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1/27/2021

Orbits of planets and spacecraft

First, consider the orbit of a single planet around a star. Let

M = mass of the star

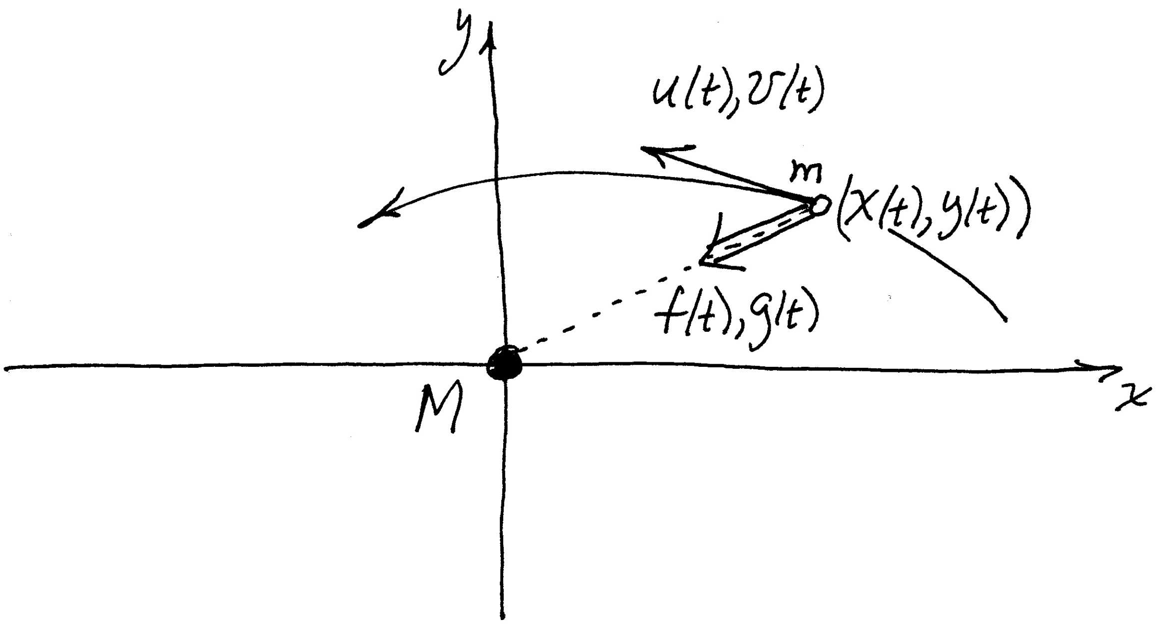
m = mass of the planet

We assume that

$$m << M$$

so we can think of the star as being at rest at the origin.

The orbit takes place in a plane, so we can describe the motion in terms of two coordinates.



Let

$x(t), y(t)$ = position of planet at time t

$u(t), v(t)$ = velocity of planet at time t

Taken together, these four quantities define the <u>state</u> of the planet at time t.

According to Newton's law that force = mass × acceleration, we have the equations

(1) $$m \frac{du}{dt} = f$$

(2) $$m \frac{dv}{dt} = g$$

where (f, g) is the force of gravity acting on the planet. The <u>magnitude</u> of this force is

(3) $$\frac{GMm}{r^2}$$

where G is called the universal gravitational constant, and r is the distance from the star to the planet. By the Pythagorean theorem

(4) $$r(t) = \sqrt{x^2(t) + y^2(t)}$$

To get the vector of the gravitational force acting on the planet, we multiply the magnitude of the gravitational force by the unit vector

(5) $$-\left(\frac{x}{r}, \frac{y}{r}\right)$$

which points from the planet towards the star. Thus, equations (1) & (2) become

(6) $$m\frac{du}{dt} = -\frac{GMm}{r^2}\frac{x}{r}$$

(7) $$m\frac{dv}{dt} = -\frac{GMm}{r^2}\frac{y}{r}$$

Also, by definition of velocity, we have

(8) $$\frac{dx}{dt} = u$$

(9) $$\frac{dy}{dt} = v$$

Note that the mass m of the planet cancels out of its equation of motion (It would appear in the equation of motion of the star, but we are not considering the motion of the star.)

This very important point was first understood by Galileo, who was concerned not with planets but with objects falling on Earth. It is said that he demonstrated it by dropping a solid cannon ball and a hollow shell from the leaning tower of Pisa, and the two objects hit the ground at almost the same time. You can do the experiment with two coins, like a quarter and a dime. Hold them up high, one in each hand, and let go at the same time. You will hear <u>one</u> sound as they hit the floor, not counting subsequent bounces.

Equations (6-9) are the <u>equations of motion</u> of the planet. If the state of the planet is known at any time t, then these equations give the rate of change of the state at that time.

Note that (6-9) is a system of <u>differential equations</u> in the unknown functions $x(t)$, $y(t)$, $u(t)$, $v(t)$. (On the right-hand sides, $r(t)$ should be thought of as shorthand for $\sqrt{x^2(t) + y^2(t)}$.) The numerical solution of this system will be described next.

The key to the numerical solution of differential equations is to introduce a time step $\Delta t > 0$. Then, we make the approximation

$$\frac{du}{dt} \approx \frac{u(t+\Delta t) - u(t)}{\Delta t} \tag{10}$$

and similarly for all other variables.

An interesting question is whether we should evaluate the right-hand sides of equations (6-9) at time t or at time $t + \Delta t$. For reasons that will be discussed later, we use time t in equations (6-7) and time $t + \Delta t$ in equations (8-9).

In this way, we arrive at the following <u>numerical scheme</u> for equations (6-9):

$$(11) \qquad \frac{u(t+\Delta t) - u(t)}{\Delta t} = -GM \frac{x(t)}{r^3(t)}$$

$$(12) \qquad \frac{v(t+\Delta t) - v(t)}{\Delta t} = -GM \frac{y(t)}{r^3(t)}$$

$$(13) \qquad \frac{x(t+\Delta t) - x(t)}{\Delta t} = u(t+\Delta t)$$

$$(14) \qquad \frac{y(t+\Delta t) - y(t)}{\Delta t} = v(t+\Delta t)$$

Here, as before, $r(t) = \sqrt{x^2(t) + y^2(t)}$

Note that equations (11–14) involve only arithmetic; there is no longer any calculus involved. Moreover, these equations are easily rearranged into an algorithm for the evaluation of

(15) $$u(t+\Delta t),\ v(t+\Delta t),\ x(t+\Delta t),\ y(t+\Delta t)$$

from the given data

(16) $$u(t),\ v(t),\ x(t),\ y(t)$$

This algorithm (which we will write out below as a Matlab program) can then be applied repeatedly to generate the computed orbit of the planet starting from given initial data

(17) $$u(0),\ v(0),\ x(0),\ y(0)$$

Recall, however, that the numerical scheme (11-14) is only <u>approximately</u> equivalent to the equations of motion (6-9). Intuitively, the approximation gets better as Δt becomes smaller (and then we have to take more steps to cover the same amount of time), but how do we know when Δt is small enough?

The best way to address this issue is to run computations for a sequence of values of Δt, for example

(18) $$\Delta t = (\Delta t)_0 \,,\ \tfrac{1}{2}(\Delta t)_0 \,,\ \tfrac{1}{4}(\Delta t)_0 \,,\ \tfrac{1}{8}(\Delta t)_0 \,, \ldots$$

Note that the number of steps taken will have to be doubled every time that Δt is halved, so that the same total amount of time is covered. Then, by comparing the computed solutions at <u>corresponding physical times</u>, you will be able to see whether the computed solution has settled down to a result that is, for practical purposes, independent of Δt. That is the goal, since Δt has <u>no physical meaning</u>.

It is important to understand that how small Δt needs to be is problem-dependent. A value of Δt that works well in a simulation of the orbit of Jupiter may be too large to use in a simulation of the orbit of Mercury. More generally, when two bodies come close to each other, the gravitational force between them becomes large, and then a small time step is typically needed to resolve their interaction. If you are not careful about this, you will see things happen in your simulations that may look interesting but are not real, like a planet that comes very close to the sun and then picks up so much speed in one time step that it subsequently leaves the solar system. To check whether any simulation result can be trusted, it is important to verify that you can reproduce essentially the same result with the timestep reduced by a factor of two.

