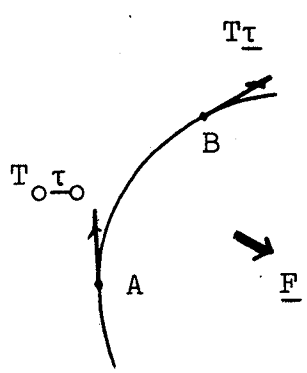
III. The Differential Geometry of the Heart

Results from Differential Geometry:*

Consider a curve in space $\underline{x}(s)$, s = arc length. The unit tangent vector to the curve is given by $\underline{\tau} = \frac{d\underline{x}}{ds}$, and the curvature is given by $k = |d\underline{\tau}/ds|$. Any vector $\underline{n}$ such that $\underline{n} \cdot \underline{\tau} = 0$ is normal to the curve, but the vector $\underline{n}_1 = \frac{d\underline{\tau}/ds}{|d\underline{\tau}/ds|}$ is called the principal normal, and the plane containing $\underline{\tau}$ and $\underline{n}_1$ is called the osculating plane to the curve.

The connection between these geometrical concepts and some aspects of mechanics can be seen in the following example. Consider the equilibrium of a fiber which is constrained to lie along a given curve in space, but is free to slip tangentially along that curve. Let T(s) be the tension in the fiber. Then the force exerted by the arc of fiber AB on the constraints is



$$\underline{F} = T\underline{\tau} - T_o\underline{\tau}_o$$

Let

$$\underline{f}(s) = \frac{d\underline{F}}{ds} = \frac{d}{ds}(T\underline{\tau}) = \frac{dT}{ds}\underline{\tau} + T\frac{d\underline{\tau}}{ds}$$

$$= \frac{dT}{ds}\underline{\tau} + k\underline{n}_1 T \quad .$$

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* see: Stoker, Differential Geometry, New York, Wiley 1969.

Because of free slip in the tangential direction

$$\underline{f} \cdot \underline{\tau} = 0 \rightarrow \frac{dT}{ds} = 0$$

and

$$\underline{f}(s) = k\underline{n}_1 T \quad .$$

Conclusions:

(1) Tension is constant.

(2) Force is in the direction of the principal normal.

(3) Force is proportional to fiber curvature.

(Remark: $k = 1/r$ where r = radius of curvature).

Curves on surfaces:

Let $\underline{x}(u,v)$ be a surface with unit normal $\underline{n}(u,v)$. As the parameters u, v are varied, the vector $\underline{x}$ sweeps out the surface and the vector $\underline{n}$ sweeps out a locus on the unit sphere. This locus is called the spherical image of the surface. A curve on a surface would be given by $\underline{x}(s) = \underline{x}(u(s), v(s))$ and the spherical image of the curve is given by $\underline{n}(s) = \underline{n}(u(s),v(s))$.

Parallel transport of a vector along a curve on a surface is defined by the differential equation

$$\left.\begin{array}{l} \dfrac{d\underline{w}}{ds} = \alpha(s)\underline{n}(s) \\ \text{where } \alpha(s) \text{ is chosen such that} \\ \underline{w} \cdot \underline{n} = 0 \end{array}\right\}$$

The first equation asserts that the changes in $\underline{w}$ are always normal to the surface, while the second asserts that $\underline{w}$ itself always lies parallel to the surface. The two equations can be combined as follows

$$0 = \frac{d}{ds}(\underline{w} \cdot \underline{n}) = \frac{d\underline{w}}{ds} \cdot \underline{n} + \underline{w} \cdot \frac{d\underline{n}}{ds}$$

$$= \alpha(s) + \underline{w} \cdot \frac{d\underline{n}}{ds}$$

Therefore:

$$\boxed{\frac{d\underline{w}}{ds} = -\left\{ \underline{w} \cdot \frac{d\underline{n}}{ds} \right\} \underline{n}(s)}$$

