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Differential geometry of surfaces: Curvature

Let

(1) $$\underline{X} = \underline{X}(u,v)$$

be the parametric description of a surface in $\mathbb{R}^3$

Then

(2) $$d\underline{X} = \frac{\partial \underline{X}}{\partial u} du + \frac{\partial \underline{X}}{\partial v} dv$$

so the arclength ds is given by

(3) $$(ds)^2 = d\underline{X}\cdot d\underline{X} = \left\|\frac{\partial \underline{X}}{\partial u}\right\|^2 (du)^2 + 2\frac{\partial \underline{X}}{\partial u}\cdot\frac{\partial \underline{X}}{\partial v}\, du\, dv$$
$$+ \left\|\frac{\partial \underline{X}}{\partial v}\right\|^2 (dv)^2$$

We introduce the notation

(4) $$E = \left\|\frac{\partial \underline{X}}{\partial u}\right\|^2, \quad F = \frac{\partial \underline{X}}{\partial u}\cdot\frac{\partial \underline{X}}{\partial v}, \quad G = \left\|\frac{\partial \underline{X}}{\partial v}\right\|^2$$

so that

(5) $$(ds)^2 = (du\ dv)\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

By the Schwarz inequality

(6) $$\left\|\frac{\partial \underline{X}}{\partial u}\right\|^2 \left\|\frac{\partial \underline{X}}{\partial v}\right\|^2 \geq \left(\frac{\partial \underline{X}}{\partial u} \cdot \frac{\partial \underline{X}}{\partial v}\right)^2$$

with equality only if $\partial \underline{X}/\partial u$ and $\partial \underline{X}/\partial v$ are aligned. We assume the coordinates (u, v) have been chosen in such a way that $\partial \underline{X}/\partial u$ and $\partial \underline{X}/\partial v$ are <u>not</u> aligned, and then $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is a positive definite matrix.

Let

(7) $$\underline{A}(u, v) = \frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}$$

Then the area of the part of the surface that is the image of a region U in the (u, v) parameter plane is given by

(8) $$\int_U \|\underline{A}(u, v)\| \, du \, dv$$

We can evaluate $\|\underline{A}\|$ as follows

(9) $$\|\underline{A}\|^2 = \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right) \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

$$= \varepsilon_{ijk} \frac{\partial X_j}{\partial u} \frac{\partial X_k}{\partial v} \varepsilon_{ilm} \frac{\partial X_l}{\partial u} \frac{\partial X_m}{\partial v}$$

$$= \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right) \frac{\partial X_j}{\partial u} \frac{\partial X_k}{\partial v} \frac{\partial X_l}{\partial u} \frac{\partial X_m}{\partial v}$$

$$= \left\|\frac{\partial \underline{X}}{\partial u}\right\|^2 \left\|\frac{\partial \underline{X}}{\partial v}\right\|^2 - \left(\frac{\partial \underline{X}}{\partial u} \cdot \frac{\partial \underline{X}}{\partial v}\right)^2$$

$$= EG - F^2 = \det\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

Here we have used the summation convention and also the totally antisymmetric tensor

(10) $$\varepsilon_{ijk} = \begin{cases} 1, & i,j,k \text{ cyclic} \\ -1, & i,j,k \text{ anticyclic} \\ 0, & \text{otherwise} \end{cases}$$

To define the curvature of the surface, we consider a curve on the surface

(11) $$u = \bar{u}(s)\,, \quad v = \bar{v}(s)$$

where s = arclength. This implies that

(12) $$E\left(\frac{d\bar{u}}{ds}\right)^2 + 2F\left(\frac{d\bar{u}}{ds}\right)\left(\frac{d\bar{v}}{ds}\right) + G\left(\frac{d\bar{v}}{ds}\right)^2 = 1$$

The unit tangent to the curve is given by

(13) $$\underline{\hat{\tau}} = \frac{\partial \underline{X}}{\partial u}\frac{d\bar{u}}{ds} + \frac{\partial \underline{X}}{\partial v}\frac{d\bar{v}}{ds}$$