Matlab code corresponding to equations (11–14) is as follows (with $dt = \Delta t$)

```
r = sqrt(x^2 + y^2)

u = u - dt * G*M* x/r^3

v = v - dt * G*M* y/r^3

x = x + dt * u

y = y + dt * v
```

Note that "=" here has a different meaning from equality in mathematics

The above statements are executed sequentially, and in each case the expression on the right-hand side is evaluated and then the result is stored in the memory location corresponding to the variable named on the left-hand side. When this happens, whatever was stored in that memory location previously is erased.

Consider, for example, the statement

$$a = a + 1$$

In mathematics, this is an equation with no solution, but in Matlab it is an instruction to increment the value of a by 1.

Notice, too, that the order of operations matters. Since the lines of code that update u and v occur before the lines that update x and y, the values of u, v that are used in the update of x, y are the new values, that is,

$$u(t+\Delta t) \quad , \quad v(t+\Delta t)$$

as in equations (13-14). If we wanted to use the old values $u(t)$, $v(t)$ we would have to save them (i.e., save their values) in some other variables before updating them, so that the old values could still be used.

The code described above does one time step. Here, we put it in a loop, and also save information in arrays for future plotting:

```
for clock = 1:clockmax
   t = clock*dt;
   r = sqrt(x^2+y^2);
   u = u-dt*G*M*x/r^3;
   v = v-dt*G*M*y/r^3;
   x = x+dt*u;
   y = y+dt*v;

   tsave(clock) = t ;
   xsave(clock) = x ;
   ysave(clock) = y ;
end
plot(0,0,'r*', xsave, ysave)
axis equal
```

Here is some explanation:

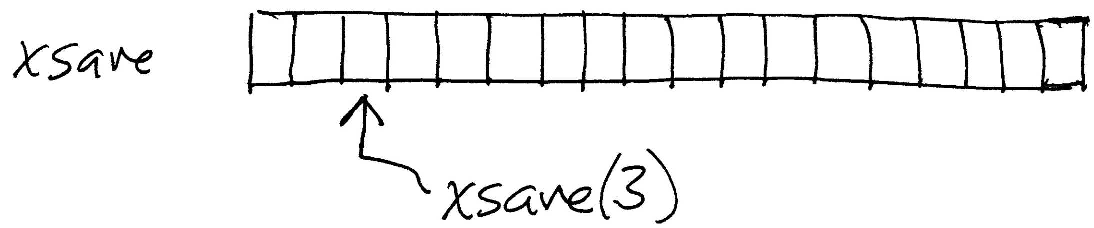
- ```
  for clock = 1:clockmax
      ⋮
  end
  ```

  creates a loop. The statements ⋮ are executed repeatedly, for the variable clock having the values 1, 2, 3, ..., clockmax.

- Note that clock is <u>not</u> time. It is a time-step counter. The actual time is given by t=clock*dt. The present program does not use t, but it is included in case you want to add some feature that depends on time, or plot some quantity as a function of time.

- semicolon at the end of a line prevents output of the result to the command window.
  ```

- tsave, xsave, ysave are one-dimensional arrays. A specific array element is referenced by putting its index in parentheses



- The plot command combines two plot statements into one. The first part plots a red * at the location (0,0)

  The second part draws a line connecting the data points

  (xsave(clock), ysave(clock))

  for clock = 1 : clockmax. This is a drawing of the orbit of the planet.

- The command "axis equal", which is given <u>after</u> the plot command,

  ensures that the same scale is used on the x and y axes. This is very important when plotting orbits. Otherwise a circle will look like an ellipse.

- The above program needs initialization, to define constants and set up initial conditions. This is described next.

  (In the following, "%" indicates the start of a comment.)

Here is an example initialization code:
(with some numbers left out for you to fill in)

```
G=       % Gravitational constant (m^3/(Kg*s^2))
M=       % Mass of Sun (Kg)
res =    % Distance of Earth from Sun (m)
tmax = 365.25*24*60*60 % one year (s)
clockmax = 1000 % number of time steps
dt = tmax/clockmax % duration of time step (s)

xsave = zeros(1, clockmax);
ysave = zeros(1, clockmax);
tsave = zeros(1, clockmax);

