

ENTROPY IN BIOLOGY

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Lecture 3 — part 1

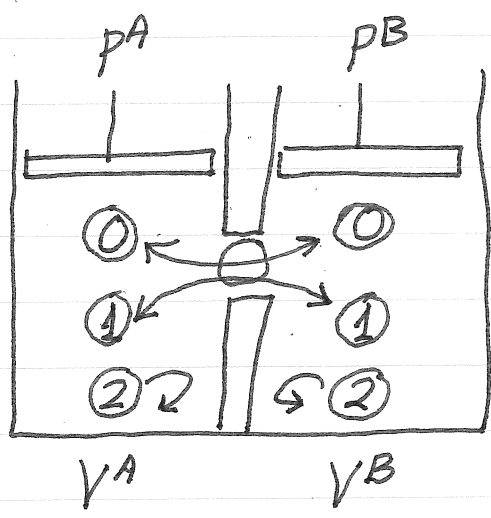
Osmotic pressure and flow in a
single-site channel with coupled
solvent and solute fluxes

Evaluation of the steady-state fluxes

Equilibrium

Steady-state fluxes near equilibrium:
The Onsager relation

Osmotic pressure and flow in the case of a one-site channel with two solutes, one of which can enter the channel and the other cannot



$T = \text{constant}$

We consider an osmotic system involving a solvent (0) and two solutes (1, 2). There are two chambers A, B at pressures $p_{A,B}$ and with volumes $V_{A,B}$. The two pressures are held constant, and the temperature T of the whole system is also held constant.

The chambers are separated by a membrane containing a single-site channel. The channel site can be occupied by a solvent molecule, or by a solute molecule of type 1, but solute molecules of type 2

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cannot get into the channel, and therefore they cannot cross the membrane.

We assume that the channel site is never empty, so its possible states are

$$(1) \quad C = 0, 1$$

in which $C=0$ means that the channel is occupied by a solvent molecule, and $C=1$ means that it is occupied by a solute molecule of type 1. We assume that these two channel states have different energies, which will be denoted U_0 and U_1 . Thus, the channel in

state C has energy U_C .

Let $N_i^{A,B}$ be the number of molecules of species i on side A or B of the membrane. For $i=2$, these numbers are constant. For $i=0,1$, these numbers are variable, and satisfy the following constraints

$$(2) \quad N_0^A + (1-C) + N_0^B = N_0$$

$$(3) \quad N_1^A + C + N_1^B = N_1$$

in which N_0 and N_1 are constants,

with N_0 being the total number of solvent molecules in the system and with N_1 being the total number of solute molecules of type 1.

Let v_i be the volume occupied by one molecule of species i . In order to avoid changes in volume of the membrane itself when the channel state changes, we assume that

$$(4) \quad v_1 = v_0$$

The volumes $V^{A,B}$ are then given by

$$(5) \quad V^{A,B} = (N_0^{A,B} + N_1^{A,B})v_0 + N_2^{A,B}v_2$$

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Ignoring constant terms, the Gibbs free energy of the system is as follows:

$$(6) \quad G = U_C - kT \log \prod_{I=A,B} \frac{(N_0^I + N_1^I + N_2^I)!}{(N_0^I)! (N_1^I)! (N_2^I)!} + \sum_{I=A,B} P^I \left((N_0^I + N_1^I) v_0 + N_2^I v_2 \right)$$

The allowed transitions of the system are right shifts and left shifts, denoted

$$(7) \quad \begin{pmatrix} R \\ i j \end{pmatrix} > \begin{pmatrix} L \\ i j \end{pmatrix}$$

where $i, j \in \{0, 1\}$. Here $C = i$ before the transition, and $C = j$ afterwards.

In the transition $\begin{pmatrix} R \\ i j \end{pmatrix}$ both of the species

i and j move to the right, with

i leaving the channel and becoming part

of the solution on the right side of the membrane and with species j leaving the solution on the left side of the membrane to occupy the channel site.

