Global well posedness for a Smoluchowski equation coupled with Navier-Stokes equations in 2D.

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Abstract

We prove global existence for a nonlinear Smoluchowski equation (a nonlinear Fokker-Planck equation) coupled with Navier-Stokes equations in 2d. The proof uses a deteriorating regularity estimate in the spirit of [5] (see also [1])

Key words Nonlinear Fokker-Planck equations, Navier-Stokes equations, Smoluchowski equation, micro-macro interactions.

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1 Introduction

Systems coupling fluids and particles are of great interest in many branches of applied physics and chemistry. The equations attempt to describe the behavior of complex mixtures of particles and fluids, and as such, they present numerous challenges, simultaneously at three levels: at the level of their derivation, the level of their numerical simulation and that of their mathematical treatment. In this paper we concentrate solely on one aspect of the mathematical treatment, the regularity of solutions. The particles in the system are described by a probability distribution $f(t, x, m)$ that depends on time t, macroscopic variable $x \in \mathbb{R}^n$, and particle configuration $m \in M$. Here M is a smooth compact Riemannian manifold without boundary. The particles are transported by a fluid, agitated by thermal noise, and interact among themselves. This is reflected in a kinetic equation for the evolution of the probability distribution of the particles ([2, 8]). The interaction between particles – a micro-micro interaction – is modeled in a mean-field fashion by a potential that represents the tendency of particles to favor certain coherent configurations. The interaction between particles occurs only when the concentration of particles is sufficiently high. Mathematically, this term is responsible for the nonlinearity of the Smoluchowski (Fokker-Planck) equation, and physically, it is responsible for nematic phase transitions. Because the particles are considerably small, and for smooth flows, the Lagrangian transport of the particles is modeled using a Taylor expansion of the velocity field. This gives rise to a drift term in the Smoluchowski equation that depends on the spatial gradient of velocity. It is a macro-micro term, and it causes mathematical difficulties in the regularity theory.

The fluid is described by the incompressible Navier-Stokes equations. The microscopic particles add stresses to the fluid. This is the micro-macro interaction and it is the most puzzling and important physical aspect of the problem. Indeed, while a macro-micro interaction can be derived, in principle, by assuming that the macroscopic entities vary little on the scale of the microscopic ones, the "scaling up" of the effect of microscopic quantities to the macroscopic level is more mysterious. A principle based on an energy dissipation balance, and that recovers familiar results in simple cases was proposed in [6], where the regularity of nonlinear Fokker-Planck systems coupled with Stokes equations in 3D was also proved. The linear Fokker-Planck system coupled with Stokes equations was considered in [19]. The nonlinear Fokker-Planck equation driven by a time averaged Navier-Stokes system in 2D was studied in [7].

An approximate closure of the linear Fokker-Planck equation reduces the description to closed viscoelastic equations for the added stresses themselves. This leads to well-known non-Newtonian fluid models that have been studied extensively. For regularity results we refer to Lions and Masmoudi [18] where the existence of global weak solutions was proved for an Oldroyd-type model. In Guillopé and Saut [13] and [14], the existence of local strong solution was proved. Also, Fernández-Cara, Guillén and Ortega [11], [10] and [12] proved local well posedness in Sobolev spaces. We also mention Lin, Liu and Zhang [16] where a formulation based on the deformation tensor is used to study the Oldroyd-B model.

An other model for the polymers is the FENE dumbbell model. From mathematical point of view, this model was studied by several authors. In particular W. E, Li and Zhang [9], Jourdain, Lelievre and Le Bris [15] and Zhang and Zhang [20] proved local well-posedness. Moreover, Lin, Liu and Zhang [17] proved global existence near equilibrium.