Note that if $\underline{w}_1$ and $\underline{w}_2$ are subjected to parallel transport along the same curve on the same surface, then

$$\frac{d}{ds}(\underline{w}_1 \cdot \underline{w}_2) = \frac{d\underline{w}_1}{ds} \cdot \underline{w}_2 + \underline{w}_1 \cdot \frac{d\underline{w}_2}{ds}$$

$$= 0$$

since

$$\underline{n} \cdot \underline{w}_1 = \underline{n} \cdot \underline{w}_2 = 0 \quad .$$

It follows that lengths of vectors are preserved during parallel transports (let $\underline{w}_1 = \underline{w}_2$), and that angles between pairs of vectors are preserved when the pair is subjected to parallel transport.

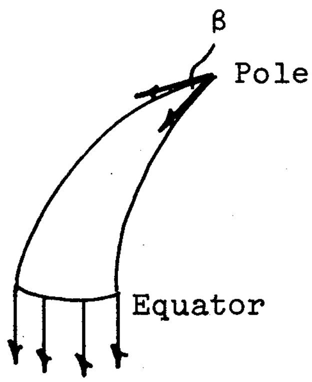
Remark: Since $\underline{n}(s)$ is the same for a curve on a surface and for the spherical image of that curve, parallel transport of a given initial vector along a curve on a surface yields the same field of vectors as parallel transport of that same initial vector along the corresponding curve on the spherical image. (Since $\underline{n}(s)$ is the same, the two fields obey the same differential equation.)

Parallel transport around a closed curve yields an angle β between the initial and final vectors. (It can be shown that β is a property of the curve and the surface but not of the starting point of the initial vector).

Examples

(1) In a plane $\beta = 0$ for all curves.

(2) On the surface of a sphere, consider the following special curve consisting of three arcs:



(i) pole to equator along a longitude

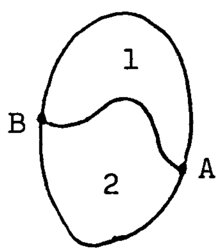
(ii) around the equator through an angle β

(iii) return to the pole along a longitude.

If we start with a vector at the pole which points tangent to the first longitude, then parallel transport consists of keeping this vector pointing south everywhere. The vector returns to the pole pointing tangent to the second longitude, and therefore the angle between the initial and final vectors is also β. Note: If the sphere is a unit sphere then the area of surface enclosed is

$$A = \frac{4\pi}{2}\frac{\beta}{2\pi} = \beta \ .$$

In general, if we have a region on a surface bounded by a simple closed curve and we partition the region into two parts we get $\beta = \beta_1 + \beta_2$. To prove this pick an arbitrary vector at A and construct its parallel transport around region 1 (counterclockwise). Then take the final vector from the transport around region 1 as initial vector for region 2, which is also circumnavigated in the counterclockwise direction. Thus we have defined a field of vectors on each of three arcs. The two fields on the interior arc are identical since parallel transport is reversible. It follows that the fields on the exterior arcs join up without a discontinuity at point β and therefore that together they constitute a parallel field on the exterior closed curve. If we call this field $\underline{w}$, then we have



$$\underline{w}^{\text{initial}} = \underline{w}_1^{\text{initial}}$$

$$\underline{w}^{\text{final}} = \underline{w}_2^{\text{final}}$$

$$\underline{w}_1^{\text{final}} = \underline{w}_2^{\text{initial}} \ .$$

It follows that $\beta = \beta_1 + \beta_2$,

since

$$\beta = \sphericalangle\,(\underline{w}^{\text{initial}}, \underline{w}^{\text{final}})$$

$$\beta_1 = \sphericalangle\,(\underline{w}_1{}^{\text{initial}}, \underline{w}_1{}^{\text{final}})$$

$$\beta_2 = \sphericalangle\,(\underline{w}_2{}^{\text{initial}}, \underline{w}_2{}^{\text{final}}) \;.$$

Since the formula $\beta = \beta_1 + \beta_2$ holds for an arbitrary partition of an arbitrary region into two parts it follows by induction that for a general subdivision into many parts that $\beta = \sum_i \beta_i$. This shows that β has the form $\beta = \int Kda$ where K is a function of position on the surface, and where the integration extends over the area enclosed by the curve in question.

