On uniqueness for the critical wave equation

Nader Masmoudi
Courant Institute of Mathematical Sciences,
251 Mercer Street, New York NY 10012
masmoudi@cims.nyu.edu
and
Fabrice Planchon
Laboratoire Analyse, Géométrie & Applications
UMR 7539, Institut Galilée
Université Paris 13, 99 avenue J.B. Clément
93430 Villetaneuse FRANCE
fab@math.univ-paris13.fr

Abstract

We prove the uniqueness of weak solutions to the critical defocusing wave equation in 3D under a local energy inequality condition. More precisely, we prove the uniqueness of $u \in L^\infty_t(\dot{H}^1) \cap \dot{W}^{1,\infty}_t(L^2)$, under the condition that u verifies some local energy inequalities.

1 Introduction and statement of result

We consider the defocusing quintic wave equation in 3D,

$$\begin{cases} \Box u + u^5 = 0, \\ u(t=0) = u_0, \quad u_t(t=0) = u_1. \end{cases} \tag{1}$$

Existence of global weak solutions goes back to Segal ([9], under milder assumptions on the nonlinearity). Existence of global smooth solutions was proved by Grillakis ([3]), while global solutions in the energy space $C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$ were constructed by Shatah and Struwe [11]. Uniqueness was proved only under an additional space-time integrability of Strichartz type, which is a crucial ingredient to the proof of the existence result. Indeed, local existence can be proved using a fixed point argument in some Banach space which can be taken to be $B = C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2) \cap L^5_{loc}(\mathbb{R}; L^{10})$. More recently, uniqueness was obtained under a different set of conditions in [13], using the energy inequality, but still with a space-time integrability condition. One should also mention [1] where the smooth solutions are proved to be globally in $L^5_t(L^{10}_x)$ and stability under weak limits is proved.

In this paper, we intend to give a more physical condition which yields the uniqueness in the energy space. This condition can be easily understood in terms of finite speed of propagation and is satisfied by the solutions constructed by Shatah and Struwe [11].

We consider two solutions $u, v \in L^\infty(\dot{H}^1) \cap \dot{W}^{1,\infty}_t(L^2)$ to the wave equation (1), with the same (real) initial data $\phi_0 \in \dot{H}^1$, $\phi_1 \in L^2$, namely

$$u(t=0) = v(t=0) = \phi_0 \qquad \partial_t u(t=0) = \partial_t v(t=0) = \phi_1,$$

(note that the second condition, on $\partial_t u$ makes sense since a solution which is in $L^\infty(\dot{H}^1) \cap \dot{W}^{1,\infty}_t(L^2)$ is also in $C^1(\dot{H}^{-1})$).

The two solutions u and v are supposed to be weak solutions, i.e. equation (1) holds in the sense of distributions: for any $\phi \in C^\infty_0([0,\infty); \mathbb{R}^3)$,

$$\int_{\mathbb{R}\times\mathbb{R}^3} u\Box\phi + u^5\phi \, dxdt = \int_{\mathbb{R}^3} -\phi_1(x)\phi(0,x) + \phi_0(x)\partial_t\phi(0,x)\, dx$$

and the same equation holds for v.

1.1 The local energy condition

Let us state the local energy or the finite speed of propagation condition. Let (t_0, x_0) be the vertex of a backward cone K, $K = \{|x - x_0| = t_0 - t\}$ and $e(u) = |\partial u|^2/2 + u^6/6$ be the energy density (here and thereafter ∂ denotes the full space-time gradient). Then we assume that for all $0 \leq s \leq t \leq t_0$

$$\int_{B(x_0, t_0-t)} e(u(t,x))dx \leq C \int_{B(x_0,\alpha(t_0-s))} e(u(s,x))dx. \tag{2}$$

where C and α are some constants, $C \geq 1$ and $\alpha \geq 1$.

Similarly, consider the forward cone K_1 of vertex (t_1, x_1), namely $K_1 = \{|x - x_1| = t - t_1\}$, and let $t' \geq t_1$, denote by ∂_{K_1} the tangential derivatives, we assume that

$$\frac{1}{\sqrt{2}}\int_{t_1}^{t'}\int_{\partial B(x_1,\tau-t_1)}\frac{|\partial_{K_1}u(\tau)|^2}{2}+\frac{u(\tau)^6}{6}\quad d\sigma d\tau \tag{3}$$
$$\leq C\int_{B(x_1,\alpha(t'-t_1))}e(u(t',x))dx.$$

We insist on the fact that both constants C and α are supposed to be uniform with respect to the vertex.