1.1 The model

Consider the system

$$\left\{\begin{array}{rcll} \frac{\partial v}{\partial t} + v\cdot\nabla v - \nu\Delta v + \nabla p &=& \nabla\cdot\tau & \text{in} \quad \Omega\times(0,T) \\ \frac{\partial f}{\partial t} + v\cdot\nabla f + \mathrm{div}_g(G(v,f)f) - \Delta_g f &=& 0 & \text{in} \quad \Omega\times(0,T) \\ \mathrm{div} v &=& 0 & \text{in} \quad \Omega\times(0,T), \end{array}\right. \tag{1}$$

where $\tau_{ij} = \int_M \gamma_{ij}^{(1)}(m)f(t,x,m)dm + \int_M\int_M \gamma_{ij}^{(2)}(m_1,m_2)f(t,x,m_1)f(t,x,m_2)dm$. We denote $G(v,f) = \nabla_g U + W$ where $W = c_\alpha^{ij}\partial_j v_i$ and $U = Kf$ is a potential given by

$$U(t,x,m) = \int_M K(m,q)f(t,x,q)\,dq \tag{2}$$

with a kernel K which is a smooth, time and space independent symmetric function $K : M\times M \to \mathbb{R}$. We also take $\Omega = \mathbb{R}^2$.

1.2 Statement of the result

Theorem 1.1 *Take $v(0) \in W^{1+\varepsilon_0,r} \cap L^2(\mathbb{R}^2)$ and $f(0) \in W^{1,r}(H^{-s})$, for some $r > 2$ and $\varepsilon_0 > 0$ and $f \geq 0$, $\int_M f_0 \in L^1 \cap L^\infty$. Then (1) has a global solution in $v \in L^\infty_{loc}(W^{1,r}) \cap L^2_{loc}(W^{2,r})$ and $f \in L^\infty_{loc}(W^{1,r}(H^{-s}))$. Moreover, for $T > T_0 > 0$, we have $v \in L^\infty((T_0,T); W^{2-\varepsilon,r})$.*

1.3 Preliminaries

We define $\mathcal{C}$ to be the ring of center 0, of small radius 1/2 and great radius 2. There exist two nonnegative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0,1))$ and to $\mathcal{D}(\mathcal{C})$ so that

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \tag{3}$$

$$|p-q| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-\mathrm{q}}\cdot) \cap \text{Supp } \varphi(2^{-\mathrm{p}}\cdot) = \emptyset. \tag{4}$$

For instance, one can take $\chi \in \mathcal{D}(B(0,1))$ such that $\chi \equiv 1$ on $B(0,1/2)$ and take

$$\varphi(\xi) = \chi(\xi/2) - \chi(\xi).$$

Then, we are able to define the Littlewood-Paley decomposition. Let us denote by $\mathcal{F}$ the Fourier transform on $\mathbb{R}^d$. Let $h,\ \widetilde{h},\ \Delta_q,\ S_q\ (q \in \mathbb{Z})$ be defined as follows:

$$\begin{aligned} h = \mathcal{F}^{-1}\varphi \quad &\text{and} \quad \widetilde{h} = \mathcal{F}^{-1}\chi, \\ \Delta_q u = \mathcal{F}^{-1}(\varphi(2^{-q}\xi)\mathcal{F}u) &= 2^{qd} \int h(2^q y) u(x-y) dy, \\ S_q u = \mathcal{F}^{-1}(\chi(2^{-q}\xi)\mathcal{F}u) &= 2^{qd} \int \widetilde{h}(2^q y) u(x-y) dy. \end{aligned}$$

We use the para-product decomposition of Bony ([3])

$$uv = T_u v \ + \ T_v u \ + \ R(u,v)$$

where

$$T_u v = \sum_{q \in \mathbb{Z}} S_{q-1} u \Delta_q v \quad \text{and} \quad R(u,v) = \sum_{|q-q'| \leq 1} \Delta_{q'} u \Delta_q v.$$

We define the inhomogeneous and homogeneous Besov spaces by

Definition 1.2 *Let s be a real number, p and r two real numbers greater than 1. Then we define the following norm*

$$\|u\|_{\widetilde{B}^s_{p,r}} \overset{def}{=} \|S_0 u\|_{L^p} + \left\| (2^{qs} \|\Delta_q u\|_{L^p})_{q \in \mathbb{N}} \right\|_{\ell^r(\mathbb{N})}$$

and the following semi-norm

$$\|u\|_{B^s_{p,r}} \overset{def}{=} \left\| (2^{qs} \|\Delta_q u\|_{L^p})_{q \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}.$$