On the unit sphere, by symmetry, K = constant, and we have already shown by a specific example that the constant is 1.

Therefore

$$\beta = \int K\,da \qquad \text{in general}$$

$$\beta = \int da \qquad \text{unit sphere .}$$

Now make use of the remark made above that parallel transport is the <u>same</u> for a curve and for its spherical image. Letting ' denote the spherical image we have

$$\beta = \int K\,da = \int da'$$

or

$$K = \frac{da'}{da}$$

since the integral equality holds for arbitrary regions. The quantity K is called Gaussian curvature.

Geodesics

Consider a fiber under tension which is constrained to lie on a given surface but may slip freely parallel to the surface. Since (see above)

$$\underline{f}(s) = k\underline{n}_1 T$$

we require for equilibrium that

$$\underline{n}_1 = \underline{n}$$

where $\underline{n}$ is the surface normal. This yields

$$\left.\begin{aligned} \frac{d\underline{\tau}}{ds} &= k\underline{n}_1 = k\underline{n} \\ \underline{\tau}\cdot\underline{n} &= 0 \end{aligned}\right\}$$

which is the pair of equations for parallel transport of the vector $\underline{\tau}$. We now define a geodesic as a curve whose unit tangent forms a parallel field and we see at once that fibers under tension on a surface form geodesics.

Some examples of geodesics:

(1) A straight line on a plane.

(2) A helix on a cylinder.

(3) A great circle* on a sphere.

(4) The shortest curve joining any pair of points in any surface (if there is a shortest curve).

Differential Geometry of Heart Valves:

Valve leaflets are surfaces under tension; the load on the closed valve is approximately a constant pressure.

(1) The line of closure. Valve leaflets touch along a curve called the line of closure. The non-zero tension in the leaflet must be supported along this line (which is approximately the edge of the leaflet.) Let T be the tension in the line of closure $\underline{x}(s)$ with unit tangent $\underline{\tau}$, and let $\underline{f}(s)$ be the force per unit length exerted by the membrane on the line of closure. Then $\underline{f}(s)$ is parallel to the membrane. The equilibrium of the line of closure is

$$\frac{d}{ds}(T\underline{\tau}) + \underline{f}(s) = 0$$

$$T\frac{d\underline{\tau}}{ds} + \frac{dT}{ds}\underline{\tau} + \underline{f}(s) = 0$$

The vectors $\underline{\tau}$ and $\underline{f}(s)$ are in the tangent plane of the membrane, so $\frac{d\underline{\tau}}{ds}$ must also lie in this plane. But $\frac{d\underline{\tau}}{ds}$ is the principal normal to the line of closure. Thus the line of closure has the property that its principal normal lies parallel to the surface. Such a

* A circle cut by a plane which includes the center of the sphere.

curve is called an asymptotic line on the surface, and such a curve can be constructed only on a surface where the Gaussian curvature, $K \leq 0$. This theorem, that the edge cables supporting a membrane under tension form asymptotic lines is proved in [*]. Here we note the following consequence: If two membranes under tension touch and are separately supported along the same curve (the line of closure of two heart valve leaflets) then the membranes not only touch along this curve but are tangent there.

Proof: The common line of closure is an asymptotic line on both surfaces. Therefore its osculating plane is tangent to both surfaces and hence the surfaces are tangent to each other.

Remark: The material stress (force/unit area) at the edge of the leaflet is of different (larger) order of magnitude than in the interior. Let

p = pressure load on the leaflet

R = typical radius of curvature of the leaflet

r = typical radius of curvature of the free edge

δ = thickness of the leaflet.