sr=      % initial speed relative to speed of circular orbit

x = res
y = 0
u = 0
v = sr * sqrt(G*M/res)
```

By setting sr=1, you should get a circular orbit. Think about what you expect to happen if sr < 1 or sr > 1, and then run the program to find out what actually happens.

Units

You can use any consistent system of units, but be careful not to mix different systems. For example, if G is in International Units, which are based on the kilogram, meter, and second, then length has to be expressed in meters (not kilometers) and velocity in meters/second (not kilometers/second).

You can even choose units that are nice for the problem you are doing. For example, you could set G, M, and r_{ES} all equal to 1. Then the speed of the Earth in its circular orbit would be equal to 1, so the duration of the year would be 2π units of time.

We have seen that it is natural from a programming point of view to use $u(t+\Delta t)$, $v(t+\Delta t)$ in the update of x and y. There is an additional and much deeper reason why this is a good idea, however:

In equations (13-14) we can replace t by $t-\Delta t$, and then these equations become

$$(19) \qquad \frac{x(t) - x(t-\Delta t)}{\Delta t} = u(t)$$

$$(20) \qquad \frac{y(t) - y(t-\Delta t)}{\Delta t} = v(t)$$

which is just another way of writing (13-14)

Now we can use (13) & (19) to eliminate the variable u from equation (11), and similarly, we can (14) & (20) to eliminate the variable v from equation (12). In this way, we get a re-statement of our scheme in terms of x and y alone:

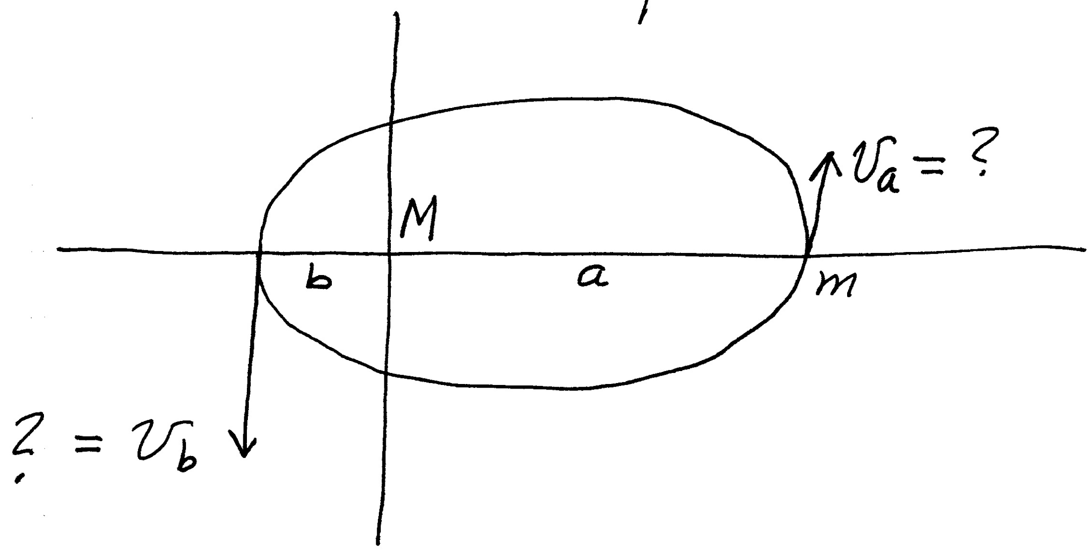
$$\text{(21)} \quad \frac{x(t+\Delta t) - 2x(t) + x(t-\Delta t)}{(\Delta t)^2} = -GM \frac{x(t)}{r^3(t)}$$

$$\text{(22)} \quad \frac{y(t+\Delta t) - 2y(t) + y(t-\Delta t)}{(\Delta t)^2} = -GM \frac{y(t)}{r^3(t)}$$

Note that these equations are perfectly symmetrical with respect to the past and the future. This reflects an important property of Newtonian mechanics (in the absense of friction). The equations of motion are reversible. A movie of the solar system running in reverse looks like the motion of a physically possible solar system. Our scheme has this property as well, and it would not if we had used $u(t)$ and $v(t)$ in equations (13-14).

How to design an orbit

We seek an orbit in which the maximum and minimum distances of the planet from the star are specified



Let these given distances be denoted a, b

We would like to solve for the corresponding speeds v_a and v_b of the planet in its orbit when it is at these two particular points.

What is special about the points of maximum and minimum distance is that these are the only points at which the radius vector and the velocity vector are perpendicular. This makes the formula for angular momentum be simply the mass times the velocity times the radius. Thus, conservation of angular momentum gives the equation

(23) $$m v_a a = m v_b b$$

We can get another equation from conservation of energy. Gravitational potential energy is negative, and when two bodies of masses M, m are separated by a distance r, the gravitational potential energy is

(24) $$-\frac{GMm}{r}$$

This is an increasing function of r, since it becomes less negative as r increases. Note that the derivative of (24) with respect to r is the magnitude of the gravitational force.

From conservation of total energy, we therefore have the equation

(25) $$\frac{1}{2} m v_a^2 - \frac{GMm}{a} = \frac{1}{2} m v_b^2 - \frac{GMm}{b}$$

Equations (23) & (25) are a pair of equations in the two unknowns v_a and v_b.

Note that m cancels out of both equations. Also, we can square both sides of (23) and then we have

(26) $$\begin{cases} a^2 v_a^2 - b^2 v_b^2 = 0 \\ v_a^2 - v_b^2 = 2GM\left(\frac{1}{a} - \frac{1}{b}\right) \end{cases}$$

This is a pair of linear equations in the unknowns v_a^2 and v_b^2. The solution (for example, by determinants) is

(27) $$v_a^2 = \frac{b^2\, 2GM\left(\frac{1}{a}-\frac{1}{b}\right)}{b^2-a^2}$$

$$= \frac{b}{a}\, 2GM\, \frac{b-a}{b^2-a^2} = \frac{b}{a}\, \frac{2GM}{a+b}$$

(28) $$v_b^2 = \frac{a^2\, 2GM\left(\frac{1}{a}-\frac{1}{b}\right)}{b^2-a^2}$$

$$= \frac{a}{b}\, 2GM\, \frac{b-a}{b^2-a^2} = \frac{a}{b}\, \frac{2GM}{a+b}$$

Thus, finally,

(29) $$v_a = \sqrt{\frac{b}{a}\left(\frac{2GM}{a+b}\right)}$$

(30) $$v_b = \sqrt{\frac{a}{b}\left(\frac{2GM}{a+b}\right)}$$

Substituting (29) & (30) into (23), we see that (23) is indeed satisfied since both sides are equal to

(31) $$L = m\sqrt{ab\,\frac{2GM}{a+b}}$$

and this gives a formula for the total angular momentum of the orbit.

Similarly, we can substitute (29) & (30) into (25). On the left-hand side, we get

(32) $$\frac{1}{2}m\,\frac{b}{a}\left(\frac{2GM}{a+b}\right) - \frac{GMm}{a}$$

$$= \frac{GMm}{a}\left(\frac{b}{a+b} - 1\right) = -\frac{GMm}{a+b}$$

and on the right-hand side

(33) $$\frac{1}{2}m\,\frac{a}{b}\,\frac{2GM}{a+b} - \frac{GMm}{b}$$

$$= \frac{GMm}{b}\left(\frac{a}{a+b} - 1\right) = -\frac{GMm}{a+b}$$

Thus, both sides are equal to

(34) $$E = -\frac{GMm}{a+b}$$

and this is a formula for the total energy of the orbit.

The special case $a = b = r$ gives results for a circular orbit:

(35) $$v_a = v_b = v = \sqrt{\frac{GM}{r}}$$

(36) $$L = m\sqrt{GMr} = mvr$$

(37) $$E = -\frac{GMm}{2r}$$

$$= +\frac{GMm}{2r} - \frac{GMm}{r}$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

It is interesting that we have been able to obtain results for the circular case, since the method we are using here fails if we try to set $a=b$ at the outset. With $a=b$, all that we can deduce from equations (23) & (25) is that $v_a = v_b$. Thus we have to use the indirect method of solving for v_a and v_b with $a \neq b$ and then setting $a=b$ at the end. This is related to the cancelation of $b-a$ in the last steps of (27) & (28).

Results for the circular case can be easily obtained, however, by balancing the gravitational force against the centrifugal force, and this is a good way to check equation (35) for the speed of a circular orbit. We leave this as an exercise for the reader.

We can evaluate the period of the orbit in the following way

We know that L is constant, and

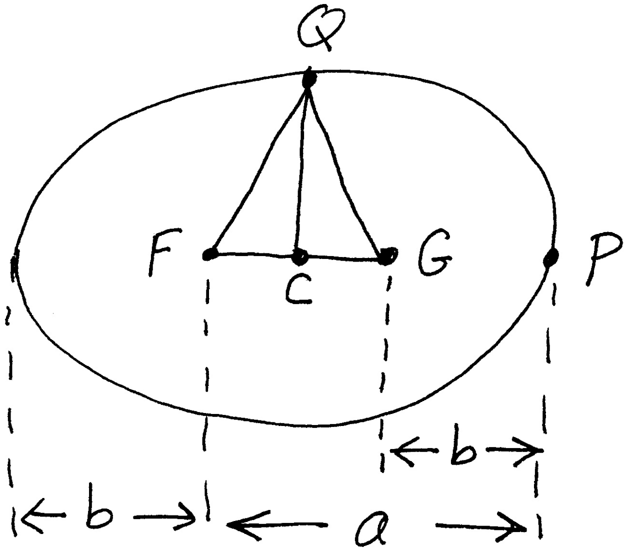
(38) $$L = mr^2 \frac{d\theta}{dt}$$

But the rate at which area is swept by the moving radius line is

(39) $$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} \frac{L}{m} = \frac{1}{2}\sqrt{ab\frac{2GM}{a+b}}$$

The fact that dA/dt is constant is one of Kepler's laws.

If we assume that the orbit is an ellipse (another Kepler law), we can find its area geometrically as follows



$F, G =$ foci , $C =$ center

$P =$ point on major axis
$Q =$ point on minor axis

Then $A_0 = \pi |CP|\,|CQ|$, $|CP| = \frac{a+b}{2}$.

