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Backward-Euler method for the structural mechanics problem with Hookean springs only (no dashpots)

$$M_k \frac{\underline{U}_k(t) - \underline{U}_k(t-\Delta t)}{\Delta t}$$

$$= \sum_{j \in N(k)} T_{jk}(t) \frac{\underline{X}_j(t) - \underline{X}_k(t)}{\|\underline{X}_j(t) - \underline{X}_k(t)\|}$$

$$\frac{\underline{X}_k(t) - \underline{X}_k(t-\Delta t)}{\Delta t} = \underline{U}_k(t)$$

$k = 1 \cdots n$

where

$$T_{jk}(t) = S_{jk}\left(\|\underline{X}_j(t) - \underline{X}_k(t)\| - R^0_{jk}\right)$$

Solve for $\cdots \underline{X}_k(t) \cdots, \cdots \underline{U}_k(t) \cdots$

by Newton's method

Eliminating $\underline{U}_k(t)$, we can put the above problem in the form

$$(*) \quad M_k(\underline{X}_k - \underline{Z}_k) - (\Delta t)^2 \sum_{j \in N(k)} T_{jk} \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|} = 0$$

where $\underline{X}_k = \underline{X}_k(t)$ (unknown)

$\underline{Z}_k = \underline{X}_k(t-\Delta t) + \Delta t\, \underline{U}_k(t-\Delta t)$ (known)

and

$$T_{jk} = S_{jk}\left(\|\underline{X}_j - \underline{X}_k\| - R^0_{jk}\right)$$

We now have a system of $3n$ equations in the $3n$ unknowns $\underline{X}_1 \cdots \underline{X}_n$

We solve the above system by Newton's method. Given a guess $\ldots \underline{X}_k^{(m)} \ldots$ we seek

$$\ldots \delta \underline{X}_k^{(m)} \ldots$$

so that

$$\underline{X}_k^{(m+1)} = \underline{X}_k^{(m)} + \delta \underline{X}_k^{(m)}$$

satisfies (*) to first order in $\ldots \delta \underline{X}_k^{(m)} \ldots$

We need to evaluate δT_{jk} and

$$\delta \frac{\underline{X}_j - \underline{X}_k}{\| \underline{X}_j - \underline{X}_k \|} \quad :$$

We have

$$\delta T_{jk} = S_{jk} \, \delta \| \underline{X}_j - \underline{X}_k \|$$

$$= S_{jk} \frac{\underline{X}_j - \underline{X}_k}{\| \underline{X}_j - \underline{X}_k \|} \cdot \left(\delta \underline{X}_j - \delta \underline{X}_k \right)$$

and therefore

$$\delta T_{jk} \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|} = S_{jk} P_{jk} \left(\delta \underline{X}_j - \delta \underline{X}_k\right)$$

where, $\overbrace{P_{jk}}^{\text{for each } jk}$ is the 3x3 matrix

representing projection onto the link direction

$$P_{jk} = \left(\frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|}\right)\left(\frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|}\right)^T$$

Note that $P_{jk} = P_{kj}$

Also

$$\delta \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|} =$$

$$\frac{\delta \underline{X}_j - \delta \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|} - \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|^2} \delta \|\underline{X}_j - \underline{X}_k\|$$

$$= \frac{\delta \underline{X}_j - \delta \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|} - \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|^2} \frac{\underline{X}_j - \underline{X}_k}{\|\underline{X}_j - \underline{X}_k\|} \cdot (\delta \underline{X}_j - \delta \underline{X}_k)$$

$$= \frac{1}{\|\underline{X}_j - \underline{X}_k\|} (I_3 - P_{jk})(\delta \underline{X}_j - \delta \underline{X}_k)$$

Note that $I_3 - P_{jk}$ is the 3×3 matrix representing projection onto the plane perpendicular to the link (j,k).

Thus

$$\delta\left(T_{jk}\frac{\underline{X}_j-\underline{X}_k}{\|\underline{X}_j-\underline{X}_k\|}\right)$$

$$=\left(\underbrace{S_{jk}P_{jk}}_{\text{stretching}}+\underbrace{\frac{T_{jk}}{\|\underline{X}_j-\underline{X}_k\|}(I_3-P_{jk})}_{\text{rotation}}\right)\left(\delta\underline{X}_j-\delta\underline{X}_k\right)$$

Now we substitute

$$\underline{X}_k^{(m)}+\delta\underline{X}_k^{(m)} \quad \text{into } (*)$$

and get

$$M_k\left(\underline{X}_k^{(m)}-\underline{Z}_k\right)-(\Delta t)^2\sum_{j\in N(k)} T_{jk}^{(m)}\frac{\underline{X}_j^{(m)}-\underline{X}_k^{(m)}}{\|\underline{X}_j^{(m)}-\underline{X}_k^{(m)}\|}$$

$$+M_k\,\delta\underline{X}_k^{(m)}$$

$$-(\Delta t)^2\sum_{j\in N(k)}\left(S_{jk}P_{jk}^{(m)}+\frac{T_{jk}^{(m)}}{\|\underline{X}_j^{(m)}-\underline{X}_k^{(m)}\|}\left(I_3-P_{jk}^{(m)}\right)\right)$$

$$\left(\delta\underline{X}_j^{(m)}-\delta\underline{X}_k^{(m)}\right)=0$$

This is of the form

$$\sum_{j=1}^{n} A_{kj}^{(m)} \delta \underline{X}_j^{(m)} = \underline{B}_k^{(m)}$$

for $k = 1 \cdots n$, where

$$\underline{B}_k^{(m)} = -\left(M_k \left(\underline{X}_k^{(m)} - \underline{Z}_k \right) - (\Delta t)^2 \sum_{j \in N(k)} T_{jk}^{(m)} \frac{\underline{X}_j^{(m)} - \underline{X}_k^{(m)}}{\| \underline{X}_j^{(m)} - \underline{X}_k^{(m)} \|} \right)$$

(Note that if $\underline{B}_k^{(m)} = 0$ for all k, the problem is solved.) For $j \neq k$

$$A_{kj}^{(m)} = A_{jk}^{(m)} =$$

$$-(\Delta t)^2 \left(S_{jk} P_{jk}^{(m)} + \frac{T_{jk}^{(m)}}{\| \underline{X}_j^{(m)} - \underline{X}_k^{(m)} \|} \left(I - P_{jk}^{(m)} \right) \right)$$

$$A_{kk}^{(m)} = M_k I_3 - \sum_{j=1}^{n} A_{jk}^{(m)}$$

code to setup the matrix A

```
DX = X(jj, :) - X(kk, :)
R = sqrt(sum(DX.^2, 2))
DXN = DX./[R, R, R]
T = S.*(R - Rzero)
TR = T./R

A = diag(M3) % diagonal matrix with each M
             repeated 3 times on diagonal

for l = 1:lmax
    P = (DXN(l, :))' * DXN(l, :)
    AA = -(dt^2)*(S(l)*P + TR(l)*(eye(3)-P))
    j = (3*jj(l)-2):(3*jj(l))
    k = (3*kk(l)-2):(3*kk(l))
    A(j,k) = AA
    A(k,j) = AA
    A(j,j) = A(j,j) - AA
    A(k,k) = A(k,k) - AA
end
```

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code to set up M3 during initialization:

```
M3 = zeros(3*n, 1)
for j = 1:n
    M3(3*j-2 : 3*j) = M(j)
end
```