Then the stress in the leaflet is of order pR/δ, while the stress in the free edge is of order pRr/δ^2 which is larger by the factor r/δ. Two way to support the stress at the edge:

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[*] F. Otto, Tensile Structures M.I.T. Press 1967.

(1) provide a thickened cable-like structure near the free edge (aortic valve)

(2) break up the free edge into a sequence of sharply curved arches (i.e., reduce r). The mitral valve has this kind of structure.

(2) Load-bearing fibers.

Suppose that the load is borne by a single family of fibers covering the surface. Introduce parameters u, v where v = constant is the equation of a fiber and the parameter u measures arc length along fibers. The unit vector $\underline{\tau} = \frac{\partial \underline{x}}{\partial u}$ points in the fiber direction.

Consider a patch du, dv. The load on such a patch due to the pressure is

$$p(\underline{\tau} \times \frac{\partial \underline{x}}{\partial v}) du\, dv \ .$$

If $T\underline{\tau}dv$ is the force in the collection of fibers dv, then the net force on the patch due to the fibers is $\frac{\partial}{\partial u}(\underline{\tau}T)du\, dv$. Hence the equation of equilibrium

$$\frac{\partial T}{\partial u}\underline{\tau} + T\frac{\partial \underline{\tau}}{\partial u} + p(\underline{\tau} \times \frac{\partial \underline{x}}{\partial v}) = 0 \quad .$$

The last two terms above are normal to $\underline{\tau}$. Therefore

$$\frac{\partial T}{\partial u} = 0 \rightarrow T = T(v).$$

It follows that

$$\left.\begin{aligned} \frac{\partial \underline{\tau}}{\partial u} &= \frac{p}{T(v)}\left(\frac{\partial \underline{x}}{\partial v} \times \underline{\tau}\right) \\ \frac{\partial \underline{x}}{\partial u} &= \underline{\tau} \end{aligned}\right\}$$

This pair determines a surface if an initial curve is given together with initial fiber directions and the function T(v). Since $\frac{\partial \underline{x}}{\partial v} \times \underline{\tau}$ is normal to the surface, we conclude that the fibers are geodesics. If the initial curve is the line of closure then the two leaflets which meet there are both generated by the differential equation if we insert a ± sign in front of the pressure.

(3) Leaflet motion:

Suppose that there are one or more families of fibers covering the surface and that during the motion of the leaflet (opening and closing) when the load is small these can be regarded as inextensible. If there is more than one family, assume that the fibers of one family do not slip through those of another. We examine the constraints imposed on the surface metric by these inextensible fibers. Arc length is given by

$$ds^2 = E\,du^2 + 2F du\,dv + G\,dv^2$$

(i) A single family of inextensible fibers: Let u = arc length along fibers which are given by v = constant. Then E = 1 but F and G are arbitrary. Such parameters can be introduced on an arbitrary surface, so there is no restriction on the shape of the surface in space.

(ii) Two families of inextensible fibers: Let

$u = \text{constant} \rightarrow$ fibers of one family

$v = \text{constant} \rightarrow$ fibers of other family

Then E, G are given functions of (u, v) but F is arbitrary. (This is a net of fibers with fixed intersections but variable angles between fibers - a special case is a Tchebychef net which has E = G = 1).