We point out that (2) and (3) are weak versions of the local energy equalities which are recalled in the next section. Indeed, for smooth solutions, one can prove that (2) and (3) hold with $C=\alpha=1$. We also notice that the left hand side of (3) does not make sense (actually can be a priori infinite) if we only assume that $u\in L^\infty(\dot{H}^1)\cap\dot{W}_t^{1,\infty}(L^2)$. Let us explain the meaning of (3). Let $\rho(x)\in C_0^\infty(\mathbb{R}^3)$ be such that $\rho\geq 0$, $\int\rho=1$ and define $\rho_n(x)=n^3\rho(nx)$, then we define $u_n=u*\rho_n$ a (space) regularization of u. Since $u\in\dot{W}_t^{1,\infty}(L^2)$, we deduce that u_n is continuous in both space and time variables. Condition (3) can be understood as

$$\limsup_{n\to\infty}\int_{t_1}^{t'}\int_{\partial B(x_1,\tau-t_1)}\frac{|\partial_{K_1}u_n(\tau)|^2}{2}+\frac{u_n(\tau)^6}{6}\quad d\sigma d\tau \tag{4}$$
$$\leq C\int_{B(x_1,\alpha(t'-t_1))}e(u(t',x))dx.$$

1.2 An important example

Let us prove that these conditions hold for any weak solution which also satisfies the local energy identity, namely

$$\partial_t e(u(t,x))-div(\partial_t u\nabla u)=0. \tag{5}$$

Let us prove that (2) holds. We denote $M_s^t=\{(\tau,x)|s<\tau<t,\,|x-x_0|<t_0-\tau\}$.

Integrating (5) over M_s^t, we formally get (7). Let us prove this rigorously. Let $\rho(x)\in C_0^\infty(\mathbb{R}^3)$ be such that $\rho\geq 0$, $\int\rho=1$ and define $\rho_n(x)=n^3\rho(nx)$. Hence

$$\partial_t e(u(t,.))*\rho_n-div((\partial_t u\nabla u)*\rho_n)=0. \tag{6}$$

Using the fact that $\partial_t u \nabla u \in L_t^\infty(L_x^1)$, we deduce that $(\partial_t u \nabla u) * \rho_n \in L_t^\infty(C_x^\infty)$. Hence, $e(u(t,.)) * \rho_n \in W_t^{1,\infty}(C_x^\infty)$. Integrating (6) in M_s^t, we get

$$\int_{B(x_0,t_0-t)} e(u) * \rho_n \, dx + \frac{1}{\sqrt{2}} \int_s^t \int_{\partial B(x_0,t_0-\tau)} [e(u) - (\partial_t u \nabla u)] * \rho_n \, d\sigma d\tau$$
$$= \int_{B(x_0,t_0-s)} e(u) * \rho_n \, dx,$$

Taking the limit when n goes to infinity, we see that the first and third terms converge to the corresponding terms in (7). For the second term, we rewrite $e(u) - (\partial_t u \nabla u)$ as $e(u) - (\partial_t u \nabla u) = \frac{|\partial_K u|^2}{2} + \frac{u^6}{6}$. Then, using Jensen inequality, we deduce that

$$\frac{|\partial_K u_n(\tau)|^2}{2} + \frac{u_n(\tau)^6}{6} \leq [\frac{|\partial_K u(\tau)|^2}{2} + \frac{u(\tau)^6}{6}] * \rho_n.$$

Hence,

$$\int_{B(x_0,t_0-t)} e(u) \, dx + \limsup_{n\to\infty} \int_s^t \int_{\partial B(x_0,t_0-\tau)} \frac{|\partial_K u_n(\tau)|^2}{2} + \frac{u_n(\tau)^6}{6} \, d\sigma d\tau$$
$$\leq \int_{B(x_0,t_0-s)} e(u) \, dx.$$

Arguing in the same way for the forward cone K_1, we deduce that (4) holds with $C = \alpha = 1$.

1.3 The main result

We now state our main result.