The transition $\binom{L}{ij}$ is the mirror image of $\binom{R}{ij}$, and the transition $\binom{L}{ji}$

is the inverse of the transition $\binom{R}{ij}$.

During a transition $\binom{R}{ij}$ the state of the system as a whole changes as follows

$$(8) \quad \left\{ \begin{array}{l} N_i^B \rightarrow N_i^B + 1 \\ N_j^A \rightarrow N_j^A - 1 \\ C \rightarrow C - i + j \end{array} \right.$$

and therefore

$$(9) \quad (\Delta G)_{ij}^R = U_j - U_i + (P^B - P^A) v_0$$

$$-kT \log \frac{(N_0^A + N_1^A + N_2^A - 1)!}{\left(\frac{(N_0^A)! (N_1^A)! (N_2^A)!}{N_j^A} \right)} \frac{(N_0^B + N_1^B + N_2^B + 1)!}{(N_0^B)! (N_1^B)! (N_2^B)! (N_i + 1)}$$

$$+ kT \log \frac{(N_0^A + N_1^A + N_2^A)!}{(N_0^A)! (N_1^A)! (N_2^A)!} \frac{(N_0^B + N_1^B + N_2^B)!}{(N_0^B)! (N_1^B)! (N_2^B)!}$$

$$= U_j - U_i + (P^B - P^A) v_0$$

$$- kT \log \left(\frac{N_j^A}{N_0^A + N_1^A + N_2^A} \right) + kT \log \left(\frac{N_i^B + 1}{N_0^B + N_1^B + N_2^B + 1} \right)$$

$$= U_j - U_i + (P^B - P^A) v_0 + kT \log \frac{(X_i^B)_{\text{final}}}{(X_j^A)_{\text{initial}}}$$

Similarly,

$$(10) \quad (\Delta G)_{ij}^L = U_j - U_i + (P^A - P^B)v_0 + kT \log \frac{(X_i^A)_{\text{final}}}{(X_j^B)_{\text{initial}}}$$

Now suppose that our system is so large that the mole fractions of all species on both sides of the membrane can be regarded as constants. Then we can drop the labels "final" and "initial", and we then have the following:

$$(11) \quad (\Delta G)_{ij}^R = U_j - U_i + (P^B - P^A)v_0 + kT \log \frac{X_i^B}{X_j^A}$$

$$(12) \quad (\Delta G)_{ij}^L = U_j - U_i + (P^A - P^B)v_0 + kT \log \frac{X_i^A}{X_j^B}$$

By interchanging (i,j) in (12) and comparing the result with (11), we see that

$$(13) \quad (\Delta G)_{ji}^L = -(\Delta G)_{ij}^R$$

Remark:

When the system is regarded as finite, its set of possible values is the lattice

(14) $C = 0, 1$

(15) $N_0^B = 0, \dots, N_0 - (C - 1)$

(16) $N_1^B = 0, \dots, N_1 - C$

A particular type of transition such as $\binom{R}{ij}$ has a ΔG value that depends on

which pair of lattice points it connects,

hence the labels "final" and "initial"

in equations (9-10). When we consider the

limit in which the volumes of the chambers

V^A and V^B are infinite, but with finite

and constant mole fractions of all species,

this situation changes drastically. Now, all of the transitions $(R)_{ij}$ have $(\Delta G)_{ij}^R$ given by equation (11), and all of the transitions $(L)_{ij}$ have $(\Delta G)_{ij}^L$ given by equation (12).

We can, if we like, think of the system as evolving on a lattice

$$(17) \quad (C, N_0^B, N_1^B) \in \{0, 1\} \times \mathbb{Z} \times \mathbb{Z}$$

where \mathbb{Z} is the set of all integers

(positive, negative, and zero), and where

N_i^B is the net number of molecules of

species i that has entered the reservoir

on side B of the membrane since $t=0$.

