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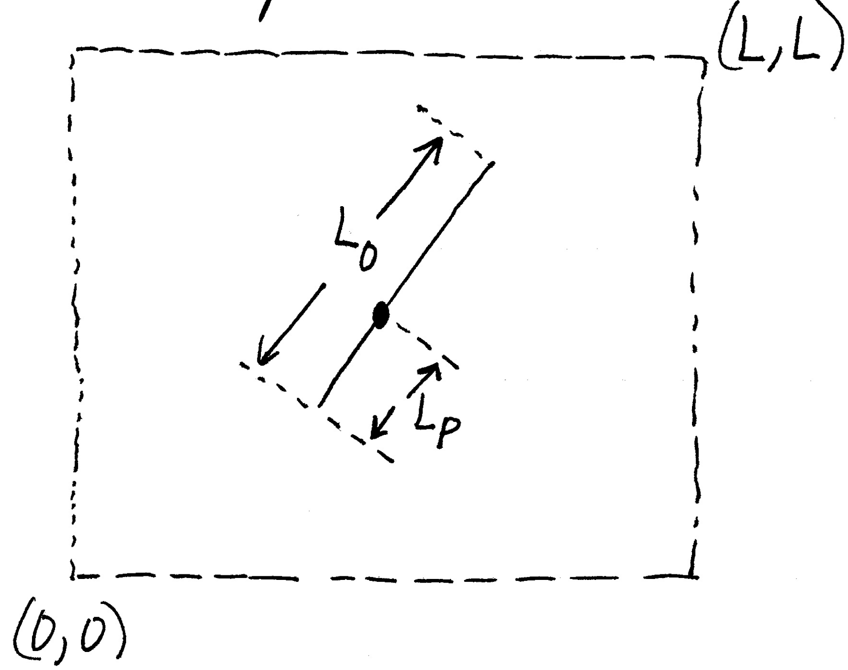
Homework:

Use the rigid pIB method in 2D to simulate an immersed boundary in the form of a straight line with uniform mass density and with a pivot point on the line that is held at a fixed location in space



Let the domain be $(0,L)\times(0,L)$ with periodic boundary conditions. Let the flow be driven by an applied force density that is constant in space and time

$$\underline{f}^0(\underline{x},t) = (f^0, 0) \tag{1}$$

The pivot point is fixed in space at

(2) $$(L/2,\ L/2)$$

and the line rotates freely about this pivot point

Let s denote distance along the line measured from one end. The line has length L_0; so

(3) $$0 \le s \le L_0$$

The pivot point is located at $s = L_p$.

The rigid line has only one degree of freedom $\theta(t)$, so its spatial configuration at time t is given by

(4) $$\underline{x} = \underline{Z}(s,t) = \left(\frac{L}{2} + (s-L_p)\cos\theta(t),\ \frac{L}{2} + (s-L_p)\sin\theta(t)\right)$$

Because the line boundary has a fixed pivot point, it is better not to make any reference to its center of mass, and instead to define angular momentum, moment of inertia, angular velocity, and torque with respect to the pivot point. Also, because we are in 2D, we have the simplification that these are all scalar quantities. In particular, the moment of inertia is a number, not a matrix, and it is constant in time even in the laboratory frame. These are major simplifications.

Let m_0 be the mass density of the boundary. Then its moment of inertia is given by

$$(5) \qquad I_0 = \int_0^{L_0} m_0 (s - L_p)^2 \, ds$$

$$= \frac{m_0}{3}\left((L_0 - L_p)^3 + L_p^3 \right)$$

The equations of motion of the whole system, in the rigid pIB formulation, may now be written as follows

(6) $$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) + \nabla p = \mu\Delta\underline{u} + \underline{f}^{0} + \underline{f}^{IB}$$

(7) $$\nabla\cdot\underline{u} = 0$$

(8) $$\underline{f}^{IB}(\underline{x},t) = \int_0^{L_0} K(\underline{Z}(s,t)-\underline{X}(s,t))\,\delta(\underline{x}-\underline{X}(s,t))\,ds$$