Theorem 1
Let u be a weak solution to (1) which satisfies (2) and (3). Then this solution is unique among all weak solutions satisfying (2) and (3).

Corollary 1
Let u be a weak solution to (1) which satisfies the local energy identity (5). *Then this solution is unique among all weak solutions satisfying the local energy identity* (5).

This unique solution is actually equal to the solution constructed in [11], but we will not use this fact in the proof, unlike for higher dimensions where a strong-weak uniqueness argument is used ([8] and remark at the end of the present paper), see also [6] for a similar uniqueness result.

It does seem fairly reasonable for weak solutions to assume that (2) and (3) hold: certainly one is willing to have at least the weak energy inequality, namely $\int e(\phi)dx \leq \int e(\phi_0)dx$, and in light of the finite speed of propagation,both (2) and (3) are not really stronger requirements. At any rate, control of the flux is already an essential tool in order to prove regularity for smooth data ([3, 10]).

A weak solution to (1) satisfying in addition (3) and (2) can be considered as a *suitable weak solution*. This is similar in spirit to the notion of suitable weak solutions for the Navier-Stokes system introduced in [2]. Indeed, both conditions are local versions of the energy inequalities.

To prove theorem 1, we introduce a dual problem as was done in [5]. Then, we prove the existence of a smooth solution to a regularized version of this dual problem. This solution is used as a test function in the weak formulation. Passing to the limit, we deduce that $u = v$.

In the next section, we recall the energy identities on backward and forward cones. In section 3, we give the proof of theorem 1. We will start by a formal argument and then explain the regularization procedure.

2 Finite speed of propagation

Let us recall that a smooth solution of the wave equation (1) satisfies the following energy identity on each backward cone : let again (t_0, x_0) be the vertex of such a backward cone K, $K = \{|x - x_0| = t_0 - t\}$ and $e(u) = |\partial u|^2/2 + u^6/6$ be the energy density. Then we have for all $s \leq t \leq t_0$

$$\int_{B(x_0,t_0-t)} e(u(t,x))\,dx + \frac{1}{\sqrt{2}}\int_s^t \int_{\partial B(x_0,t_0-\tau)} \frac{|\partial_K u(\tau)|^2}{2} + \frac{u(\tau)^6}{6} \quad d\sigma d\tau = \int_{B(x_0,t_0-s)} e(u(s,x))\,dx, \tag{7}$$

where we recall that ∂_K denotes the derivatives tangent to the backward cone K. The second term on the left-hand side is usually referred to as the (outgoing) flux through the cone K.

Moreover, the solution verifies the same inequality for forward cones as well: specifically, consider the forward cone K_1 of vertex (t_1, x_1), namely $K_1 = \{|x - x_1| = t - t_1\}$, and let $t' \geq t_1$, we have

$$\frac{1}{\sqrt{2}} \int_{t_1}^{t'} \int_{\partial B(x_1, \tau - t_1)} \frac{|\partial_{K_1} u(\tau)|^2}{2} + \frac{u(\tau)^6}{6} \quad d\sigma d\tau = \int_{B(x_1, t'-t_1)} e(u(t', x))\, dx. \tag{8}$$

The left-hand side is usually referred to as the (incoming) flux through the cone K_1.

REMARK 1
Of course (8) is only a special case of the backward version of (7): we could have stated an inequality between the two space-like surfaces $t = t'$ and $t = t''$ with $t_1 \leq t'' \leq t'$. Here we chose to take $t'' = t_1$ as this is what will actually be needed later in the proof.

The conditions (3) and (2) which imply the uniqueness are weaker versions of (8) and (7). Indeed, the equality is replaced by an inequality and we can even allow the presence of fixed constants C and α.

Alternatively, one could rephrase both equalities in terms of only one equality, if one is willing to replace space balls by annuli (or even, say, domains with reasonably smooth boundaries). Then, if Σ is the boundary of the backward domain of influence, one would ask the sum of the energy in our space domain at time T and the outgoing flux through Σ between times T and $S \leq T$ to be equal to the energy at time S in the space domain $\Sigma \cap \{t = S\}$. Such equalities and their weaker counterparts are a reasonable way to quantify the finite speed of propagation which one expects from any physically meaningful solutions to the equation.