From this we derive

(14) $$\frac{d\hat{\tau}}{ds} = \frac{\partial^2 \underline{X}}{\partial u^2}\left(\frac{d\bar{u}}{ds}\right)^2 + 2\frac{\partial^2 \underline{X}}{\partial u \partial v}\frac{d\bar{u}}{ds}\frac{d\bar{v}}{ds}$$

$$+ \frac{\partial^2 \underline{X}}{\partial v^2}\left(\frac{d\bar{v}}{ds}\right)^2 + \frac{\partial \underline{X}}{\partial u}\frac{d^2\bar{u}}{ds^2} + \frac{\partial \underline{X}}{\partial v}\frac{d^2\bar{v}}{ds^2}$$

Now let

$$\underline{n} = \frac{\underline{A}}{\|\underline{A}\|} = \frac{\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}}{\left\| \frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v} \right\|} \tag{15}$$

Then $\underline{n}$ is orthogonal to $\partial \underline{X}/\partial u$ and $\partial \underline{X}/\partial v$, so

$$\underline{n} \cdot \frac{d\underline{\tau}}{ds} = L\left(\frac{du}{ds}\right)^2 + 2M\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + N\left(\frac{dv}{ds}\right)^2 \tag{16}$$

where

$$L = \underline{n} \cdot \frac{\partial^2 \underline{X}}{\partial u^2}, \quad M = \underline{n} \cdot \frac{\partial^2 \underline{X}}{\partial u \partial v}, \quad N = \underline{n} \cdot \frac{\partial^2 \underline{X}}{\partial v^2} \tag{17}$$

The quantity $n \cdot \frac{d\tau}{ds}$ with $s =$ arclength is called the <u>normal curvature</u> of the curve. Note that it depends, for a given point on the surface, only on the local <u>direction</u>

of the curve as given by its tangent vector.

We now derive equations for $\partial \underline{n}/\partial u$ and $\partial \underline{n}/\partial v$
We have

$$(18) \qquad \underline{n} \cdot \frac{\partial \underline{X}}{\partial u} = 0 \qquad \underline{n} \cdot \frac{\partial \underline{X}}{\partial v} = 0 \qquad \underline{n} \cdot \underline{n} = 1$$

Differentiation of these equations with respect to u and v gives

$$(19) \qquad L + \frac{\partial \underline{n}}{\partial u} \cdot \frac{\partial \underline{X}}{\partial u} = 0, \quad M + \frac{\partial \underline{n}}{\partial u} \cdot \frac{\partial \underline{X}}{\partial v} = 0, \quad \underline{n} \cdot \frac{\partial \underline{n}}{\partial u} = 0$$

$$(20) \qquad M + \frac{\partial \underline{n}}{\partial v} \cdot \frac{\partial \underline{X}}{\partial u} = 0, \quad N + \frac{\partial \underline{n}}{\partial v} \cdot \frac{\partial \underline{X}}{\partial v} = 0, \quad \underline{n} \cdot \frac{\partial \underline{n}}{\partial v} = 0$$

Since $\partial \underline{n}/\partial u$ and $\partial \underline{n}/\partial v$ are orthogonal to $\underline{n}$, they can be written in the form

(21) $$\frac{\partial \underline{n}}{\partial u} = B_{uu}\, \underline{n} \times \frac{\partial \underline{X}}{\partial u} + B_{uv}\, \underline{n} \times \frac{\partial \underline{X}}{\partial v}$$

(22) $$\frac{\partial \underline{n}}{\partial v} = B_{vu}\, \underline{n} \times \frac{\partial \underline{X}}{\partial u} + B_{vv}\, \underline{n} \times \frac{\partial \underline{X}}{\partial v}$$

and from these equations together with (19-20) we derive

(23) $$-L = B_{uv}\left(\underline{n} \times \frac{\partial \underline{X}}{\partial v}\right) \cdot \frac{\partial \underline{X}}{\partial u} = -B_{uv}\, \underline{n} \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

(24) $$-M = B_{uu}\left(\underline{n} \times \frac{\partial \underline{X}}{\partial u}\right) \cdot \frac{\partial \underline{X}}{\partial v} = B_{uu}\, \underline{n} \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

(25) $$-M = B_{vv}\left(\underline{n} \times \frac{\partial \underline{X}}{\partial v}\right) \cdot \frac{\partial \underline{X}}{\partial u} = -B_{vv}\, \underline{n} \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

(26) $$-N = B_{vu}\left(\underline{n} \times \frac{\partial \underline{X}}{\partial u}\right) \cdot \frac{\partial \underline{X}}{\partial v} = B_{vu}\, \underline{n} \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