(iii) Three or more families of inextensible fibers. Pick two families of fibers and introduce parameters u, v as above. This fixes E(u, v) G(u, v). A third family of fibers will then have the form u(r, s), v(r, s) where r identifies the fiber and s measures arc length along it. Taking differentials along such a fiber we have

$$1 = E\left(\frac{du}{ds}\right)^2 + 2F\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + G\left(\frac{dv}{ds}\right)^2$$

Now E(u, v) and G(u, v) are fixed, and

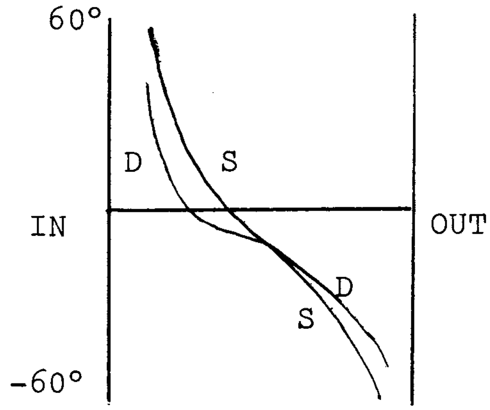
$$\frac{du}{ds} = \left(\frac{\partial u}{\partial s}\right)\Big|_{r=\text{constant}} \quad \text{and} \quad \frac{dv}{ds} = \left(\frac{\partial v}{\partial s}\right)\Big|_{r=\text{constant}}$$

are known as functions of (r, s) and hence of (u, v). Therefore F is also determined unless $\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) = 0$. The latter would imply that the third family of fibers was parallel to one of the first two. With three or more families of fibers the surface metric is completely determined, and the motions of the surface are those which leave <u>all</u> lengths of arcs invariant.

<u>Fiber Architecture of the Heart Wall</u>:

Each point of the muscular heart wall can be characterized by a <u>direction</u> in which the muscle fibers run at that point.

These directions vary smoothly throughout the heart wall; they have been measured by Streeter* in the equitorial plane of the left ventricle with roughly these results. The fibers



are circular near the center of the wall but become progressively more inclined with respect to the equitorial plane as one moves toward either the inner or the outer surface of the heart. In the reference cited the authors use this fiber distribution to calculate the distribution of pressure and fiber stress in the heart wall.** To do this they make the assumption that the fibers lie along surfaces which are elipsoids of revolution.

Here we shall take the opposite point of view and see whether a differential equation for the shape of the heart can be derived from the equilibrium of its muscle fibers.

Consider a region of heart wall bounded by a surface S.

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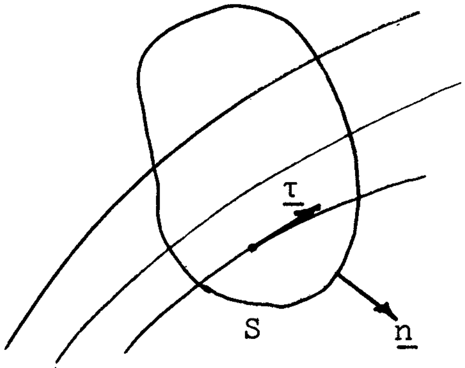
* Streeter, D.D., Vaishnav, Patel, D.J., Spotnitz, H.M., Ross, J., Sonnenblick, E.H., "Stress Distribution in the Canine Left Ventricle During Diastole and Systole" Biophysical J. <u>10</u>, 345 (1970).

**This and other efforts to compute the stress distribution are reviewed and compared in Stroot, M.T., Douglas, M.A., and Bailey, J.J., "An Analysis of Myocardial Stress Formulations in the Human Left Ventricle" Preprint (National Institutes of Health).



Let

$\underline{E}$ = Force on the region bounded by S

$\underline{n}$ = Unit normal on S

$\underline{\tau}$ = Unit tangent to fiber direction

T = Tension per fiber

σ = Fibers per unit area

$\underline{F}$ = Force per unit volume due to the fibers.

v = Region bounded by the surface S

$$\underline{E} = \int_S \sigma T \underline{\tau} \; (\underline{\tau} \cdot \underline{n}) da \quad .$$

The component of E in the space direction k is

$$E_k = \int_S \sigma T \tau_k \; (\underline{\tau} \cdot \underline{n}) da = \int_v \nabla \cdot (\sigma T \tau_k \underline{\tau}) dv$$

$$= \int_v [\nabla \cdot (\sigma \underline{\tau}) T \tau_k + \sigma \underline{\tau} \cdot \nabla (T \tau_k)] dv$$

But $\nabla \cdot (\sigma \underline{\tau}) = 0$ (conservation of fibers). And $\underline{\tau} \cdot \nabla = \frac{d}{ds}$ where s measures arc length along fibers, and where $\frac{d}{ds}$ denotes the derivative with respect to arc length in the fiber direction.