3 Proof of Theorem 1

Assume that u and v are two solutions of (1). Taking ϕ to be an admissible test function $\phi \in C_0^2([0, \infty), \mathbb{R}^3)$, we have

$$\int (u - v)\Box\phi + (u^5 - v^5)\phi = 0, \tag{9}$$

which can be rewritten as

$$\int (u-v)(\Box\phi + (u^4+4u^3v+6u^2v^2+4uv^3+v^4)\phi) = 0. \tag{10}$$

We intend to solve the following (dual) problem: let $F \in C_0^\infty((0,T)\times\mathbb{R}^3)$ and ϕ be the solution of the following backward wave equation

$$\begin{cases} \Box\phi + V\phi = F, \\ \phi(T) = \partial_t\phi(T) = 0, \end{cases} \tag{11}$$

where we define $V = u^4+4u^3v+6u^2v^2+4uv^3+v^4$ and $T>0$ is small enough, to be fixed later. Provided we solve (11) and prove that ϕ is regular enough to be used as a test function in (9), we will have uniqueness for our problem. All is required is for ϕ to be an admissible test function, in order to justify the integration by parts. Actually, this will turn out to be untrue, but one may still proceed using a smoothing and a limiting procedure which will be explained later.

Proposition 1
Provided T is small enough, there exists a (compactly supported) smooth solution ϕ_n to (a regularized version of) the dual problem (11), such that ϕ_n is uniformly bounded in $L^\infty_{t,x}$.

3.1 Formal proof

Let us start by a formal proof. We will need a regularization of (11) to make it rigorous. We denote $K(z_0)$ the forward cone with vertex $z_0 = (t_0, x_0)$ and time $t \leq T$ i.e. $K(z_0) = \{(t,x) |\ |x-x_0| = t-t_0,\ t_0 \leq t \leq T\}$. Then, the solution of (11) is given by, taking advantage of the explicit space representation of the fundamental solution to the 3D wave equation,

$$\phi(t_0,x_0) = \int_{K(z_0)} \frac{F(z)-V\phi(z)}{|z-z_0|} d\sigma(z), \tag{12}$$

with $z=(t,x)$ and where σ is the surface measure on forward cones. Then, we proceed as Jörgens ([4]), with

$$\|\phi\|_{L^\infty((0,T)\times\mathbb{R}^3)} \leq C(F) + \|\phi\|_{L^\infty((0,T)\times\mathbb{R}^3)} \sup_{z_0} \int_{K(z_0)} \frac{|V(z)|}{|z-z_0|} d\sigma(z), \tag{13}$$

and as $|V| \lesssim u^4 + v^4$, we use

$$\int_K \frac{|u|^4}{|z-z_0|} d\sigma(z) \sim \int_{B(0,T-t_0)} \frac{|w(y)|^4}{|y|} dy,$$

where $w(y) = u(t_0 + |y|, x_0 + y)$. This in turn yields

$$\int_K \frac{|u|^4}{|z-z_0|} d\sigma(z) \lesssim \int_{B(0,T-t_0)} \frac{|w(y)|^2}{|y|^2} dy + \int_{B(0,T-t_0)} |w|^6. \tag{14}$$

By an appropriate local version of Hardy's inequality (see e.g. [12]), the first term in (14) is controlled:

$$\int_{B(0,T-t_0)} \frac{|w(y)|^2}{|y|^2} dy \lesssim \int_{B(0,T-t_0)} |\nabla_y w|^2 dy + (\int_{B(0,T-t_0)} |w(y)|^6 dy)^{2/6}. \tag{15}$$

We then recognize the flux,

$$\frac{1}{2} \int_{B(0,t_0)} |\nabla_y w|^2 + \frac{1}{6} \int_{B(0,t_0)} |w|^6 \, dy = \text{flux},$$

as

$$|\nabla_y w|^2 = |\nabla u - \frac{y}{|y|} \partial_t u|^2,$$

and recall

$$\text{flux} = \int_K \frac{1}{2} \left| \nabla u - \frac{y}{|y|} \partial_t u \right|^2 + \frac{|u|^6}{6} d\sigma.$$

Hence (15) becomes

$$\int_K \frac{|u|^4}{|z-z_0|} d\sigma(z) \lesssim \text{flux} + \text{flux}^{\frac{1}{3}}. \tag{16}$$