Definition 1.3

- *Let s be a real number, p and r two real numbers greater than 1. We denote by $\widetilde{B}^s_{p,r}$ the space of tempered distributions u such that $\|u\|_{\widetilde{B}^s_{p,r}}$ is finite.*

- *If $s < d/p$ or $s = d/p$ and $r = 1$ we define the homogeneous Besov space $B^s_{p,r}$ as the closure of compactly supported smooth functions for the norm $\|\cdot\|_{B^s_{p,r}}$.*

We refer to [4] for the proof of the following results and for the multiplication law in Besov spaces.

Lemma 1.4

$$\|\Delta_q u\|_{L^b} \leq 2^{d(\frac{1}{a}-\frac{1}{b})q}\|\Delta_q u\|_{L^a} \quad \text{for } b \geq a \geq 1$$

$$\|e^{t\Delta}\Delta_q u\|_{L^b} \leq C2^{-ct2^{2q}}\|\Delta_q u\|_{L^b}$$

The following corollary is straightforward.

Corollary 1.5 *If $b \geq a \geq 1$, then, we have the following continuous embeddings*

$$B^s_{a,r} \subset B_{b,r}^{s-d\left(\frac{1}{a}-\frac{1}{b}\right)}.$$

Definition 1.6 *Let p be in $[1,\infty]$ and r in $\mathbb{R}$; the space $\widetilde{L}^p_T(C^r)$ is the space of distributions u such that*

$$\|u\|_{\widetilde{L}^p(0,T;C^r)} \stackrel{def}{=} \sup_q 2^{qr}\|\Delta_q u\|_{L^p_T(L^\infty)} < \infty.$$

We will use the following theorem from [5]

Theorem 1.7 *Let v be the solution in $L^2_T(H^1)$ of the two dimensional Navier-Stokes system*

$$(NS_\nu)\left\{\begin{array}{rcl} \dfrac{\partial v}{\partial t} + v\cdot\nabla v - \nu\Delta v &=& -\nabla p + f \\ \mathrm{div} v &=& 0 \\ v|_{t=0} &=& v_0. \end{array}\right.$$

with an initial data in L^2 and an external force f in $L^1_T(C^{-1}) \cap L^2_T(H^{-1})$; then, for any ε, a T_0 in the interval $]0,T[$ exists such that

$$\|\nabla v\|_{\widetilde{L}^1_{[T_0,T]}(C^0)} \leq \varepsilon.$$

2 A deteriorating regularity estimate

The main part of this section is the proof of a deteriorating regularity estimate for transport equations in the spirit of [1] and [5]. After this proof, we will apply this estimate in order to prove Theorem 1.1.

We also denote $H = (-\Delta_g + I)^{-s/2}$ with $s > d/2 + 1$.

Theorem 2.1 *Let σ and β be two elements of $]0,1[$ such that $\sigma+\beta < 1$. A constant C exists that satisfies the following properties. Let T and λ be two positive numbers and v a smooth divergence free vector field so that*

$$\sigma - \lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)} \geq \beta \quad \text{and} \quad \sigma + \lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)} \leq 1-\beta. \tag{5}$$

Consider two smooth functions f and v so that f is the solution of

$$\left\{\begin{array}{rcl} \partial_t f + v\cdot\nabla f + \mathrm{div}_g(G(v,f)f) - \Delta_g f &=& 0 \\ f_{|t=0} &=& f_0. \end{array}\right. \tag{6}$$