Note that in this setting, the mole fractions

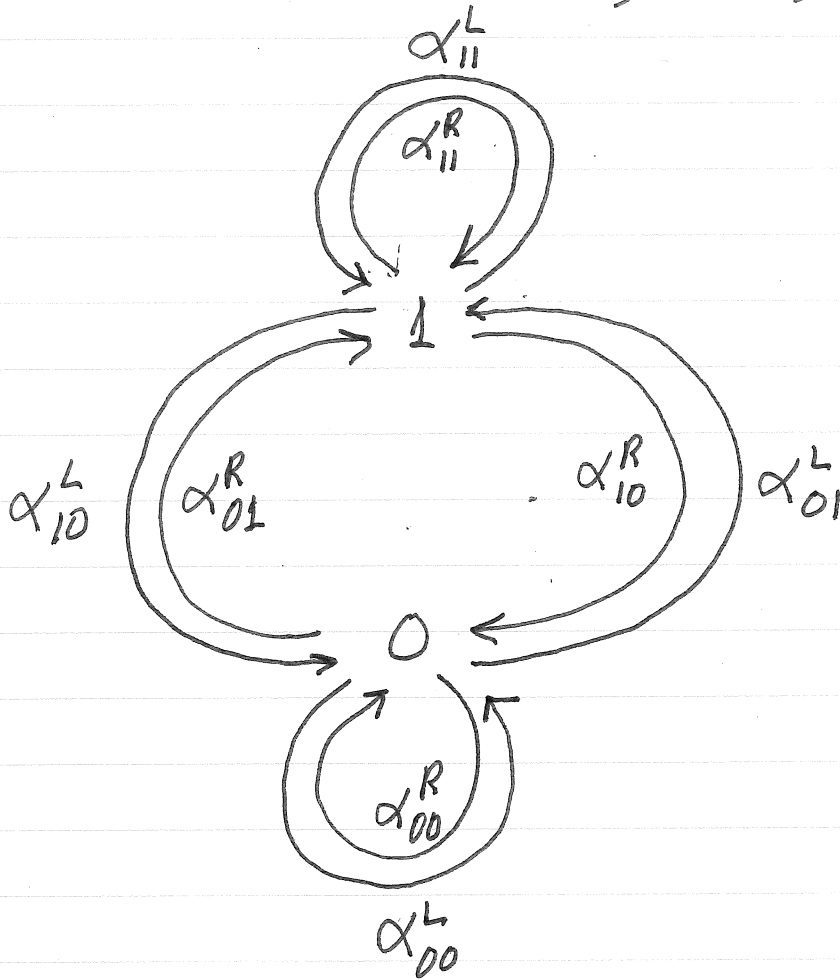
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are fixed, and the finite numbers N_i^B do not alter them. For this reason, all of the lattice points with $C=0$ are equivalent to each other, and all of the lattice points with $C=1$ are equivalent to each other.

What the system does is a translation-invariant random walk in this lattice.

Because of the translation invariance we can work entirely with the reduced state C as if it were the state of the system.

The continuous-time Markov chain for the two-state process with the states $C=0, 1$ is shown in the following diagram



The right shifts (R) are shown as clockwise arrows, and the left shifts (L) are shown as counterclockwise arrows.

There are 4 pairs of transitions in the above diagram with net flux to the right given by

$$(18) \quad f_{ij} = P_i \alpha_{ij}^R - P_j \alpha_{ji}^L$$

for $i=0,1$ and $j=0,1$, where P_i is the probability that $C=i$. In a steady state

$$(19) \quad f_{01} = f_{10}$$

so

$$(20) \quad P_0 \alpha_{01}^R - P_1 \alpha_{10}^L = P_1 \alpha_{10}^R - P_0 \alpha_{01}^L$$

Rearranging, this we get

$$(21) \quad P_0 (\alpha_{01}^R + \alpha_{01}^L) = P_1 (\alpha_{10}^R + \alpha_{10}^L)$$

and this, together with

$$(22) \quad p_0 + p_1 = 1$$

determines p_0 and p_1 as follows

$$(23) \quad p_0 = \frac{\alpha_{10}^R + \alpha_{10}^L}{\alpha_{10}^R + \alpha_{10}^L + \alpha_{01}^R + \alpha_{01}^L}$$

$$(24) \quad p_1 = \frac{\alpha_{01}^R + \alpha_{01}^L}{\alpha_{10}^R + \alpha_{10}^L + \alpha_{01}^R + \alpha_{01}^L}$$

We would like to evaluate the net, ^{Steady-state} rates at which solvent molecules and solute molecules of type 1 are transported across the membrane from left to right.