To find $|CQ|$, note that every point on the ellipse has the same sum of distances to the two foci. By considering the point P, we see that this sum is $(a+b)$. Therefore

$|FQ| = \frac{a+b}{2}$, and we also have $|FC| = \frac{a-b}{2}$, so

$$|CQ| = \left(\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2\right)^{1/2} = (ab)^{1/2}$$

We have shown that the area of the ellipse is

(40) $$A_0 = \pi\left(\frac{a+b}{2}\right)(ab)^{1/2}$$

Since $\frac{dA}{dt}$ is constant, the period T is given by

(41) $$T\,\frac{dA}{dt} = A_0$$

Combining (39), (40), and (41), we then get

(42) $$T = \frac{\pi\left(\frac{a+b}{2}\right)(ab)^{1/2}}{\frac{1}{2}\left(ab\,\frac{2GM}{a+b}\right)^{1/2}}$$

$$= \frac{\pi}{(2GM)^{1/2}}\,(a+b)^{3/2}$$

Thus, the period of the orbit is proportional to the major diameter to the 3/2 power, and all orbits with the same major diameter have the same period, regardless of their shape. (This is yet another of Kepler's laws)

The independence of shape is the truly remarkable feature of this law. (Similarly, the energy of an orbit only depends on the major diameter, see equation (34).)

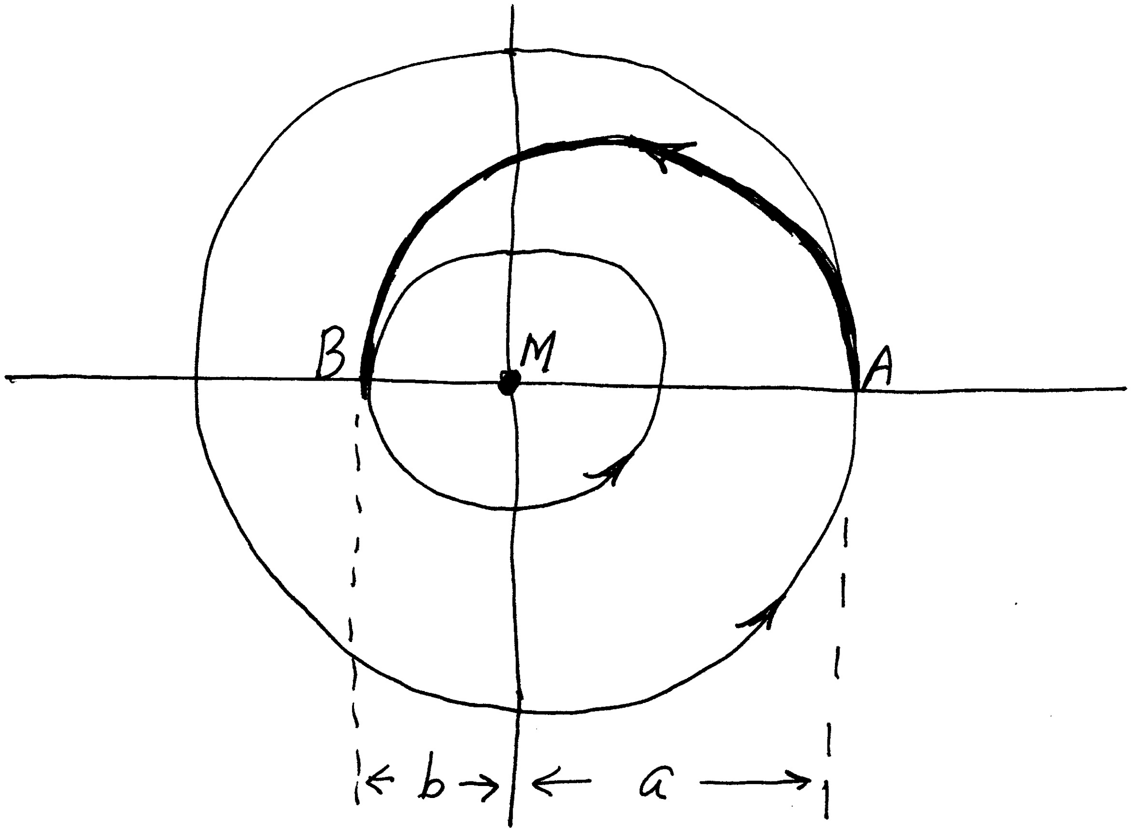
If we set $a=b=r$ in (42), we should get a formula for the period of a circular orbit:

$$(43) \qquad T = \frac{\pi (2r)^{3/2}}{(2GM)^{1/2}} = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$$

and indeed this is correct, since $2\pi r$ is the circumference and $\sqrt{\frac{Gm}{r}}$ is the speed of the circular orbit.

Although equation (42) can be derived using only Newton's equations of motion, we did not quite do that here, since we made use of some facts we did not derive, namely that the orbit is an ellipse with the star at one focus.

Equation (43) is a lot easier to remember than equation (42), but if you just recall that the major diameter determines the period, then you can derive (42) from (43) simply by setting $r = \frac{a+b}{2}$.



As an application, consider the transition from a circular orbit of radius a to a circular orbit of radius b. Assume that $a > b$, as in the figure.

The transition can be made via a half-ellipse that connects the points A and B. This requires a "burn" at A and another burn at B to change the speed (but not the direction) of the spacecraft.

Except for the sudden changes in speed at A and B, the motion will be ballistic, i.e., governed by Newton's laws. There are four velocities that we need to consider

$$v_a^{circ} = \sqrt{\frac{GM}{a}} \tag{44}$$

$$v_a^{ellipse} = \sqrt{\frac{b}{a}\left(\frac{2GM}{a+b}\right)} \tag{45}$$

$$v_b^{ellipse} = \sqrt{\frac{a}{b}\left(\frac{2GM}{a+b}\right)} \tag{46}$$

$$v_b^{circ} = \sqrt{\frac{GM}{b}} \tag{47}$$

Note that

(48) $$v_a^{ellipse} < v_a^{circ}$$

since $\frac{2b}{a+b} = \frac{b+b}{a+b} < 1$, and also

(49) $$v_b^{circ} < v_b^{ellipse}$$

since $\frac{2a}{a+b} = \frac{a+a}{a+b} > 1$.

Thus, the spacecraft has to <u>slow down</u> at A to enter the elliptical orbit, and it has to <u>slow down</u> again at B to leave the elliptical orbit.

Despite these two reductions in speed, the final speed $v_b^{circ} = \sqrt{\frac{GM}{b}}$ is <u>greater</u> than the initial speed $v_a^{circ} = \sqrt{\frac{GM}{a}}$.

How is this possible?

The amount of time spent during the transition is half of the period of the elliptical orbit. It is therefore equal to $T/2$, where T is given by equation (42). It would be useful to know this if the goal is to arrive at the inner orbit at the right time to intercept another spacecraft that is already on that orbit.

Fuel consumption of a burn

To change its velocity, a spacecraft has to eject some mass. The energy to do so comes from burning fuel, and the burnt fuel is the substance that is ejected. We model this process as one that occurs at an instant of time and produces a sudden change in velocity

$$\overrightarrow{\Delta v}$$

Because the process is instantaneous, it happens with the spacecraft in a particular location, and therefore there is no change in the gravitational potential energy of the spacecraft or fuel during the burn.

To analyze the burn, we are free to use any inertial frame of reference, and we choose a frame in which the spacecraft is at rest prior to the burn.

Since $\vec{\Delta v}$ is the same in any reference frame, the end result will not be affected by the frame that we choose. (Note that we are using here the relativity of Galileo and Newton, not that of Einstein.)

Let

m = mass of the spacecraft (including fuel) before the burn

m_f = mass of the fuel ejected

$\mathcal{E}$ = available energy per unit mass of the fuel

The word "available" in the definition of $\mathcal{E}$ means that $\mathcal{E}$ does not include the energy that is converted into heat when the fuel is burnt.

Since the spacecraft was at rest before the burn (in our chosen frame of reference), its velocity after the burn is $\vec{\Delta v}$. By conservation of momentum we then have

$$(m - m_f)\,\vec{\Delta v} + m_f\,\vec{v}_f = 0 \tag{50}$$

where $\vec{v}_f$ is the velocity of the ejected fuel. Thus,

$$\vec{v}_f = -\left(\frac{m - m_f}{m_f}\right)\vec{\Delta v} \tag{51}$$

The kinetic energy of the system was zero before the burn (in our chosen frame of reference), so conservation of energy now gives the equation

(52) $$m_f \mathcal{E} = \frac{1}{2}(m-m_f)\|\vec{\Delta v}\|^2 + \frac{1}{2} m_f \|\vec{v_f}\|^2$$

$$= \frac{1}{2}(m-m_f)\|\vec{\Delta v}\|^2\left(1 + \frac{m-m_f}{m_f}\right)$$

$$= \frac{1}{2}(m-m_f)\|\vec{\Delta v}\|^2 \frac{m}{m_f}$$

This can be rearranged into a quadratic equation for m_f/m :

(53) $$\frac{2\mathcal{E}}{\|\vec{\Delta v}\|^2}\left(\frac{m_f}{m}\right)^2 + \left(\frac{m_f}{m}\right) - 1 = 0$$

In these equations $\|\vec{\Delta v}\|$ denotes the length of the vector $\vec{\Delta v}$.

which is solved by

$$(54) \qquad \frac{m_f}{m} = \frac{-1 \pm \sqrt{1 + \frac{8\mathcal{E}}{\|\vec{\Delta v}\|^2}}}{\frac{4\mathcal{E}}{\|\vec{\Delta v}\|^2}}$$

Since masses have to be positive, only the + solution makes sense, and it is indeed positive. What is not immediately obvious from the above expression with the + sign is that $m_f/m < 1$, as it has to be, since m is the mass of the spacecraft <u>including fuel</u> before the burn, and m_f is the mass of the fuel that is burned. Also, it is not immediately obvious whether m_f/m is an increasing or a decreasing function of $\mathcal{E}/\|\vec{\Delta v}\|^2$, since the numerator and denominator (with the + sign chosen) are both increasing.

We can get a much nicer (and equivalent) formula by rationalizing the numerator, i.e., by multiplying the numerator and the denominator by

$$\sqrt{1+\frac{8\mathcal{E}}{\|\overrightarrow{\Delta v}\|^2}} + 1 \tag{55}$$

The result is