By choosing T small enough, we can make the local energy $\int_{B(x_0,\alpha T)} e(u(T,x))dx$ smaller than a fixed constant ε_0, uniformly in x_0: we simply use the energy inequality (2), fixing T such that $\int_{B(x_0,\alpha(\alpha+1)T)} e(u(0,x))dx$ is (uniformly) small enough, which in turn is a trivial consequence of the initial data being in $\dot{H}^1 \times L^2$. Then we deduce that the flux through the forward cone which is needed in the construction of ϕ can be made smaller than $1/2$ by using (3) and choosing ε_0 such that $C\varepsilon_0 = \frac{1}{2}$. Next, we can perform a contraction argument in $L^\infty_{t,x}$ to obtain ϕ.

Remark 2
Note that the whole argument is local in space-time. Hence, the assumptions on the data could be relaxed to $\dot{H}^1_{loc} \times L^2_{loc}$, and one could consider local in time weak solutions. We elected to keep $\dot{H}^1 \times L^2$ data and global in time solutions for simplicity.

3.2 Rigorous proof

Let us explain the regularization procedure which yields a rigorous proof of (13) and the proposition. Recall that $\rho(x) \in C_0^\infty(\mathbb{R}^3)$ is such that $\rho \geq 0$, $\int \rho = 1$ and $\rho_n(x) = n^3\rho(nx)$, then we define $u_n = u * \rho_n$, $v_n = v * \rho_n$ and $V_n = u_n^4 + 4u_n^3 v_n + 6u_n^2 v_n^2 + 4u_n v_n^3 + v_n^4$. We intend to solve

$$\begin{cases} \Box \phi_n + V_n \phi_n = F, \\ \phi_n(T) = \partial_t \phi_n(T) = 0, \end{cases} \tag{17}$$

by a fixed point argument. Considering

$$\begin{cases} \Box \psi + V_n \tilde{\psi} = F, \\ \psi(T) = \partial_t \psi(T) = 0, \end{cases} \tag{18}$$

for smooth ψ and $\tilde{\psi}$, we have

$$\psi(t_0, x_0) = \int_{K(z_0)} \frac{F(z) - (V_n \tilde{\psi})(z)}{|z - z_0|} d\sigma(z), \tag{19}$$

from which we infer that

$$\|\psi\|_{L^\infty((0,T)\times\mathbb{R}^3)} \leq C(F) + \|\tilde{\psi}\|_{L^\infty((0,T)\times\mathbb{R}^3)} \sup_{z_0} \int_{K(z_0)} \frac{|V_n|}{|z - z_0|} d\sigma(z). \tag{20}$$

Now, we can proceed as in the formal proof and choose T small enough so that

$$\sup_n \sup_{z_0} \int_{K(z_0)} \frac{|V_n|}{|z - z_0|} d\sigma(z) \leq \frac{1}{2}.$$

Notice that given we are solving a linear problem, estimating ψ or $\psi - \phi$ is identical, where ϕ solves

$$\begin{cases} \Box \phi + V_n \tilde{\phi} = F, \\ \phi(T) = \partial_t \phi(T) = 0. \end{cases} \tag{21}$$

Hence we deduce from the previous computations that

$$\|\psi - \phi\|_{L^\infty_{t,x}} \leq \frac{1}{2}\|\tilde{\psi} - \tilde{\phi}\|_{L^\infty_{t,x}}.$$

This estimate allows a fixed point argument in C^0 to be carried out. Therefore we have constructed a solution ϕ_n to the equation (17). Moreover, we recover an estimate on $\|\phi_n\|_{L^\infty_{t,x}}$ which is uniform with respect to n, thanks to (3). Furthermore, ϕ_n is smooth, as the regularity can as usual be carried along the iterates which yield ϕ_n: for any derivative ∂, we have

$$\|\partial\psi - \partial\phi\|_{L^\infty_{t,x}} \leq \frac{1}{2}\|\partial\tilde{\psi} - \partial\tilde{\phi}\|_{L^\infty_{t,x}} + \|\tilde{\psi} - \tilde{\phi}\|_{L^\infty_{t,x}} C(\partial V_n).$$

We do not get good control of norms, as they involve derivative of V_n, but we will not need it. Moreover, ϕ_n is compactly supported, by finite speed of propagation (again, all iterates are in a uniform way). This ends the proof of proposition 1.