It follows that

$$E_k = \int_v \sigma \frac{d}{ds}(T \tau_k) dv$$

$$\underline{E} = \int_v \sigma \frac{d}{ds}(T \tau) dv$$

$$\underline{F} = \sigma \frac{d}{ds}(T\underline{\tau}) = \text{force per unit volume due to the fibers.}$$

It we assume that the fibers are free to slip tangentially, then the tangential component of force $\sigma \frac{dT}{ds} \underline{\tau}$ must vanish and we have

$$\boxed{\frac{dT}{ds} = 0 \qquad \underline{F} = \sigma T \frac{d\underline{\tau}}{ds}}$$

Now assume that the fibers of the heart wall are always approximately in equilibrium with the pressure gradiant in the wall. This allows us to write

$$\sigma T \frac{d\underline{\tau}}{ds} = \nabla p \quad .$$

Therefore

$$\underline{\tau}\cdot\nabla p = \sigma T \underline{\tau}\cdot \frac{d\underline{\tau}}{ds} = \frac{\sigma T}{2} \frac{d}{ds} |\underline{\tau}|^2 = 0$$

so the fibers lie on the surfaces p = constant. Moreover they are geodesics on these surfaces since their curvature vector $\frac{d\underline{\tau}}{ds}$ lies parallel to p and hence normal to the surface p = constant.

<u>Remark</u>: The foregoing explains the sense in which it can be true that fibers in the <u>interior</u> of a tissue can be geodesics (without being straight lines). They are geodesics on the surfaces of constant pressure. It has often been mentioned that fibers in the walls of blood vessels are helical, and a helix is a geodesic on a cylinder. Of course, the wall is a

thick cylinder, not a surface, but the surfaces of constant pressure should be concentric with the inner and outer surfaces of the vessel wall.

The results derived above suggest the introduction of a special coordinate system as follows. Use the parameters (s,v,p)

$$s = \text{arc length along fiber}$$

$$p = \text{pressure}$$

$$p,v = \text{constant along each fiber.}$$

Note that:

$$\sigma\underline{\tau}\cdot\left(\frac{\partial \underline{\underline{x}}}{\partial v} \times \frac{\partial \underline{\underline{x}}}{\partial p}\right) dvdp = \#\text{ of fibers in the collection } dvdp\ .$$

Therefore the quantity λ defined by

$$\frac{1}{\lambda} = T\sigma\underline{\tau}\cdot\left(\frac{\partial \underline{\underline{x}}}{\partial v} \times \frac{\partial \underline{\underline{x}}}{\partial p}\right)$$

is constant along each fiber. $\lambda = \lambda(v,p)$. Since $\underline{\tau} = \frac{\partial \underline{\underline{x}}}{\partial s}$, the quantity $\underline{\tau}\cdot\left(\frac{\partial \underline{\underline{x}}}{\partial v} \times \frac{\partial \underline{\underline{x}}}{\partial p}\right)$ can also be written

$$\frac{\partial \underline{\underline{x}}}{\partial s} \cdot \left(\frac{\partial \underline{\underline{x}}}{\partial v} \times \frac{\partial \underline{\underline{x}}}{\partial p}\right) = \frac{\partial(x,y,z)}{\partial(s,v,p)}$$

the Jacobian of the change of coordinates. The quantity

$$\nabla p\, \frac{\partial(x,y,z)}{\partial(s,v,p)} = \nabla p\, \frac{\partial \underline{\underline{x}}}{\partial p} \cdot \left(\frac{\partial \underline{\underline{x}}}{\partial s} \times \frac{\partial \underline{\underline{x}}}{\partial v}\right)$$

But $\frac{\partial \underline{\underline{x}}}{\partial s}$, $\frac{\partial \underline{\underline{x}}}{\partial v}$ both lie on the surface p = constant and therefore $\left(\frac{\partial \underline{\underline{x}}}{\partial s} \times \frac{\partial \underline{\underline{x}}}{\partial v}\right)$ lies normal to this surface and hence parallel to ∇p.