These fluxes can be evaluated by considering transport across either face of the membrane, and of course the result should be the same either way. We get

$$(25) \quad f_0 = f_{00} + f_{01} = f_{00} + f_{10}$$

$$(26) \quad f_1 = f_{11} + f_{10} = f_{11} + f_{01}$$

Equation (19) guarantees that the two forms of f_0 are the same and that the two forms of f_1 are the same. Here, f_0 is the flux of solvent across the membrane, and f_1 is the flux of solute of type 1.

Substituting (23) & (24) into (18), we have formulae for all 4 of the f_{ij} in terms of the 8 parameters α_{ij}^R and α_{ij}^L . These in turn can be expressed in terms of the corresponding changes in Gibbs free energy by making the assumption that

$$(27) \quad \frac{\alpha_{ij}^R}{\alpha_{ji}^L} = \exp\left(-\frac{(\Delta G)_{ij}^R}{kT}\right)$$

$$(28) \quad \alpha_{ij}^R - \alpha_{ji}^L = -\gamma \frac{(\Delta G)_{ij}^R}{kT}$$

for $i=0,1$ and $j=0,1$, and where γ is a constant with units of 1/time. Equations (27-28) imply that-

$$(29) \quad \alpha_{ij}^R = \gamma \frac{\frac{(\Delta G)_{ij}^R}{kT}}{\exp\left(\frac{(\Delta G)_{ij}^R}{kT}\right) - 1}$$

$$(30) \quad \alpha_{ji}^L = \gamma \frac{\frac{(\Delta G)_{ij}^R}{kT}}{1 - \exp\left(-\frac{(\Delta G)_{ij}^R}{kT}\right)}$$

Note that the fluxes f_0 and f_1 are now fully determined, see equations (25-26), (18), (23-24), (29-30), and (11).

Instead of writing out formulae for f_0 and f_1 , however, we now consider the behavior of these fluxes near a state of thermodynamic equilibrium. First, we determine the equilibrium state.

According to the principle of detailed balance, all four of the individual ^{net} fluxes f_{ij} given by equation (18) must be equal to zero. This gives the equations

$$(32) \quad P_0 \alpha_{00}^R - P_0 \alpha_{00}^L = 0$$

$$(33) \quad P_0 \alpha_{01}^R - P_1 \alpha_{10}^L = 0$$

$$(34) \quad P_1 \alpha_{10}^R - P_0 \alpha_{01}^L = 0$$

$$(35) \quad P_1 \alpha_{11}^R - P_1 \alpha_{11}^L = 0$$

Note that P_0 cancels out of (32) and P_1 cancels out of (35), and then by making use of (27) & (11), these equations imply

$$(36) \quad 0 = (\Delta G)_{00}^R = (P^B - P^A) v_0 + kT \log \frac{X_0^B}{X_0^A}$$

$$(37) \quad 0 = (\Delta G)_{11}^R = (P^B - P^A) v_0 + kT \log \frac{X_1^B}{X_1^A}$$

Equations (33) & (34) determine the ratio P_0/P_1 , and we need to check that they are consistent. From (33), (27), and (11), we get

$$(38) \quad \frac{P_1}{P_0} = \frac{\alpha_{01}^R}{\alpha_{10}^L} = \exp\left(-\frac{(\Delta G)_{01}^R}{kT}\right)$$

$$= \exp\left(-\frac{1}{kT} \left(U_1 - U_0 + (P^B - P^A) v_0 + kT \log \frac{X_0^B}{X_1^A} \right) \right)$$

$$= \exp\left(-\frac{1}{kT} \left(U_1 - U_0 + kT \log \frac{X_0^B}{X_1^A} \right) \right)$$

In the last step, we used (36) to eliminate $(P^B - P^A) v_0$.