(9) $$\frac{\partial \underline{X}}{\partial t}(s,t) = \int_{(0,L)^2} \underline{u}(\underline{x},t)\,\delta(\underline{x}-\underline{X}(s,t))\,d\underline{x}$$

(10) $$I_0\,\frac{d\omega}{dt} = (-\sin\theta, \cos\theta)\cdot\int_0^{L_0} K(\underline{X}(s,t)-\underline{Z}(s,t))(s-L_p)\,ds$$

(11) $$\frac{d\theta}{dt} = \omega$$

In these equations $\underline{f}^0$ is given by (1), $\underline{Z}(s,t)$ is given by (4), and I_0 is given by (5). In particular, at any time t, $\underline{Z}(s,t)$ is determined by the value of $\theta(t)$.

The curve $\underline{X}(s,t)$ is held close to $\underline{Z}(s,t)$ by the spring-like force density

(12) $$K\left(\underline{Z}(s,t) - \underline{X}(s,t)\right)$$

and at the same time $\underline{X}(s,t)$ is moving at the local fluid velocity. This enforces the no-slip condition in the limit $K \to \infty$. The same force density (12) but with opposite sign acts on the rigid line, and the resulting torque about the pivot is evaluated on the right-hand side of (10).

A note on units:

When doing fluid mechanics in 2D, there are two ways to think about the problem:

1) We can think about it as if we are really in a 2D universe. In that case the units are as follows

$\rho \sim$ mass/area

$\mu \sim$ (mass/area) (length^2/time)

$u \sim$ length/time

$p \sim$ force/length

$f \sim$ force/area

$m_0 \sim$ mass/length

$K \sim$ (force/length)/length

2) Alternatively, we can think about the x, y plane of our computation as a cross-section of a 3D problem in which every plane perpendicular to the z axis is the same and in which there is no velocity in the z direction.

From this point of view, our rigid "line" is actually the cross section of a rigid planar strip that is infinitely long and has width L_0, and the pivot "point" is actually a pivot axis that is an infinitely long line running parallel to the z-axis.

The equations are exactly the same from either point of view, but the units are different. From the 3D perspective we have more familiar units for fluid quantities, but some quantities like the moment of inertia become the moment of inertia per unit length, where the length in question is in the z direction.

From the 3D point of view, we have the following units

$\rho \sim$ mass/volume

$\mu \sim$ (mass/volume)(length^2/time)

$p \sim$ force/area

$f \sim$ force/volume

$m_0 \sim$ mass/area

$K \sim$ (force/length)/area

$I_0 \sim$ moment of inertia / length

$= (\text{mass})(\text{length}^2) / \text{length}$

$= \text{mass} \cdot \text{length}$

$= \left(\dfrac{\text{mass}}{\text{area}}\right) \cdot \text{length}^3$

(see equation 5)

Your task for this homework is to simulate the above system and observe its behavior for different positions of the pivot on the line (including the center as one special case) and for different boundary mass densities (but note that zero mass density is not possible without a change of methodology*, which you might think about, but of course this is optional) *see Appendix

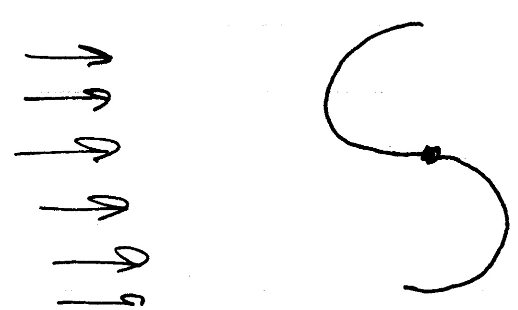
The 2D Matlab code on the course website is a good starting point for doing this homework, since most of the code can be used as is, and the modifications that need to be made are not very extensive. A small detail to watch out for is that the part of the code that plots vorticity contours sets parameters based on the initial vorticity field, and it will fail if there is no initial vorticity. You can set these parameters in some other way. Also note that the upper/lower limits on vorticity contours that get plotted may leave you with blank regions of your flow field unless you extend these limits. You could also consider other ways to visualize the flow besides vorticity contours.