Therefore we can interchange the role of ∇p and this vector in the foregoing to obtain the identity

$$\nabla p \, \frac{\partial(x,y,z)}{\partial(s,v,p)} = \nabla p \cdot \frac{\partial \underline{x}}{\partial p} \left(\frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial v}\right)$$

$$= \left(\frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial v}\right) = \left(\underline{\tau} \times \frac{\partial \underline{x}}{\partial v}\right)$$

where we have used

$$\nabla p \cdot \frac{\partial \underline{x}}{\partial p} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial p}{\partial z}\frac{\partial z}{\partial p} = 1 \quad .$$

Now take the equation of equilibrium

$$\sigma T \frac{d\underline{\tau}}{ds} = \nabla p$$

and multiply both sides by $\frac{\partial(x,y,z)}{\partial(s,v,p)}$ to obtain

$$\boxed{\frac{1}{\lambda}\frac{d\underline{\tau}}{ds} = \left(\underline{\tau} \times \frac{\partial \underline{x}}{\partial v}\right)}$$

where λ is independent of s.

Next, we apply this equation to the heart wall. We make Streeter's assumption of axial symmetry but otherwise we don't assume the shape of the heart in advance. Introduce cylindrical coordinates r, θ, z and the corresponding unit vectors $\underline{\hat{r}}$, $\underline{\hat{\theta}}$, $\underline{\hat{z}}$. Note that $\underline{\hat{r}}$ and $\underline{\hat{\theta}}$ depend on θ, with $\frac{d}{d\theta}\,\underline{\hat{r}} = \underline{\hat{\theta}}$, $\frac{d}{d\theta}\,\underline{\hat{\theta}} = -\underline{\hat{r}}$. The unit tangent to a fiber is given by

$$\underline{\tau} = \frac{dr}{ds}\,\underline{\hat{r}} + r\,\frac{d\theta}{ds}\,\underline{\hat{\theta}} + \frac{dz}{ds}\,\underline{\hat{z}}$$

where

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1$$

The equation of equilibrium is

$$\frac{d\underline{\tau}}{ds} = \lambda \left(\frac{\partial \underline{x}}{\partial v} \times \underline{\tau}\right) .$$

Here we let $v = \theta$ in the plane $z = 0$. Then v remains constant along a fiber so that in some other plane $v = \theta + \theta_o$. It follows that $\frac{\partial \underline{x}}{\partial v} = r\underline{\hat{\theta}}$ and the equation of equilibrium for the case of axial symmetry becomes

$$\begin{aligned}\frac{d\underline{\tau}}{ds} &= \lambda \left(r\underline{\hat{\theta}} \times \underline{\tau}\right) \\ &= -\lambda r \frac{dr}{ds}\,\underline{\hat{z}} + \lambda r \frac{dz}{ds}\,\underline{\hat{r}} .\end{aligned}$$

Here λ is independent of s (as before) and also of θ because of the symmetry. Thus $\lambda = \lambda(p)$. Differentiating the formula for $\underline{\tau}$ (From here on $'$ indicates $\frac{d}{ds}$) we get

$$\frac{d\underline{\tau}}{ds} = [r'' - r(\theta')^2]\underline{\hat{r}} + [(r\theta')' + r'\theta']\underline{\hat{\theta}} + z''\underline{\hat{z}} .$$

Comparison with the equation of equilibrium yields

$$\left.\begin{aligned} r'' &= r(\theta')^2 + \lambda r z' \\ \theta'' &= -\frac{2}{r}\, r'\theta' \\ z'' &= -\lambda r r' \end{aligned}\right\}$$

and one can verify that

$$[(r')^2 + r^2(\theta')^2 + (z')^2]' = 0$$

as required by the interpretation of s as arc length.