$$\frac{m_f}{m} = \frac{2}{1+\sqrt{1+\frac{8\mathcal{E}}{\|\overrightarrow{\Delta v}\|^2}}} \tag{56}$$

From this form of the solution it is clear that $m_f/m < 1$ and also that m_f/m is a decreasing function of $\mathcal{E}/\|\overrightarrow{\Delta v}\|^2$ and hence an increasing function of $\|\overrightarrow{\Delta v}\|^2$.

Equation (56) gives the cost in fuel of a burn in which the change in velocity of the spacecraft is given by the vector $\overrightarrow{\Delta v}$.

Drag on a spacecraft in a very thin atmosphere

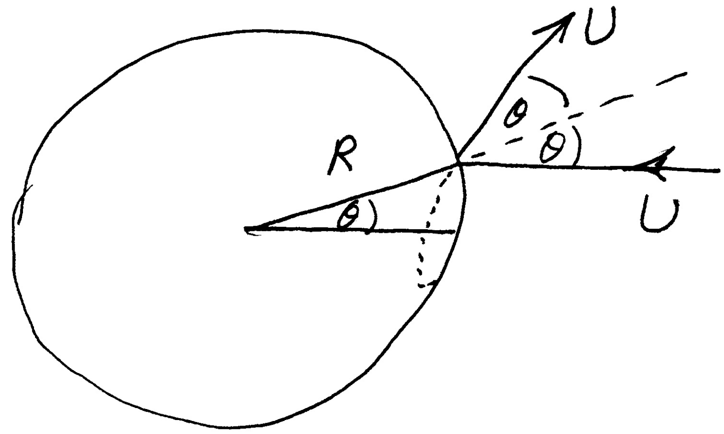
Since the atmosphere is very thin, we can consider the collisions of the spacecraft with individual air molecules, and not worry about the interactions of the air molecules with each other.

We think of the air molecules as being at rest until the spacecraft hits them. (Random motion of the molecules does not produce any net force on the spacecraft, so it can be ignored.)

Let the spacecraft be a sphere of radius R that moves with velocity U in the x-direction. We work in a frame in which the spacecraft is at rest, so the air molecules are moving at velocity $-U$ in the x-direction, prior to their collisions with the spacecraft.

Let each air molecule have mass m, and let their number per unit volume be denoted σ.

Given a point on the front half of the sphere, where a collision might occur, let θ be the angle between the radius vector to that point and the x-axis



There is a ring of points on the front half of the sphere, all of which have the same value of θ. The circumference of this ring is

(57) $$2\pi R \sin\theta$$

The domain of θ corresponding to the front half of the sphere is

(58) $$(0, \pi/2)$$

In the frame of reference with the spacecraft at rest, the air molecules bounce off of it in such a way that the angle of incidence equals the angle of reflection, as shown in the figure on the previous page. When the collision occurs at a point with angle θ, the change in x-momentum of the air molecule is

$$(59) \qquad mU\cos(2\theta) - (-mU)$$

$$= mU(\cos(2\theta) + 1)$$

$$= mU(\cos^2\theta - \sin^2\theta + 1)$$

$$= 2mU\cos^2\theta$$

Next, we need to evaluate the number of collisions per unit time that occur in $(\theta, \theta + d\theta)$. The forward-facing area of the strip $(\theta, \theta+d\theta)$ is given by

$$(60) \qquad 2\pi R\sin\theta \; d(R\sin\theta)$$

$$= 2\pi R^2 \sin\theta\cos\theta\, d\theta$$

To get the number of collisions per unit time that occur within $(\theta, \theta + d\theta)$ we multiply the forward-facing area by $U\sigma$.

Thus, the number of collisions per unit time within $\theta, \theta + d\theta$ is

(61) $$U\sigma\, 2\pi R^2 \sin\theta \cos\theta \, d\theta$$

Since <u>force</u> is <u>momentum per unit time</u>, and since any momentum gained by an air molecule is lost by the spacecraft, the x-component of force on the spacecraft from collisions that occur in $(\theta, \theta + d\theta)$ is minus the product of (59) and (61), that is

(62) $$-2m\sigma U^2 \, 2\pi R^2 \cos^3\theta \sin\theta \, d\theta$$

Note that

(63) $$\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta = -\frac{1}{4} \int_0^{\pi/2} \frac{d}{d\theta}\left(\cos^4\theta\right) d\theta = \frac{1}{4}$$

Therefore, by integrating (62) over $(0, \pi/2)$, we get

(64) $$F_x = -m\sigma U^2 \pi R^2$$
$$= -\rho U^2 \pi R^2$$

Where

(65) $$\rho = m\sigma$$

So that ρ is the mass per unit volume of the air.

The above result is for the special case in which the spacecraft is moving in the positive x-direction at speed U. More generally, if the spacecraft has velocity $\vec{U}$ relative to the air through which it is moving, then the drag force that it feels is

(66) $$\vec{\mathbf{F}} = -\rho \|\vec{U}\|^2 \pi R^2 \frac{\vec{\mathbf{U}}}{\|\vec{U}\|}$$

Note that (66) can be written more simply as

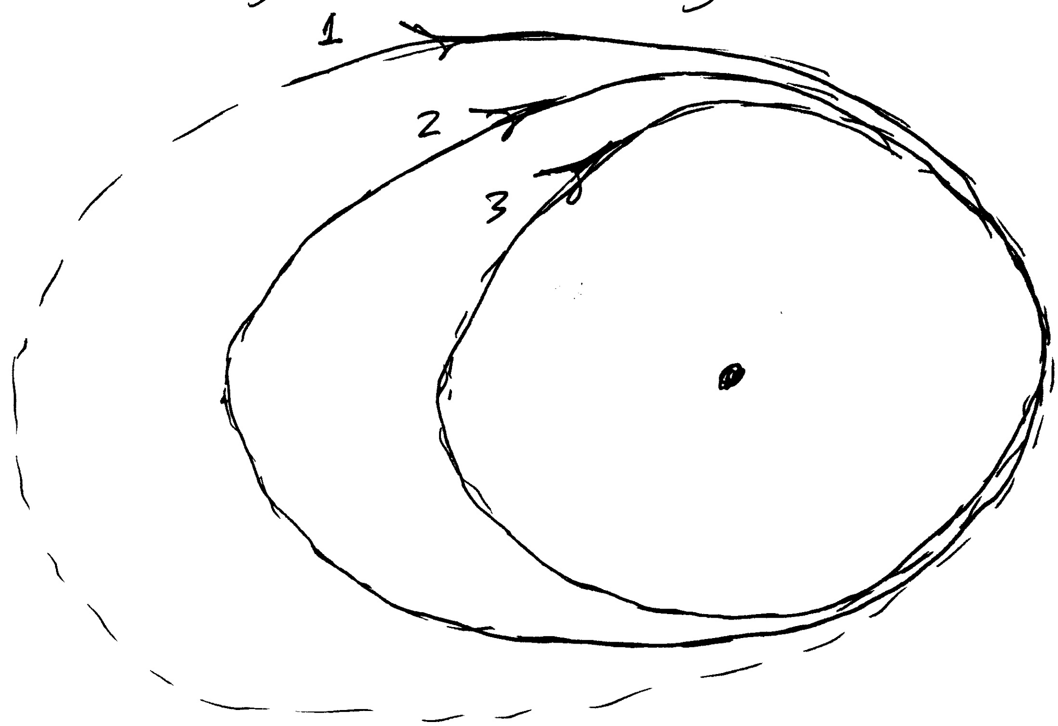
$$\vec{F} = -\rho \vec{U} \|\vec{U}\| \pi R^2 \tag{67}$$

This can be added to the gravitational force to simulate atmospheric drag.

(To do so, however, it is necessary to know how ρ varies with position.)

When a spacecraft approaches a planet, it typically needs to slow down a lot to enter orbit around the planet, and fuel can be saved by using atmospheric drag for this purpose. This is called <u>aerobraking</u>.

Aerobraking looks something like this:



with successive orbits becoming more circular, and with most of the aerobraking happening at the point of closest approach to the planet, where the (still very thin) atmosphere is most dense.

The n-body problem

Now we consider a system of n bodies moving under the influence of their mutual gravitational attraction. Let

m_i = mass of the i^{th} body

(x_i, y_i, z_i) = position of the i^{th} body

(u_i, v_i, w_i) = velocity of the i^{th} body

All of these quantities are defined for $i = 1 \cdots n$. The masses are constants, but the positions and velocities are functions of time.

Let $r_{ij} = r_{ji}$ be the distance between bodies i and j. This is given by the 3D version of the Pythagorean theorem:

(68) $$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

The equations of motion are as follows

$$(69) \qquad m_i \frac{du_i}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j}{r_{ij}^2} \frac{x_j - x_i}{r_{ij}}$$

$$(70) \qquad m_i \frac{dv_i}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j}{r_{ij}^2} \frac{y_j - y_i}{r_{ij}}$$

$$(71) \qquad m_i \frac{dw_i}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j}{r_{ij}^2} \frac{z_j - z_i}{r_{ij}}$$

$$(72) \qquad \frac{dx_i}{dt} = u_i$$

$$(73) \qquad \frac{dy_i}{dt} = v_i$$

$$(74) \qquad \frac{dz_i}{dt} = w_i$$

for $i = 1..n$.

These equations can be written more succinctly in vector notation. Let

(75) $$\vec{X}_i = (x_i, y_i, z_i)$$

(76) $$\vec{U}_i = (u_i, v_i, w_i)$$

for $i = 1 \cdots n$. Then

(77) $$m_i \frac{d\vec{U}_i}{dt} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j}{r_{ij}^2} \frac{\vec{X}_j - \vec{X}_i}{r_{ij}}$$

(78) $$\frac{d\vec{X}_i}{dt} = \vec{U}_i$$

for $i = 1 \cdots n$

Note that $\dfrac{\vec{X}_j - \vec{X}_i}{r_{ij}}$ is a unit vector that points <u>from</u> the i^{th} body <u>to</u> the j^{th} body.

The momentum, angular momentum, and energy of the whole system are given by

$$\vec{P} = \sum_{i=1}^{n} m_i \vec{U}_i \tag{79}$$

$$\vec{L} = \sum_{i=1}^{n} m_i \left(\vec{X}_i \times \vec{U}_i \right) \tag{80}$$

$$E = \frac{1}{2} \sum_{i=1}^{n} m_i \vec{U}_i \cdot \vec{U}_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j}{r_{ij}} \tag{81}$$

A good exercise is to show that all of these quantities are conserved. Apply $\frac{d}{dt}$ and use the equation of motion.

The main loop of a code for the n-body problem looks like this:

```
for clock = 1:clockmax
  for i = 1:n
     for j = 1:n
        if (i~=j)
           dx = x(j) - x(i)
           dy = y(j) - y(i)
           dz = z(j) - z(i)
           r = sqrt(dx^2 + dy^2 + dz^2)
           u(i) = u(i) + dt*G*m(j)*dx/r^3
           v(i) = v(i) + dt*G*m(j)*dy/r^3
           w(i) = w(i) + dt*G*m(j)*dz/r^3
        end
     end
  end
  for i = 1:n
     x(i) = x(i) + dt*u(i)
     y(i) = y(i) + dt*v(i)
     z(i) = z(i) + dt*w(i)
  end
end
```

An important detail in the above program is that we do not update any positions until we have updated all of the velocities. That is why we end the first loop over i and open a new one, rather than merging the two loops into one by deleting the lines

```
end
for i = 1:n
```

If we were to update the position of each body as soon as we have updated its velocity, then we would be (slightly) violating Newton's law of equal but opposite forces, and we would start to see unphysical effects in our simulation.

If results need to be saved for future plotting, statements like this can be added following the loop that updates x, y, z:

```
xsave(:, clock) = x
ysave(:, clock) = y
zsave(:, clock) = z
```

Here x, y, z are $n \times 1$

xsave, ysave, zsave are $n \times$ clockmax

and the notation : means the full range of the index to which it refers.

Thus xsave(:, clock) refers to one column of the array xsave, namely, the column with index clock.

The above code can be made more efficient by looking at each pair of bodies once, insted of twice, and updating the velocities of both bodies of the pair. We have to be careful, though, about the masses and the signs. Here is just the loop that updates velocities

```
for i = 1:(n-1)               % look at each pair
  for j = (i+1):n             % only once
      dx = x(j) - x(i)
      dy = y(j) - y(i)
      dz = z(j) - z(i)
      r = sqrt(dx^2 + dy^2 + dz^2)

      u(i) = u(i) + dt*G*m(j)*dx/r^3
      v(i) = v(i) + dt*G*m(j)*dy/r^3
      w(i) = w(i) + dt*G*m(j)*dz/r^3