Similarly, from (34), (27), and (11),

$$\begin{aligned}
 (39) \quad \frac{p_0}{p_1} &= \frac{\alpha_{10}^R}{\alpha_{01}^L} = \exp\left(-\frac{(\Delta G)_{10}^R}{RT}\right) \\
 &= \exp\left(-\frac{1}{RT}(U_0 - U_1 + (p^B - p^A)v_0^* + kT \log \frac{x_1^B}{x_0^A})\right) \\
 &= \exp\left(-\frac{1}{RT}(U_0 - U_1 + kT \log \frac{x_1^A}{x_0^A})\right)
 \end{aligned}$$

To get the last line of (39), we used (37) to eliminate $(p^B - p^A)v_0^*$.

Equations (38) & (39) are indeed consistent, since the product of their right-hand sides is equal to 1.

Note that equation (38) can also be written

$$(40) \quad \frac{p_1}{p_0} = \frac{x_1^A}{x_0^A} \exp\left(-\frac{U_1 - U_0}{RT}\right)$$

and since $P_0 + P_1 = 1$, this implies

$$(41) \quad P_0 = \frac{1}{1 + \frac{X_1^A}{X_0^A} \exp\left(-\frac{U_1 - U_0}{kT}\right)}$$

$$(42) \quad P_1 = \frac{\frac{X_1^A}{X_0^A} \exp\left(-\frac{U_1 - U_0}{kT}\right)}{1 + \frac{X_1^A}{X_0^A} \exp\left(-\frac{U_1 - U_0}{kT}\right)}$$

Note that the foregoing equations are equally valid with A replaced by B , since equations (36) & (37) imply

$$(43) \quad \frac{X_0^B}{X_0^A} = \frac{X_1^B}{X_1^A} \quad \Leftrightarrow \quad \frac{X_1^A}{X_0^A} = \frac{X_1^B}{X_0^B}$$

Now that the state of equilibrium has been determined, we ask what happens to the fluxes in a nearby steady state. Considering a first-order perturbation in (18), we have

$$(44) \quad \delta f_{ij} = \delta P_i (\alpha_{ij}^R)^* - \delta P_j (\alpha_{ji}^L)^* \\ + p_i^* \delta \alpha_{ij}^R - p_j^* \delta \alpha_{ji}^L$$

Here and in the following, $*$ denotes an equilibrium value.

The two cases in which $i=j$ in (44) are especially simple:

$$(45) \quad \delta f_{ii} = \delta P_i \left((\alpha_{ii}^R)^* - (\alpha_{ii}^L)^* \right) \\ + p_i^* \left(\delta \alpha_{ii}^R - \delta \alpha_{ii}^L \right)$$

But $(\alpha_{ii}^R)^* = (\alpha_{ii}^L)^*$, so the first term is zero. Then, from (28),

$$\begin{aligned}
 (46) \quad \delta f_{ii} &= p_i^* \delta (\alpha_{ii}^R - \alpha_{ii}^L) \\
 &= p_i^* \delta \left(-\gamma \frac{(\Delta G)_{ii}^R}{kT} \right) \\
 &= p_i^* \left(-\gamma \frac{(\Delta G)_{ii}^R}{kT} \right)
 \end{aligned}$$

since $((\Delta G)_{ii}^R)^* = 0$, so

$$(47) \quad \delta (\Delta G)_{ii}^R = (\Delta G)_{ii}^R - ((\Delta G)_{ii}^R)^* = (\Delta G)_{ii}^R$$