Simplify the foregoing system of equations as follows:

$$z'' = -\lambda r r' = -\frac{1}{2}\lambda(r^2)'$$

$$\boxed{z' = -\frac{1}{2}\lambda r^2 + K}$$

$$\begin{aligned} rr'' &= r^2(\theta')^2 + \lambda r^2 z' \\ &= 1 - (r')^2 - (z')^2 + \lambda r^2 z' \\ &= 1 - (r')^2 - (K - \tfrac{1}{2}\lambda r^2)^2 + \lambda r^2(K - \tfrac{1}{2}\lambda r^2) \end{aligned}$$

But

$$\begin{aligned} \tfrac{1}{2}(r^2)'' &= (rr')' = rr'' + (r')^2 \\ &= (1 - K^2) + 2K\lambda r^2 - \tfrac{3}{4}\lambda^2 r^4 \end{aligned}$$

Let $\rho = r^2$. Then we have the following differential equation for ρ

$$\tfrac{1}{2}\rho'' = (1-K^2) + 2k\lambda\rho - \tfrac{3}{4}\lambda^2\rho^2 = f(\rho)$$

$$\tfrac{1}{2}\rho''\rho' = f(\rho)\rho'$$

$$\tfrac{1}{4}(\rho')^2 = F(\rho) + a$$

where

$$F(\rho) = (1-K^2)\rho + K\lambda\rho^2 - \tfrac{1}{4}\lambda^2\rho^3$$

so that

$$\frac{dF}{d\rho} = f \quad .$$

Suppose that at $\rho = \rho_o$, $\rho' = 0$. (This will be in the equitorial plane $z = 0$). Then $a = -F(\rho_o)$ and we have

$$\rho' = \pm 2\sqrt{F(\rho)-F(\rho_o)}$$

But we already have

$$z' = K - \frac{1}{2}\lambda\rho$$

Therefore

$$\frac{dz}{d\rho} = \frac{K - \frac{1}{2}\lambda\rho}{\pm 2\sqrt{F(\rho)-F(\rho_o)}}$$

where

$$\rho = r^2$$

$$F(\rho) = (1 - K^2)\rho + K\lambda\rho^2 - \frac{1}{4}\lambda^2\rho^3 \quad .$$

This differential equation defines a three parameter family of surfaces of revolution which are possible shapes for the surfaces p = constant in the heart wall. The parameters are λ, ρ_o, K. The parameter ρ_o fixes the initial radius but does not affect the shape of the surface if λ is varies according to $\lambda = \frac{\lambda_o}{\rho_o}$ where λ_o is dimensionless. Thus there are essentially only two parameters K and λ_o. The parameter K is easily interpreted in terms of λ_o as follows. The quantity $\frac{dz}{ds}$ in the equitorial plane is the sine of the fiber angle measured by Streeter. Therefore

$$\sin\theta = K - \frac{1}{2}\lambda\rho_o = K - \frac{1}{2}\lambda_o \quad .$$

Thus we can write $K = \frac{1}{2}\lambda_o + \sin\theta$. For each initial fiber angle θ we get a one parameter family of surfaces, the parameter being λ_o. It would be extremely interesting at this point if some "design criterion" could be invoked to predict Streeter's distribution of initial angles θ through the wall and also the distribution of λ_o through the wall and hence to determine the architecture of the myocardium uniquely.

Such a criterion might involve uniformity of σ or T through the wall, or perhaps the necessity that the whole structure hang together as it is deformed during its motions.