We now return to the proof of Theorem 1 and explain why the smoothing procedure which yields ϕ_n still allows for the heuristic argument to be carried out. In fact, we use ϕ_n as a test function: for all n, we have

$$\int (u - v)(\Box\phi_n + V\phi_n) = 0. \tag{22}$$

This translates into

$$\int (u - v)(F + (V - V_n)\phi_n) = 0. \tag{23}$$

We know that $u - v \in L^\infty_t(L^6)$, and that V_n converges to V strongly in $L^\infty_{t,\mathrm{loc}} L^{3/2}$. Hence, $(V - V_n)\phi_n$ converges toward zero in $L^1_{t,\mathrm{loc}} L^{6/5}_{x,\mathrm{loc}}$, given that ϕ_n is uniformly in $L^\infty_{t,x}$ and is compactly supported; this ultimately gives the desired equality:

$$\int (u - v)F = 0,$$

from which we deduce that $u = v$ on the interval $[0, T]$. Now, we can argue by contradiction, choosing the initial time to be t_0 where

$$t_0 = \inf\{t,\, t \geq 0,\, u(t) \neq v(t)\}.$$

Using the fact that u and v are continuous in time with values in $\dot{H}^{-1}$, we deduce that $u(t_0) = v(t_0)$. Then we have $u = v$ on some interval $[t_0, t_0 + \eta]$, which proves that no such t_0 exists. Hence, we deduce that $u = v$ on $[0, \infty)$ which achieves the proof of the main theorem.

Finally, we make some comments on the case $n \geq 4$. In higher dimensions, one cannot rely on Jörgens estimate. However, uniqueness was proven under the assumption $\phi \in C_t(\dot{H}^1) \cap C_t^1(L^2)$ in [8], for both focusing and defocusing critical wave equation, with $n \geq 4$. In the defocusing case, assuming only the local energy identity (2), one can easily get rid of the continuity in time and obtain uniqueness as in Theorem 1. We refer the interested reader to [7] for further discussions in a similar (albeit more complicated) setting.

4 Acknowledgments

The authors would like to thank Jalal Shatah for many discussions about this work. The first author was partially supported by an NSF grant and by an Alfred Sloan Fellowship. Part of this work was done while the second author was visiting the Courant Institute, which he would like to thank for its hospitality.

References

[1] Hajer Bahouri and Patrick Gérard. High frequency approximation of solutions to critical nonlinear wave equations. *Amer. J. Math.*, 121(1):131–175, 1999.

[2] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831, 1982.

[3] Manoussos G. Grillakis. Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity. *Ann. of Math. (2)*, 132(3):485–509, 1990.

[4] Konrad Jörgens. Das Anfangswertproblem im Grossen für eine Klasse nichtlinearer Wellengleichungen. *Math. Z.*, 77:295–308, 1961.

[5] P.-L. Lions and N. Masmoudi. Uniqueness of mild solutions of the Navier-Stokes system in L^N. *Comm. Partial Differential Equations*, 26(11-12):2211–2226, 2001.

[6] Nader Masmoudi and Kenji Nakanishi. Uniqueness of finite energy solutions for Maxwell-Dirac and Maxwell-Klein-Gordon equations. *Comm. Math. Phys.*, 243(1):123–136, 2003.

[7] Nader Masmoudi and Fabrice Planchon. On uniqueness for wave maps. preprint, 2003.

[8] Fabrice Planchon. On uniqueness for semilinear wave equations. *Math. Z.*, 244(3):587–599, 2003.

[9] I. E. Segal. The global Cauchy problem for a relativistic scalar field with power interaction. *Bull. Soc. Math. France*, 91:129–135, 1963.

[10] Jalal Shatah and Michael Struwe. Regularity results for nonlinear wave equations. *Ann. of Math. (2)*, 138(3):503–518, 1993.

[11] Jalal Shatah and Michael Struwe. Well-posedness in the energy space for semilinear wave equations with critical growth. *Internat. Math. Res. Notices*, (7):303ff., approx. 7 pp. (electronic), 1994.

[12] Jalal Shatah and Michael Struwe. *Geometric wave equations*, volume 2 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1998.

[13] Michael Struwe. Uniqueness for critical nonlinear wave equations and wave maps via the energy inequality. *Comm. Pure Appl. Math.*, 52(9):1179–1188, 1999.