Then we have, if $\lambda \geq 3C$,

$$M_\lambda^\sigma(f) \leq 3\|f_0\|_{B^\sigma_{p,\infty}(H^{-s})} + \frac{3C}{\lambda} M_\lambda^{\sigma+1}(v) \tag{7}$$

where

$$M_\lambda^\sigma(v) \overset{def}{=} \sup_{t\in[0,T],q} 2^{q\sigma - \Phi_{q,\lambda}(t)} \|\Delta_q v(t)\|_{L^p} \quad or \tag{8}$$

$$M_\lambda^\sigma(f) \overset{def}{=} \sup_{t\in[0,T],q} 2^{q\sigma - \Phi_{q,\lambda}(t)} \|\Delta_q f(t)\|_{L^p(H^{-s})} \quad with \tag{9}$$

$$\Phi_{q,\lambda}(t,t') \overset{def}{=} \lambda \int_{t'}^{t} (\|S_{q-1}\nabla v(t'')\|_{L^\infty} + 1)dt'', \;\; \Phi_{q,\lambda}(t) = \Phi_{q,\lambda}(t,0). \tag{10}$$

We will use the notation $f_q \overset{\text{def}}{=} \Delta_q f$. Applying the operator Δ_q to the transport equation (6), we get

$$\begin{cases} \partial_t f_q + S_{q-1}v \cdot \nabla f_q + \mathrm{div}_g(G(S_{q-1}v, S_{q-1}f)f_q) - \Delta_g f_q + R_q(v,f) = 0 \\ \hfill f_{q|t=0} = \Delta_q f_0. \end{cases} \tag{11}$$

where R_q is a rest term.

We denote

$$N_q^2(t,x) = \int_M |Hf_q|^2 dm \tag{12}$$

Applying H to (11) and taking the L^2 norm on M, we get

$$\partial_t N_q^2 + S_{q-1}v \cdot \nabla N_q^2 + V(S_{q-1}v, S_{q-1}f, f_q) + |\nabla_g H f_q|^2 + \int_M Hf_q(HR_q(v,f))dm = 0 \tag{13}$$

where

$$V(v,h,f) = \partial_j v_i \int_M (H\mathrm{div}_g(c_\alpha^{ij} f))(Hf)dm. + \int_M (H\mathrm{div}_g(\nabla_g hf))(Hf)dm. \tag{14}$$

Hence, arguing as in [7], we have $|V(S_{q-1}v, f_q)| \leq C(|\nabla S_{q-1}v| + ||S_{q-1}f||_{L^2(M)})N_q^2$.

We will use now the following lemma, postponing its proof:

Lemma 2.2 *$R_q(v,f)$ satisfies*

$$\begin{aligned} 2^{q\sigma - \Phi_{q,\lambda}(t)} \|HR_q(v(t),f(t))\|_{L^p(L^2)} \leq C e^{C\lambda\|\nabla v\|_{\tilde{L}^1_T(C^0)}} \\ \times \Big(M_\lambda^{\sigma+1}(v) + \Big(1 + \|S_q \nabla v(t)\|_{L^\infty} + \sum_{|q'-q|\leq N} \|\Delta_{q'}\nabla v(t)\|_{L^\infty}\Big) M_\lambda^\sigma(f)\Big). \end{aligned} \tag{15}$$

Taking the L^p norm of N_q, we get

$$\|N_q(t)\|_{L^p} \leq \|N_q(0)\|_{L^p} + \int_0^t \|HR_q(v(t'),f(t'))\|_{L^p(L^2)} + \|\nabla S_q v(t')\|_{L^\infty}\|N_q(t')\|_{L^p} dt'.$$

After multiplication by $2^{q\sigma - \Phi_{q,\lambda}(t)}$, we get

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|N_q(t)\|_{L^p} \le 2^{q\sigma}\|N_q(0)\|_{L^p} + \int_0^t 2^{-\Phi_{q,\lambda}(t,t')}2^{q\sigma-\Phi_{q,\lambda}(t')}\|\nabla S_q v(t')\|_{L^\infty}\|N_q\|_{L^p}dt' + \int_0^t 2^{-\Phi_{q,\lambda}(t,t')}2^{q\sigma-\Phi_{q,\lambda}(t')}\|HR_q(v(t'),f(t'))\|_{L^p(L^2)}dt'.$$

