually independent of t, and it can also be made diagonal by the proper choice of axes. This is especially useful for studying the free motions of an isolated rigid body, but the formulation given here, in which everything is done in a fixed system of coordinates (the laboratory frame of reference) seems better adapted to the situation in which the rigid body is part of a larger system possibly involving other rigid bodies or non-rigid components.