      u(j) = u(j) - dt*G*m(i)*dx/r^3
      v(j) = v(j) - dt*G*m(i)*dy/r^3
      w(j) = w(j) - dt*G*m(i)*dz/r^3
  end
end
```

Note that m(j) affects u(i), v(i), w(i)

m(i) affects u(j), v(j), w(j)

Binary star

An interesting project would be to simulate a binary star system with a planet. The first step would be to set up the binary star system, and for that the following theory should be helpful.

The equations of motion of a pair of stars with masses m_1 and m_2 are as follows

(82) $$m_1 \frac{d\vec{U}_1}{dt} = \frac{G m_1 m_2}{r^2} \frac{\vec{X}_2 - \vec{X}_1}{r}$$

(83) $$m_2 \frac{d\vec{U}_2}{dt} = \frac{G m_1 m_2}{r^2} \frac{\vec{X}_1 - \vec{X}_2}{r}$$

(84) $$\frac{d\vec{X}_1}{dt} = \vec{U}_1$$

(85) $$\frac{dX_2}{dt} = \vec{U}_2$$

(86) $$r = \| \vec{X}_2 - \vec{X}_1 \|$$

let

(87) $$\vec{X}_{CM} = (m_1 \vec{X}_1 + m_2 \vec{X}_2)/(m_1 + m_2)$$

(88) $$\vec{U}_{CM} = (m_1 \vec{U}_1 + m_2 \vec{U}_2)/(m_1 + m_2)$$

Then it follows easily from the equations of motion that

(89) $$\frac{d\vec{U}_{CM}}{dt} = 0$$

(90) $$\frac{d\vec{X}_{CM}}{dt} = \vec{U}_{CM}$$

Thus, the center of mass $\vec{X}_{CM}$ moves at the constant velocity $\vec{U}_{CM}$. We can always choose a frame of reference in which $\vec{U}_{CM} = 0$ and then $\vec{X}_{CM}$ is constant, and we can choose our coordinates so that the center of mass of the system is at the origin.

Once the above choices have been made, we have the equations

$$m_1 \vec{X}_1 + m_2 \vec{X}_2 = 0 \tag{91}$$

$$m_1 \vec{U}_1 + m_2 \vec{U}_2 = 0 \tag{92}$$

Now let

$$\vec{X} = \vec{X}_2 - \vec{X}_1 \tag{93}$$

$$\vec{U} = \vec{U}_2 - \vec{U}_1 \tag{94}$$

After canceling m_1 in the first equation of motion and canceling m_2 in the second equation, we subtract corresponding equations and obtain

$$\frac{d\vec{U}}{dt} = -\frac{G(m_1+m_2)}{r^2}\frac{\vec{X}}{r} \tag{95}$$

$$\frac{d\vec{X}}{dt} = \vec{U} \tag{96}$$

$$r = \|\vec{X}\| \tag{97}$$

These are the equations of a single body (of any mass) that orbits a fixed star of mass (m_1+m_2).

Note that the distance r that appears in the equations for $(\vec{X},\vec{U})$ is the distance between the two stars in the binary system; it is not the distance of either one of them from the center of mass. A common mistake is to think that the two stars can be replaced by a single star of mass (m_1+m_2) located at the center of mass. This is an approximation that one can make when considering the gravitational force on a third body that is far away from the binary system, but it is not valid within the binary system.

In the equation for $d\vec{U}/dt$, it is conventional to multiply both sides by

(98) $$\frac{m_1 m_2}{m_1+m_2}$$

which is called the reduced mass.

Then the equation becomes

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right)\frac{d\vec{U}}{dt} = -\frac{G m_1 m_2}{r^2}\frac{\vec{X}}{r} \tag{99}$$

The conceptual advantage of doing this is that the usual formula for the gravitational force appears on the right hand side, and on the left hand side we have an effective mass that plays the role of the inertial mass in Newton's equation of motion. Note that the reduced mass is smaller than either of the two masses m_1, m_2, and also that it is approximately equal to the smaller one when the two masses are very unequal.

The reduced mass is only mentioned here because you may have seen it before and may be wondering why it did not appear in our equation for $d\vec{U}/dt$ as we derived it first. Equation (95) is simpler and more useful than (99).

To every solution of the single-body equations (95 – 97), there corresponds a motion of the binary star system. To find this motion, we only need to solve the following pair of equations

$$m_1 \vec{X}_1(t) + m_2 \vec{X}_2(t) = 0 \tag{100}$$

$$-\vec{X}_1(t) + \vec{X}_2(t) = \vec{X}(t) \tag{101}$$

The solution is

$$\vec{X}_1(t) = -\frac{m_2}{m_1+m_2} \vec{X}(t) \tag{102}$$

$$\vec{X}_2(t) = \frac{m_1}{m_1+m_2} \vec{X}(t) \tag{103}$$

In exactly the same way, or by differentiation with respect to time, we get

$$\vec{U}_1(t) = -\frac{m_2}{m_1+m_2} \vec{U}(t) \tag{104}$$

$$\vec{U}_2(t) = \frac{m_1}{m_1+m_2} \vec{U}(t) \tag{105}$$

Thus, $\vec{X}_1(t)$ and $\vec{X}_2(t)$ are scaled versions of $\vec{X}(t)$. Note that the scale factor is $m_2/(m_1+m_2)$ in the case of $\vec{X}_1(t)$ and it is $m_1/(m_1+m_2)$ in the case of $\vec{X}_2(t)$. Also the signs are opposite, so the line joining the two stars always goes through the origin, which is where we have put the center of mass. The same algebraic relationships hold for the velocity vectors as for the position vectors.

We can use the above result to set up initial conditions that will create a binary star system with specified properties

Suppose we want the masses of the two stars to be m_1, m_2, and also we want the distance between them at maximum separation to be a, and the distance between them at minimum separation to be b. To create this binary system, we first think about an elliptical orbit of a single body about a fixed star of mass $m_1 + m_2$ in which the maximum and minimum distances of the body from the fixed star are a and b. We will get such an orbit if we start from the initial conditions

(106) $$\vec{X}(0) = (a, 0, 0)$$

(107) $$\vec{U}(0) = (0, v_a, 0)$$

where v_a is given by equation (29) with $M = m_1 + m_2$, that is,

(108) $$v_a = \sqrt{\frac{b}{a}\,\frac{2G(m_1+m_2)}{a+b}}$$

It follows that we will get the desired binary star system if we choose the initial conditions

(109) $$\vec{X}_1(0) = -\frac{m_2}{m_1+m_2}(a,0,0)$$

(110) $$\vec{U}_1(0) = -\frac{m_2}{m_1+m_2}(0,v_a,0)$$

(111) $$\vec{X}_2(0) = \frac{m_1}{m_1+m_2}(a,0,0)$$

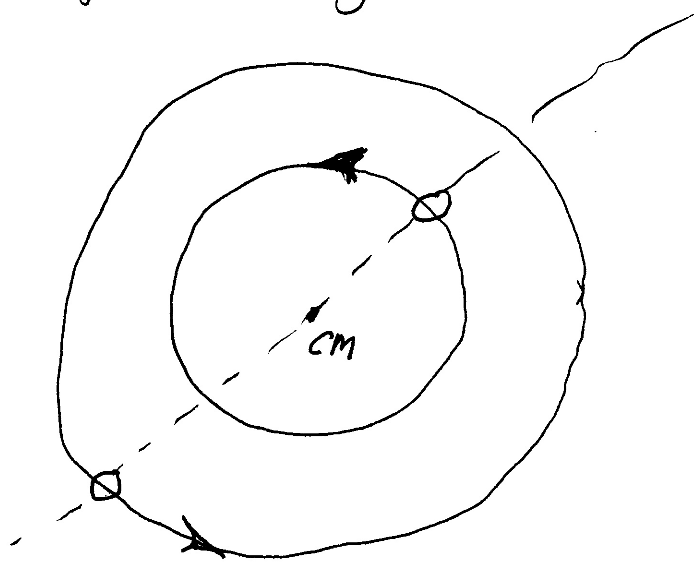
(112) $$\vec{U}_2(0) = \frac{m_1}{m_1+m_2}(0,v_a,0)$$

In the special case $a=b=r$, the stars maintain a constant distance from each other, which is equal to r. Each star moves in a circular orbit about the center of mass. The radii of the two circles are

$$r_1 = \frac{m_2}{m_1+m_2}\, r \tag{113}$$

$$r_2 = \frac{m_1}{m_1+m_2}\, r \tag{114}$$

It is a good exercise to find the speeds of the two stars in their orbits by balancing centrifugal force against gravity. In doing this exercise, it is important to remember that the relevant distance for the gravitational force is r, whereas the relevant radius for the centrifugal force on each star is the radius of the circle on which is moving, r_1 or r_2. A hint is that the two stars have the same <u>angular</u> velocity, since they are always on opposite sides of the origin



The star with larger mass is on the smaller circle, and the two circles coincide if the masses are equal.
Once you have a binary star system, you can try putting planets into it.
There are a lot of possibilities,

especially in 3D !

Correction on Fuel Consumption during a Burn

My previous analysis of fuel consumption during a burn involved the assumption that all of the ejected fuel has a specific velocity $\vec{v}_f$. This would be true if the spacecraft changed its velocity by shooting out a single solid projectile, so my analysis is applicable to that situation. Then m_f is the mass of the ejected projectile, and $\mathcal{E} m_f$ is the gain in kinetic energy of the whole system when the projectile is ejected.

A rocket engine, however, ejects mass continuously, at a prescribed velocity <u>relative to the spacecraft</u>. Thus, as seen by an observer in an inertial frame, the velocity of the ejected gas varies during a burn, since the velocity of the spacecraft is changing. This is true even in the limit in which the burn happens instantaneously, and it needs to be taken into account.