Then, using the inequality (15) and taking the sup over q, we get

$$M_\lambda^\sigma(f) \le \|f_0\|_{B^\sigma_{p,\infty}(H^{-s})} + e^{C\lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)}} \sup_{t\in[0,T],q}\int_0^t 2^{-\Phi_{q,\lambda}(t,t')} \tag{16}$$

$$\times\Big(M_\lambda^{\sigma+1}(v) + M_\lambda^\sigma(f)\Big(1+2\|S_q\nabla v(t')\|_{L^\infty} + \sum_{|q'-q|\le N}\|\Delta_{q'}\nabla v(t')\|_{L^\infty}\Big)\Big)dt'. \tag{17}$$

As $\lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)}$ is smaller than $(\sigma-\beta)$, we have

$$e^{C\lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)}} \le e^{C(\sigma-\beta)}.$$

Moreover, by definition of $\Phi_{q,\lambda}(t,t')$, it is obvious that

$$\int_0^t 2^{-\Phi_{q,\lambda}(t,t')}(\|S_q\nabla v(t')\|_{L^\infty}+1)dt' \le \frac{1}{\lambda\log 2}.$$

Then, we obtain that

$$\begin{aligned} M_\lambda^\sigma(f) &\le \|f_0\|_{B^\sigma_{p,\infty}(H^{-s})} + \frac{C}{\lambda}M_\lambda^{\sigma+1}(v) + C\|\nabla v\|_{\widetilde{L}^1_T(C^0)}M_\lambda^\sigma(f) + \frac{C}{\lambda}M_\lambda^\sigma(f)\\ &\le \|f_0\|_{B^\sigma_{p,\infty}(H^{-s})} + \frac{C}{\lambda}M_\lambda^{\sigma+1}(v) + \frac{2C}{\lambda}M_\lambda^\sigma(f). \end{aligned}$$

This proves the theorem of course if we prove the estimate (15) of the lemma. First of all, let us decompose the operator R_q. We have

$$\begin{aligned}
R_q(v,f) &= \sum_{\ell=1}^{6} R_q^\ell(v,f) \quad \text{with}\\
R_q^1(v,f) &= \sum_{j=1}^{d} \Delta_q(T_{\partial_j f} v^j),\\
R_q^2(v,f) &= \sum_{j=1}^{d} [\Delta_q, T_{v^j}\partial_j] f,\\
R_q^3(v,f) &= \sum_{j=1}^{d} \Delta_q \partial_j R(v^j,f) + \Delta_{q-1} v^j \partial_j \Delta_{q+1} f_q - \Delta_{q-2} v^j \partial_j \Delta_{q-1} f_q\\
R_q^4(v,f) &= \sum_{i,j=1}^{d} \operatorname{div}_g(c_\alpha^{ij} \Delta_q(T_f \partial_j v^i)) + \operatorname{div}_g(\Delta_q(T_f \nabla_g U)),\\
R_q^5(v,f) &= \sum_{i,j=1}^{d} \operatorname{div}_g(c_\alpha^{ij}[\Delta_q, T_{\partial_j v^i}] f) + \operatorname{div}_g([\Delta_q, T_{\nabla_g U}] f)\\
R_q^6(v,f) &= \sum_{i,j=1}^{d} \operatorname{div}_g\Big(c_\alpha^{ij}(R(\partial_j v^i, f) + \Delta_{q-1}\partial_j v^i \Delta_{q+1} f_q - \Delta_{q-2}\partial_i v^j \Delta_{q-1} f_q)\Big)\\
&+ \sum_{i,j=1}^{d} \operatorname{div}_g\Big(R(\nabla_g U, f) + \Delta_{q-1}\nabla_g U \Delta_{q+1} f_q - \Delta_{q-2}\nabla_g U \Delta_{q-1} f_q\Big)
\end{aligned}$$

Indeed,

$$\begin{aligned}
\Delta_q(v\cdot\nabla f) &= \Delta_q(\sum_{j=1}^{d} T_{\partial_j f} v^j + T_{v^j}\partial_j f + R(v^j, \partial_j f))\\
&= \sum_{\ell=1}^{2} R_q^\ell(v,f) + \sum_{j=1}^{d} T_{v^j}\partial_j \Delta_q f + \Delta_q R(v^j, \partial_j f),
\end{aligned}$$