Next, we consider the two cases in which $i \neq j$.
Equation (44) can be rewritten as follows

$$(48) \quad \delta f_{ij} = \frac{\delta p_i}{p_i^*} p_i^* (\alpha_{ij}^R)^* - \frac{\delta p_j}{p_j^*} p_j^* (\alpha_{ji}^L)^* \\ + p_i^* (\alpha_{ij}^R)^* \frac{\delta \alpha_{ij}^R}{(\alpha_{ij}^R)^*} - p_j^* (\alpha_{ji}^L)^* \frac{\delta \alpha_{ji}^L}{(\alpha_{ji}^L)^*}$$

Let

$$(49) \quad a_{ij}^* = p_i^* (\alpha_{ij}^R)^* = p_j^* (\alpha_{ji}^L)^*$$

in which the second equality is the principle of detailed balance. Then (48) becomes

$$(50) \quad \delta f_{ij} = a_{ij}^* \left(\frac{\delta p_i}{p_i^*} - \frac{\delta p_j}{p_j^*} + \frac{\delta \alpha_{ij}^R}{(\alpha_{ij}^R)^*} - \frac{\delta \alpha_{ji}^L}{(\alpha_{ji}^L)^*} \right)$$

But

$$\begin{aligned}
 (51) \quad \frac{\delta \alpha_{ij}^R}{(\alpha_{ij}^R)^*} - \frac{\delta \alpha_{ji}^L}{(\alpha_{ji}^L)^*} &= \delta \log \frac{\alpha_{ij}^R}{\alpha_{ji}^L} \\
 &= \delta \log \exp \left(- \frac{(\Delta G)_{ij}^R}{kT} \right) \\
 &= \delta \left(- \frac{(\Delta G)_{ij}^R}{kT} \right) = - \frac{\delta (\Delta G)_{ij}^R}{kT}
 \end{aligned}$$

Now make use of this in (50), divide both sides by a_{ij}^* , and write out separately the two cases $(i,j) = (0,1)$ and $(i,j) = (1,0)$:

$$(52) \quad \frac{\delta f_{01}}{a_{01}^*} = \frac{\delta p_0}{p_0^*} - \frac{\delta p_1}{p_1^*} - \frac{\delta (\Delta G)_{01}^R}{kT}$$

$$(53) \quad \frac{\delta f_{10}}{a_{10}^*} = \frac{\delta p_1}{p_1^*} - \frac{\delta p_0}{p_0^*} - \frac{\delta (\Delta G)_{10}^R}{kT}$$

Recall that any steady state has $f_{01} = f_{10}$
 and therefore $\delta f_{01} = \delta f_{10}$. Thus, by adding
 (52) & (53) and then solving for δf_{01} or δf_{10} ,
 we get

$$(54) \quad \delta f_{01} = \delta f_{10} = \frac{-\frac{1}{kT} \delta \left((\Delta G)_{01}^R + (\Delta G)_{10}^R \right)}{\frac{1}{a_{01}^*} + \frac{1}{a_{10}^*}}$$

From equation (11)

$$(55) \quad (\Delta G)_{01}^R + (\Delta G)_{10}^R =$$

$$2(p^B - p^A) + kT \log \left(\frac{x_0^B}{x_1^A} \frac{x_1^B}{x_0^A} \right) =$$

$$2(p^B - p^A) + kT \log \left(\frac{x_0^B}{x_0^A} \frac{x_1^B}{x_1^A} \right) =$$

$$(\Delta G)_{00}^R + (\Delta G)_{11}^R$$

Note that $(\Delta G)_{00}^R = (\Delta G)_{11}^R = 0$ at equilibrium, so (55) shows that $(\Delta G)_{01}^R + (\Delta G)_{10}^R$ is zero at equilibrium as well. Thus, the numerator on the right-hand side of (54) can be written without the δ as

$$(56) \quad -\frac{1}{RT} \left((\Delta G)_{00}^R + (\Delta G)_{11}^R \right)$$

We still need to evaluate a_{01}^* and a_{10}^*

From (49), by making the choice that involves p_0^* in both cases, we have

$$(57) \quad a_{01}^* = p_0^* (\alpha_{01}^R)^* = p_0^* \gamma \frac{\frac{((\Delta G)_{01}^R)^*}{kT}}{\exp\left(\frac{((\Delta G)_{01}^R)^*}{kT}\right) - 1}$$

$$(58) \quad a_{10}^* = p_0^* (\alpha_{01}^L)^* = p_0^* \gamma \frac{\frac{((\Delta G)_{10}^R)^*}{kT}}{1 - \exp\left(-\frac{((\Delta G)_{10}^R)^*}{kT}\right)}$$