Generalizations

Here are some extensions of the homework that would be possible to do as the course project:

1). Change the shape of the rigid, pivoted immersed boundary. This will make only minor changes in the formulation and code, but the behavior could be quite different.

An interesting example would be an S-shaped immersed boundary with pivot in the center:



It should spin in a flow, thus forming a 2D turbine. You could easily apply a counter-torque, as if the turbine is turning a generator, and adjust the counter-torque to extract maximum power.

2) In 3D, a rigid immersed boundary or body with a pivot <u>axis</u> that is a fixed line in space is almost as simple to simulate as the kind of 2D situation considered here. This is because the angular velocity and angular momentum are both aligned with the pivot axis and hence with each other, so the moment of inertia is a scalar that is constant in time (even in the laboratory frame) just as in 2D.

Thus, you could model a 3D turbine for example. Any shape turbine blade that you can describe by a layout of points is as easy to model as any other.

In 3D, there could be several rigid immersed boundaries or bodies, each with its own pivot axis, and with these axes not necessarily aligned and the simplifying feature mentioned above would still be applicable to each of them.

Appendix: $m_0 = 0$

If the immersed boundary is massless, then its moment of inertia is zero, and the condition that determines θ at each time t is that the torque about the pivot must be zero, see equation (10).

On the right-hand side of (10), note that

$$\text{(A1)} \quad \underline{X}(s,t) - \underline{Z}(s,t)$$

$$= \left(\underline{X}(s,t) - \left(\frac{L}{2}, \frac{L}{2}\right)\right) - \left(\underline{Z}(s,t) - \left(\frac{L}{2}, \frac{L}{2}\right)\right)$$

$$= \left(\underline{X}(s,t) - \left(\frac{L}{2}, \frac{L}{2}\right)\right) - (s - L_p)(\cos\theta, \sin\theta)$$

and the term that comes from $\underline{Z}(s,t)$ does not contribute to the torque, since the torque involves a dot product with $(-\sin\theta, \cos\theta)$. Therefore, the condition that the torque must be zero can be rewritten in the following way

(A2) $$0 = \left(-\sin\theta(t), \cos\theta(t)\right) \cdot$$

$$\int_0^{L_0} \left(\underset{\sim}{X}(s,t) - \left(\frac{L}{2}, \frac{L}{2}\right)\right)\left(s - L_p\right)\, ds$$

If $\theta(t)$ is a solution of this equation, so is $\theta(t) + \pi$, and these two solutions are <u>different</u> (even if $L_p = L_0/2$, since the two solutions put a given material point s on opposite sides of the pivot point.

We can guess, however, that one of these two solutions minimizes the elastic energy $E(\theta, t)$ of the penalty springs and the other one maximizes it.

To investigate this, we simplify the notation by dropping the variable t, which is constant throughout this discussion anyway, and also we make explicit the dependence of $\underline{Z}$ on θ. Then

(A3) $$E(\theta) = \frac{1}{2}K\int_0^{L_0} \left|\underline{X}(s) - \underline{Z}(s,\theta)\right|^2 ds$$

where $\underline{X}(s)$ is given (it does not depend on θ),

and

(A4) $$\underline{Z}(s,\theta) = \left(\frac{L}{2}, \frac{L}{2}\right) + (s - L_p)(\cos\theta, \sin\theta)$$

so

(A5) $$\frac{\partial \underline{Z}}{\partial \theta}(s,\theta) = (s - L_p)(-\sin\theta, \cos\theta)$$