But

(27) $$\underline{n} \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right) = \frac{\|\underline{A}\|^2}{\|\underline{A}\|} = \|\underline{A}\|$$

Therefore

(28) $$\frac{\partial \underline{n}}{\partial u} = -\frac{M}{\|\underline{A}\|}\left(\underline{n} \times \frac{\partial \underline{X}}{\partial u}\right) + \frac{L}{\|\underline{A}\|}\left(\underline{n} \times \frac{\partial \underline{X}}{\partial v}\right)$$

(29) $$\frac{\partial \underline{n}}{\partial v} = -\frac{N}{\|\underline{A}\|}\left(\underline{n} \times \frac{\partial \underline{X}}{\partial u}\right) + \frac{M}{\|\underline{A}\|}\left(\underline{n} \times \frac{\partial \underline{X}}{\partial v}\right)$$

and then we have

(30) $$\frac{\partial \underline{n}}{\partial u} \times \frac{\partial \underline{n}}{\partial v} = \left(-\frac{M^2}{\|\underline{A}\|^2} + \frac{LN}{\|\underline{A}\|^2}\right)\left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

$$= \frac{LN - M^2}{EG - F^2}\left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

$$= \frac{\det\begin{pmatrix} L & M \\ M & N \end{pmatrix}}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}}\left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

The quantity

$$K = \frac{\det\begin{pmatrix} L & M \\ M & N \end{pmatrix}}{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}} \tag{31}$$

is called the Gaussian curvature.

The unit normal

$$\underline{n} = \frac{\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}}{\left\| \frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v} \right\|} \tag{32}$$

associates a point on the unit sphere with each point on the surface.

Then K is the ratio of the area element on the unit sphere to the corresponding area element on the surface. That is,

$$\left(\frac{\partial \underline{n}}{\partial u} \times \frac{\partial \underline{n}}{\partial v}\right) du\, dv = K \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right) du\, dv \tag{33}$$

Remark:

The simplicity of equation (31) for the Gaussian curvature is somewhat misleading.
A better way to write the same result would be

$$K = \frac{\det\begin{pmatrix} L & M \\ M & N \end{pmatrix} \Big/ \sqrt{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}}}{\sqrt{\det\begin{pmatrix} E & F \\ F & G \end{pmatrix}}}$$

In this version, if we multiply the denominator by $du\,dv$ we get the area element on the original surface, and if we multiply the numerator by $du\,dv$ we get the corresponding area element on the sphere.

Another measure of curvature is obtained by considering the rate of change of area when the surface moves. Let the moving surface be denoted

(34) $$\underline{x} = \underline{X}(u, v, t)$$

Then

$$\|\underline{A}\|^2 = \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right) \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right)$$

(35) $$2\|\underline{A}\| \frac{\partial}{\partial t}\|\underline{A}\| = 2\left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}\right) \cdot \left(\left(\frac{\partial^2 \underline{X}}{\partial u \partial t} \times \frac{\partial \underline{X}}{\partial v}\right) + \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial^2 \underline{X}}{\partial v \partial t}\right)\right)$$

Dividing both sides by $2\|\underline{A}\|$, we get

(36) $$\frac{\partial}{\partial t}\|\underline{A}\| = \underline{n} \cdot \left(\frac{\partial^2 \underline{X}}{\partial u \partial t} \times \frac{\partial \underline{X}}{\partial v}\right) + \underline{n} \cdot \left(\frac{\partial \underline{X}}{\partial u} \times \frac{\partial^2 \underline{X}}{\partial v \partial t}\right)$$

$$= -\left(\underline{n} \times \frac{\partial \underline{X}}{\partial v}\right) \cdot \frac{\partial^2 \underline{X}}{\partial u \partial t} + \left(\underline{n} \times \frac{\partial \underline{X}}{\partial u}\right) \cdot \frac{\partial^2 \underline{X}}{\partial v \partial t}$$

Now consider a surface that is the image of a fixed region U of the (u,v) parameter plane. The area of this surface at time t is given by

$$(37)\qquad \mathcal{A}(t) = \int_U \|\underline{A}\|\, du\, dv$$

So

$$(38)\qquad \frac{d\mathcal{A}}{dt} = \int_U \frac{\partial}{\partial t}\|\underline{A}\|\, du\, dv$$

We assume that $\partial \underline{X}/\partial t$ and all of its u,v derivatives approach zero as the boundary of U is approached, so that we can integrate by parts without worrying about boundary terms.