We have previously shown (see equations (38) & (39)) that

$$(59) \quad -(\Delta G)_{10}^R * = ((\Delta G)_{01}^R)^* \\ = U_1 - U_0 + kT \log \left(\frac{X_0}{X_1} \right)^*$$

(Recall that $(X_0^A / X_1^A)^* = (X_0^B / X_1^B)^*$, so we drop the superscript A or B to emphasize this.)

It follows from (59) that the right-hand sides of (57) & (58) are the same, and that

$$(60) \quad a_{10}^* = a_{01}^* = \frac{\gamma \left(\frac{U_1 - U_0}{kT} + \log \left(\frac{X_0}{X_1} \right)^* \right)}{\left(1 + \left(\frac{X_1}{X_0} \right)^* \exp \left(-\frac{U_1 - U_0}{kT} \right) \right) \left(\left(\frac{X_0}{X_1} \right)^* \exp \left(\frac{U_1 - U_0}{kT} \right) - 1 \right)} \\ = \frac{\gamma \left(\frac{U_1 - U_0}{kT} + \log \left(\frac{X_0}{X_1} \right)^* \right)}{\left(\frac{X_0}{X_1} \right)^* \exp \left(\frac{U_1 - U_0}{kT} \right) - \left(\frac{X_1}{X_0} \right)^* \exp \left(-\frac{U_1 - U_0}{kT} \right)}$$

Since all of the net fluxes are zero at equilibrium,

$$(61) \quad df_{ij} = f_{ij} - f_{ij}^* = f_{ij},$$

and we have the following results to first order in $(\Delta G)_{00}^R$ and $(\Delta G)_{11}^R$:

$$(62) \quad f_{00} = \frac{-\gamma \frac{(\Delta G)_{00}^R}{kT}}{1 + \left(\frac{X_1}{X_0}\right)^* \exp\left(\frac{U_0 - U_1}{kT}\right)}$$

$$(63) \quad f_{11} = \frac{-\gamma \frac{(\Delta G)_{11}^R}{kT}}{1 + \left(\frac{X_0}{X_1}\right)^* \exp\left(\frac{U_1 - U_0}{kT}\right)}$$

$$(64) \quad f_{10} = f_{01} = \frac{-\frac{\gamma}{2} \frac{(\Delta G)_{00}^R + (\Delta G)_{11}^R}{kT} \left(\frac{U_1 - U_0}{kT} + \log\left(\frac{X_0}{X_1}\right)^*\right)}{\left(\frac{X_0}{X_1}\right)^* \exp\left(\frac{U_1 - U_0}{kT}\right) - \left(\frac{X_1}{X_0}\right)^* \exp\left(\frac{U_0 - U_1}{kT}\right)}$$

The above equations are of the form

$$(65) \quad f_{00} = -A (\Delta G)_{00}^R$$

$$(66) \quad f_{11} = -B (\Delta G)_{11}^R$$

$$(67) \quad f_{10} = f_{01} = -C \left((\Delta G)_{00}^R + (\Delta G)_{11}^R \right)$$

where A, B, C are positive coefficients.

Now recall that the fluxes of solvent and solute of type 1 are given by $f_0 = f_{00} + f_{01}$
 $f_1 = f_{10} + f_{11}$, so

$$(68) \quad \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} A+C & C \\ C & B+C \end{pmatrix} \begin{pmatrix} -(\Delta G)_{00}^R \\ -(\Delta G)_{11}^R \end{pmatrix}$$

The symmetry of this matrix is an Onsager relation. Note that the matrix is also positive definite, since it is diagonally dominant.