Then, we use that

$$\begin{aligned}
\sum_{j=1}^{d} T_{v^j}\partial_j f_q &= \sum_{|q-q'|\le 1} S_{q'-1} v^j \partial_j \Delta_{q'} f_q\\
&= S_{q-1} v^j \partial_j f_q + \sum_{|q-q'|\le 1} (S_{q'-1} v^j - S_{q-1} v^j)\partial_j \Delta_{q'} f_q\\
&= S_{q-1} v^j \partial_j f_q + \Delta_{q-1} v^j \partial_j \Delta_{q+1} f_q - \Delta_{q-2} v^j \partial_j \Delta_{q-1} f_q
\end{aligned}$$

Hence,

$$\Delta_q(v\cdot\nabla f) = \sum_{\ell=1}^{3} R_q^\ell(v,f) + S_{q-1} v\cdot\nabla f_q.$$

In the same way, we have

$$\Delta_q(\operatorname{div}_g(G(v,f)f)) = \sum_{\ell=4}^{6} R_q^\ell(v,f) + \operatorname{div}_g(G(S_{q-1}v, S_{q-1}f) f_q).$$

Let us estimate the six terms appearing above. We have

Let us begin with $R_q^1(v,f)$. By definition of the paraproduct, we have

$$R_q^1(v,f) = \sum_{j=1}^{d}\sum_{q'} \Delta_q(S_{q'-1}\partial_j f \Delta_{q'} v^j).$$

As, if $|q-q'|>2$ then the above term is equal to 0, we deduce that

$$\|HR_q^1(v(t),f(t))\|_{L^p(L^2)} \leq C \sum_{|q-q'|\leq 2} \|HS_{q'-1}\nabla f\|_{L^\infty(L^2)}\|\Delta_{q'}v(t)\|_{L^p}.$$

Using the fact that, if $|q-q'|\leq 2$, then $\|HS_{q'-1}\nabla f\|_{L^\infty(L^2)} \leq C2^q\|Hf(t)\|_{L^\infty(L^2)} \leq C2^q$, we infer that

$$\|HR_q^1(v(t),f(t))\|_{L^p(L^2)} \leq C2^q \sum_{|q-q'|\leq 2}\|\Delta_{q'}v(t)\|_{L^p} \leq C\sum_{|q-q'|\leq 2}\|\nabla\Delta_{q'}v(t)\|_{L^p}.$$

Hence

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR_q^1(v(t),f(t))\|_{L^p(L^2)} \leq CM_\lambda^{\sigma+1}(v)\sum_{|q-q'|\leq 2} 2^{-\lambda\int_0^t\|S_q\nabla v(t')\|_{L^\infty}dt'+\lambda\int_0^t\|S_{q'}\nabla v(t')\|_{L^\infty}dt'}.$$

But, it is obvious that

$$\int_0^t \|S_{q'}\nabla v(t')\|_{L^\infty}dt' - \int_0^t\|S_q\nabla v(t')\|_{L^\infty}dt' \leq \int_0^t \|(S_{q'}-S_q)\nabla v(t')\|_{L^\infty}dt'.$$

Using the fact that $|q-q'|\leq 2$, we get

$$\int_0^t \|S_{q'}\nabla v(t')\|_{L^\infty}dt' - \int_0^t\|S_q\nabla v(t')\|_{L^\infty}dt' \leq C|q-q'|\|\nabla v\|_{\widetilde{L}^1_T(C^0)}. \tag{18}$$

So it turns out that

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR_q^1(v(t),f(t))\|_{L^p(L^2)} \leq 2^{C\lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)}}M_\lambda^{\sigma+1}(v). \tag{19}$$

Now let us look at $R_q^2(v,f)$. By definition of the paraproduct, we have

$$\begin{aligned} R_q^2(v,f) &= -\sum_{j=1}^{d}\sum_{q'}[S_{q'-1}v^j\partial_j\Delta_{q'},\Delta_q]f \\ &= -\sum_{j=1}^{d}\sum_{q'}[S_{q'-1}v^j,\Delta_q]\partial_j\Delta_{q'}f. \end{aligned}$$