(A6) $$\frac{\partial^2 \underline{Z}}{\partial \theta^2}(s,\theta) = (s - L_p)(-\cos\theta, -\sin\theta)$$

Now we can differentiate with respect to θ in A3 as follows:

(A7) $$E'(\theta) = K\int_0^{L_0} \left(\underline{X}(s) - \underline{Z}(s,\theta)\right) \cdot \left(-\frac{\partial \underline{Z}}{\partial \theta}\right) d\theta$$

This can be rewritten by making use of (A1) & (A5). In this way we get

(A8) $$E'(\theta) = K\,(\sin\theta, -\cos\theta) \cdot \int_0^{L_0} \underline{\tilde{X}}(s)(s - L_p)\, ds$$

where

(A9) $$\underline{\tilde{X}}(s) = \underline{X}(s) - \left(\frac{L}{2}, \frac{L}{2}\right)$$

Note that $E'(\theta) = 0$ is the same as the zero-torque condition (A2), and we did not have to mention torque or cross products to derive it !

Moreover, we can differentiate once more with respect to θ in (A8) and obtain

(A10) $$E''(\theta) = K(\cos\theta, \sin\theta)\cdot\int_0^{L_0} \underline{\tilde{X}}(s)(s-L_p)\,ds$$

Since we want an energy minimum, we require $E'(\theta)=0$ and $E''(\theta)>0$. The unique direction that satisfies these conditions is given by

(A11) $$(\cos\theta, \sin\theta) = \frac{\int_0^{L_0} \underline{\tilde{X}}(s)(s-L_p)\,ds}{\left|\int_0^{L_0} \underline{\tilde{X}}(s)(s-L_p)\,ds\right|}$$

The only possible non-uniqueness that we still have to worry about is that the right-hand side of (A11) might turn out to be 0/0. This will not happen, however, if K is sufficiently large, since that will keep $\underline{X}(s)$ and $\underline{Z}(s)$ close together, and then there will always be some θ'

such that

(A12) $$\underline{\tilde{X}}(s) \approx \underline{Z}(s,\theta') - \left(\frac{L}{2}, \frac{L}{2}\right)$$

$$= (\cos\theta', \sin\theta')\,(s - L_p)$$

so

(A13) $$\int_0^{L_0} \underline{\tilde{X}}(s)(s-L_p)\,ds$$

$$\approx (\cos\theta', \sin\theta') \int_0^{L_0} (s-L_p)^2\,ds$$

and

(A14) $$\left| \int_0^{L_0} \underline{\tilde{X}}(s)(s-L_p)\,ds \right| \approx \int_0^{L_0} (s-L_p)^2\,ds$$

which is a positive constant. Thus, for K sufficiently large the right-hand side of (A11) should always be well defined.

Now we can go a step further and combine (A11) and (A1) to write a non-local force law for the massless immersed boundary that makes no reference at all to θ or $\underline{Z}(s,\theta)$:

(A15) $$\underline{F}(s,t) = K\left(\underline{X}(s,t) - \underline{Z}(s,t)\right)$$

$$= K\left(\underline{\tilde{X}}(s,t) - (s-L_p)\frac{\int_0^{L_0} \underline{\tilde{X}}(s',t)(s'-L_p)\,ds'}{\left|\int_0^{L_0} \underline{\tilde{X}}(s',t)(s'-L_p)\,ds'\right|}\right)$$

where, as before

(A16) $$\underline{\tilde{X}}(s,t) = \underline{X}(s,t) - \left(\frac{L}{2}, \frac{L}{2}\right)$$

This gives a straightforward IB method for the massless case, since we now have, as usual, a map from

(A17) $$\underline{X}(\ ,t) \longrightarrow \underline{F}(\ ,t)$$

As an <u>optional</u> part of the homework, implement the massless case as well. Note that your code does not need to make any reference to θ, ω, $\underline{Z}$ but you can evaluate $\theta(t)$ and then $\underline{Z}(s,t)$ if you like by making use of (A11).