Then, after integrating by parts, we have

(39) $$\frac{dA}{dt} = \int_U \left(\frac{\partial \underline{n}}{\partial u} \times \frac{\partial \underline{X}}{\partial v} - \frac{\partial \underline{n}}{\partial v} \times \frac{\partial \underline{X}}{\partial u} \right) \cdot \frac{\partial \underline{X}}{\partial t} \, du \, dv$$

Recall that $\partial \underline{n}/\partial u$ and $\partial \underline{n}/\partial v$ are orthogonal to $\underline{n}$, since $\underline{n}$ is a unit vector.

Also, $\partial \underline{X}/\partial u$ and $\partial \underline{X}/\partial v$ are orthogonal to $\underline{n}$ by definition of $\underline{n}$, equation (15).

Therefore, both $\frac{\partial \underline{n}}{\partial u} \times \frac{\partial \underline{X}}{\partial v}$ and $\frac{\partial \underline{n}}{\partial v} \times \frac{\partial \underline{X}}{\partial u}$

are multiples of $\underline{n}$, and we may write

(40) $$\frac{\partial \underline{n}}{\partial u} \times \frac{\partial \underline{X}}{\partial v} - \frac{\partial \underline{n}}{\partial v} \times \frac{\partial \underline{X}}{\partial u} = -H\underline{A} = -H\underline{n}\|\underline{A}\|$$

for some scalar H that remains to be determined. This makes (39) have the form

(41) $$-\frac{dA}{dt} = \int_U H \left(\underline{n} \cdot \frac{\partial \underline{X}}{\partial t} \right) \|\underline{A}\| \, du \, dv$$

Since $\|\underline{A}\|\,du\,dv$ is the element of area, this shows that H is minus the rate of change of area per unit area when the normal component of the surface velocity is equal to 1.

To evaluate H, we apply $\underline{n}\cdot$ to both sides of (40). This gives

(42) $$-H\|\underline{A}\| = \underline{n}\cdot\left(\frac{\partial \underline{n}}{\partial u}\times\frac{\partial \underline{X}}{\partial v}\right) - \underline{n}\cdot\left(\frac{\partial \underline{n}}{\partial v}\times\frac{\partial \underline{X}}{\partial u}\right)$$

$$= \left(\underline{n}\times\frac{\partial \underline{n}}{\partial u}\right)\cdot\frac{\partial \underline{X}}{\partial v} - \left(\underline{n}\times\frac{\partial \underline{n}}{\partial v}\right)\cdot\frac{\partial \underline{X}}{\partial u}$$

But $\left(\underline{n}\times\frac{\partial \underline{n}}{\partial u}\right)$ and $\left(\underline{n}\times\frac{\partial \underline{n}}{\partial v}\right)$ are easily evaluated from (28-29) by making use of

$\underline{n}\times(\underline{n}\times\underline{a}) = -\underline{a}$ for all $\underline{a}$ that are orthogonal to $\underline{n}$. In this way, we get

(43) $$\underline{n}\times\frac{\partial \underline{n}}{\partial u} = \frac{M}{\|\underline{A}\|}\frac{\partial \underline{X}}{\partial u} - \frac{L}{\|\underline{A}\|}\frac{\partial \underline{X}}{\partial v}$$

(44) $$\underline{n}\times\frac{\partial \underline{n}}{\partial v} = \frac{N}{\|\underline{A}\|}\frac{\partial \underline{X}}{\partial u} - \frac{M}{\|\underline{A}\|}\frac{\partial \underline{X}}{\partial v}$$

and (42) becomes

$$(45)\qquad -H\|\underline{A}\| = \frac{M}{\|\underline{A}\|}\frac{\partial \underline{X}}{\partial u}\cdot\frac{\partial \underline{X}}{\partial v} - \frac{L}{\|\underline{A}\|}\left\|\frac{\partial \underline{X}}{\partial v}\right\|^2$$