The key to the correct calculation of fuel consumption is called the <u>rocket equation</u>. It is derived as follows:

Consider a spacecraft with no external forces acting on it, and with mass $m(t)$. The mass is decreasing because fuel is being burned and the exhaust is being ejected. Thus

(115) $$\frac{dm}{dt} < 0$$

Let $\vec{U}(t)$ be the velocity of the spacecraft, and let $\vec{U}_{frel}(t)$ be the velocity of the ejected fuel exhaust, <u>relative</u> to that of the spacecraft. Thus the velocity of the ejected fuel exhaust as seen by an inertial observer is

(116) $$\vec{U}(t) + \vec{U}_{frel}(t)$$

Note that the time t in the above expression for the ejected fuel velocity is the time at which the fuel in question was ejected. Following ejection, the ejected fuel moves at a constant velocity, since there are no forces acting on it. Thus, the total momentum of the system (spacecraft + all of the fuel that it has ejected) at time t is given by

$$(117)\qquad m(t)\,\vec{U}(t) + \int_{-\infty}^{t} -\frac{dm}{dt}(t')\left(\vec{U}(t') + \vec{U}_{frel}(t')\right)dt'$$

The total momentum is constant, so if we differentiate with respect to time we get zero. Thus

$$(118)\qquad 0 = m(t)\,\frac{d\vec{U}}{dt}(t) + \frac{dm}{dt}(t)\,\vec{U}(t) - \frac{dm}{dt}(t)\left(\vec{U}(t) + \vec{U}_{frel}(t)\right)$$

Note the cancelation of $\frac{dm}{dt}\vec{U}(t)$.

The result can be written as

$$m(t)\,\frac{d\vec{U}}{dt}(t) = \frac{dm}{dt}(t)\,\vec{U}_{frel}(t) \tag{119}$$

and this is called the rocket equation

The right-hand side is the thrust generated by the rocket engine. Since $dm/dt < 0$, it is always in the opposite direction to the relative velocity with which the exhaust is ejected.

In the special case that $\vec{U}_{frel}(t)$ is constant over some interval of time, say (t_1, t_2), the rocket equation can be integrated to find out how the change in velocity of the spacecraft is related to the change in mass.

To figure this out, we first divide both sides of the rocket equation by $m(t)$, and then integrate over (t_1, t_2). This gives

$$\int_{t_1}^{t_2} \frac{d\vec{U}}{dt}(t)\,dt = \left(\int_{t_1}^{t_2} \frac{1}{m(t)} \frac{dm}{dt}(t)\,dt\right) \vec{U}_{frel} \tag{120}$$

or

$$\vec{U}(t_2) - \vec{U}(t_1) = \left(\log \frac{m(t_2)}{m(t_1)}\right) \vec{U}_{frel} \tag{121}$$

Here log denotes the natural logarithm, that is $\log_e$. Since $m(t_2) < m(t_1)$, $\log \frac{m(t_2)}{m(t_1)} < 0$. Therefore

$$\log \frac{m(t_2)}{m(t_1)} = -\frac{\|\vec{U}(t_2) - \vec{U}(t_1)\|}{\|\vec{U}_{frel}\|} \tag{122}$$

and this can be exponentiated to give

$$\frac{m(t_2)}{m(t_1)} = \exp\left(-\frac{\|\vec{U}(t_2) - \vec{U}(t_1)\|}{\|\vec{U}_{frel}\|}\right) \quad (123)$$

Note that this equation holds regardless of the duration of the time interval (t_1, t_2) provided only that $\vec{U}_{frel}$ is constant

during that time interval and also that there are no other forces acting. We can therefore consider the limit in which $t_2 - t_1 \to 0$, and the burn takes place at an instant of time, and the above equation is still applicable. An advantage of doing this is that other forces acting on the spacecraft (such as gravity, or the drag force of an atmosphere) will have no effect, since they act on velocity at a finite rate and there is no time available!

Thus, in an instantaneous burn (which, of course, is an idealization, but a useful one), equation (123) is applicable even if there are other forces acting.

In your simulations, you may therefore assume that a spacecraft can make arbitrary changes in velocity at particular times, but to do so it needs to burn a mass of fuel equal to $m(t_1) - m(t_2)$, and the spacecraft cannot, of course, burn more fuel than it has on board. This sets a constraint on what space missions are possible.

The energetics of the rocket equation in the absence of external forces is interesting. The total kinetic energy of rocket and its exhaust at time t is given by

(124) $$E(t) = \frac{1}{2} m(t)\, \vec{U}(t) \cdot \vec{U}(t)$$

$$+ \frac{1}{2} \int_{-\infty}^{t} - \frac{dm}{dt}(t') \left(\vec{U}(t') + \vec{U}_{frel}(t') \right) \cdot \left(\vec{U}(t') + \vec{U}_{frel}(t') \right) dt'$$

Differentiating with respect to t gives

(125) $$\frac{dE}{dt} = \frac{1}{2} \frac{dm}{dt}(t) \left\| \vec{U}(t) \right\|^2 + m(t)\, \vec{U}(t) \cdot \frac{d\vec{U}}{dt}(t)$$

$$- \frac{1}{2} \frac{dm}{dt}(t) \Big(\left\| \vec{U}(t) \right\|^2 + 2\, \vec{U}(t) \cdot \vec{U}_{frel}(t)$$

$$+ \left\| \vec{U}_{frel}(t) \right\|^2 \Big)$$

Note the cancelation of the two terms involving $\|\vec{U}\|^2$. Also, by applying $\vec{U}(t)\cdot$ to both sides of the rocket equation (119), we see that the two terms involving $\vec{U}(t)\cdot$ cancel as well.

Thus, we are left with

$$\frac{dE}{dt} = -\frac{1}{2}\frac{dm}{dt}\left\|\vec{U}_{fuel}\right\|^2 \tag{126}$$

This is exactly the same as if the rocket were held at a fixed location and ejecting burned fuel at the same rate $(-dm/dt)$ with the speed $\|\vec{U}_{fuel}\|$, but then the rocket itself would have no kinetic energy, so it is amazing that we get the same result here for the rate of change of kinetic energy of the whole system.

Relationship between the two methods for evaluating fuel consumption

Equation (126) shows that the available energy per unit mass of fuel is given by

$$\mathcal{E} = \frac{1}{2} \| \vec{U}_{frel} \|^2 \tag{127}$$

This allows a direct comparison between the rocket-equation results and our earlier result based on discrete ejection of mass. To make this comparison, we rewrite equation (123) as follows:

$$\frac{m(t_2)}{m(t_1)} = \exp\left(-\frac{\|\Delta\vec{v}\|}{\sqrt{2\varepsilon}}\right) \tag{128}$$

Now let $m = m(t_1)$, $m_f = m(t_1) - m(t_2)$, so that m is the mass of the spacecraft plus fuel before the burn, and m_f is the mass of fuel ejected during the burn. Then

$$\frac{m_f}{m} = \frac{m(t_1) - m(t_2)}{m(t_1)} = 1 - \frac{m(t_2)}{m(t_1)} = \tag{129}$$

$$1 - \exp\left(-\frac{\|\Delta\vec{v}\|}{\sqrt{2\varepsilon}}\right)$$

Equation (129) involves exactly the same variables as equation (56), so they are directly comparable.

let

(130) $$\theta = \frac{\| \Delta \vec{v} \|}{\sqrt{2\mathcal{E}}}$$

Then equation (129) becomes

(131) $$\frac{m_f}{m} = f_1(\theta) = 1 - \exp(-\theta)$$

and equation (56) becomes

(132) $$\frac{m_f}{m} = f_2(\theta) = \frac{2}{1 + \sqrt{1 + \frac{4}{\theta^2}}}$$

The functions f_1 and f_2 are very similar over $0 < \theta < \infty$, as you can see simply by plotting them both on the same axes. Both of them increase monotonically from 0 at $\theta = 0$ to 1 at $\theta = +\infty$. For all $\theta > 0$, $f_1(\theta) > f_2(\theta)$, but their maximum difference is about 0.04. For small θ,

$$(133) \quad f_1(\theta) = 1 - \left(1 - \theta + \frac{\theta^2}{2} + \cdots\right)$$

$$= \theta - \frac{\theta^2}{2} + \cdots$$

and

(134) $$f_2(\theta) = \frac{2\theta}{\theta + \sqrt{\theta^2+4}}$$

$$= \frac{2\theta}{\theta + 2\left(1 + \frac{\theta^2}{4}\right)^{1/2}} =$$

$$\frac{2\theta}{\theta + 2\left(1 + \frac{\theta^2}{8} + \cdots\right)} =$$

$$\frac{\theta}{1 + \frac{\theta}{2} + \frac{\theta^2}{8} + \cdots} = \theta\left(1 - \frac{\theta}{2} + \cdots\right)$$

$$= \theta - \frac{\theta^2}{2} + \cdots$$

so the Taylor series for f_1 and f_2 about $\theta = 0$ agree at least through second order.

Thus, the two different ways of burning fuel (continuous or in discrete explosions) have very similar fuel requirements for making the same change in velocity, and the agreement is especially good when the change in velocity is small, i.e., when the mass of fuel consumed is a small fraction of the mass of the spacecraft (including its fuel).