$$- \frac{N}{\|\underline{A}\|}\left\|\frac{\partial \underline{X}}{\partial u}\right\|^2 + \frac{M}{\|\underline{A}\|}\frac{\partial \underline{X}}{\partial u}\cdot\frac{\partial \underline{X}}{\partial v}$$

and then we have the result that

$$(46)\qquad H = \frac{LG - 2MF + NE}{EG - F^2}$$

For comparison, note that

$$(47)\qquad \operatorname{trace}\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2}\right)$$

$$= \operatorname{trace}\left(\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\right)$$

$$= \frac{1}{EG - F^2}\operatorname{trace}\left(\begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\right)$$

$$= \frac{LG - MF - MF + NE}{EG - F^2} = H$$

We claim that H is the sum of the normal curvatures of <u>any</u> two curves in the surface that cross in such a manner that their tangents are orthogonal at the point of crossing, where the normal curvatures are evaluated at the point of intersection. (Note that the two curvatures do not have to be the principal curvatures.)

To show this, let the two curves be parameterized by arclength, so that their normal curvatures are given by (16). Let

(48) $$u^i(s),\ v^i(s) \quad , \quad i=1,2$$

be the parametric description of the two curves.

At the point of intersection, we have

(49) $$E\,\frac{du^i}{ds}\frac{du^j}{ds} + 2F\,\frac{du^i}{ds}\frac{dv^j}{ds} + G\,\frac{dv^i}{ds}\frac{dv^j}{ds} = \delta_{ij}$$

which can also be written as

$$\begin{pmatrix} \frac{du^i}{ds} & \frac{dv^i}{ds} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \frac{du^j}{ds} \\ \frac{dv^j}{ds} \end{pmatrix} = \delta_{ij} \tag{50}$$

When $i=j$, equation (49) or (50) states that $s=$ arclength on each curve, and when $i \neq j$, equation (49) or (50) states that the two curves are orthogonal.

The sum of the two normal curvatures at the point of intersection is given by

$$C = \sum_{i=1}^{2} L\left(\frac{du^i}{ds}\right)^2 + 2M \frac{du^i}{ds}\frac{dv^i}{ds} + N\left(\frac{dv^i}{ds}\right)^2 \tag{51}$$

$$= \sum_{i=1}^{2} \begin{pmatrix} \frac{du^i}{ds} & \frac{dv^i}{ds} \end{pmatrix} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} \frac{du^i}{ds} \\ \frac{dv^i}{ds} \end{pmatrix}$$

let

$$(52) \qquad \begin{pmatrix} \alpha^i \\ \beta^i \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{1/2} \begin{pmatrix} \frac{du^i}{ds} \\ \frac{dv^i}{ds} \end{pmatrix}$$

for $i = 1, 2$. Then (50) becomes

$$(53) \qquad \begin{pmatrix} \alpha^i & \beta^i \end{pmatrix} \begin{pmatrix} \alpha^j \\ \beta^j \end{pmatrix} = \delta_{ij}$$

and (51) becomes

$$(54) \qquad C = \sum_{i=1}^{2} \begin{pmatrix} \alpha^i & \beta^i \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} \alpha^i \\ \beta^i \end{pmatrix}$$

$$= \operatorname{trace}\left(\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \underbrace{\sum_{i=1}^{2} \begin{pmatrix} \alpha^i \\ \beta^i \end{pmatrix} \begin{pmatrix} \alpha^i & \beta^i \end{pmatrix}}_{I} \right)$$

$= H$, as claimed.

The quantity H is often called the "mean curvature" but actually it is the sum of two curvatures (of curves running in orthogonal directions where they cross), so total curvature would be a better name. Of course, we could have introduced a factor of 2 into the definition to make the name "mean curvature" applicable. This is sometimes done, but there is little reason to do so other than the name, since the total curvature is a more useful quantity. As we have shown here, it is the total curvature that gives minus the rate of change of area per unit area when the surface is moving in the normal direction at unit speed. Also, a soap film with surface tension γ and a pressure difference across it is in equilibrium when γ times the total curvature is equal to the pressure difference.

Note that we have defined the Gaussian curvature K and the total curvature H without making any reference to the principal curvatures, which are the eigenvalues of the matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1/2}$$

In particular, it is often overlooked (or at least not mentioned) that H can be evaluated by choosing any two orthogonal directions tangent to the surface, not necessarily the directions in which the principal curvatures are obtained.