The terms of the above sum are equal to 0 except if $|q-q'|\leq 2$. Moreover, by definition of the operators Δ_q, we have

$$[S_{q'-1}v^j,\Delta_q]\partial_j\Delta_{q'}f(x) = 2^{qd}\int_{\mathbb{R}^d} h(2^q(x-y))(S_{q'-1}v^j(x)-S_{q'-1}v^j(y))\partial_j\Delta_{q'}f(y)dy.$$

So we infer that

$$\|H[S_{q'-1}v^j,\Delta_q]\partial_j\Delta_{q'}f(x)\|_{L^2(M)} \leq 2^{-q}\|\nabla S_{q'-1}v\|_{L^\infty}2^{qd}\Big(\big(2^q|\cdot|\times|h(2^q\cdot)|\big)\star\|H\partial_j\Delta_{q'}f\|_{L^2(M)}\Big)(x).$$

Hence,

$$\|H[S_{q'-1}v^j,\Delta_q]\partial_j\Delta_{q'}f(x)\|_{L^p(L^2(M))}\le 2^{-q}\|\nabla S_{q'-1}v\|_{L^\infty}\|H\partial_j\Delta_{q'}f\|_{L^p(L^2(M))}.$$

Then, we have, using Inequality (18),

$$\begin{aligned}&2^{q\sigma-\Phi_{q,\lambda}(t)}\|H[S_{q'-1}v^j,\Delta_q]\partial_j\Delta_{q'}f\|_{L^p(L^2(M))}\\&\qquad\le CM^\sigma_\lambda(f)\sum_{|q-q'|\le 2}2^{C\lambda\|v\|_{\widetilde{L}^1_T(C^1)}}(\|\nabla(S_{q'-1}-S_q)v(t)\|_{L^\infty}+\|S_qv(t)\|_{L^\infty}).\end{aligned}$$

So, we get

$$\begin{aligned}2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR^2_q(v(t),f(t))\|_{L^p(L^2)}&\le CM^\sigma_\lambda(f)2^{C\lambda\|v\|_{\widetilde{L}^1(C^1)}}\\&\quad\times\Big(\|S_q\nabla v(t)\|_{L^\infty}+\sum_{|q-q'|\le 2}\|\nabla(\Delta_{q'}v(t)\|_{L^\infty}\Big).\end{aligned}\tag{20}$$

For R^3_q, we have

$$\begin{aligned}\|HR^3_q(v,f)\|_{L^p(L^2)}&\le C\sum_{\substack{|q'-q''|\le 1\\ q'\ge q-2}}2^q\|\Delta_{q'}v\|_{L^p}\|H\Delta_{q''}f\|_{L^\infty(L^2)}\\&\le C\sum_{q'\ge q-2}2^{q-q'}\|\Delta_{q'}\nabla v\|_{L^p}\|Hf\|_{L^\infty(L^2)}.\end{aligned}$$

Hence,

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR^3_q(v,f)\|_{L^p(L^2)}\le C\sum_{q'\ge q-2}2^{(1+\sigma)(q-q')-\Phi_{q,\lambda}(t)+\Phi_{q',\lambda}(t)}M^{\sigma+1}_\lambda(v)\|Hf\|_{L^\infty(L^2)}.$$

Then, we see that the sum converges since

$$|\Phi_{q,\lambda}(t)-\Phi_{q',\lambda}(t)|\le\lambda\|\nabla v\|_{\widetilde{L}^1_T(C^0)}|q-q'|\le(\sigma-\beta)|q-q'|\tag{21}$$

and $1+\sigma-(\sigma-\beta)=1+\beta>0$. Hence, we get

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR^3_q(v,f)\|_{L^p(L^2)}\le CM^{\sigma+1}_\lambda(v)\|Hf\|_{L^\infty(L^2)}.$$

The estimate for $R^4_q(v,f)=R^{4,1}_q(v,f)+R^{4,2}_q(f)$ is the same as the estimate for $R^1_q(v,f)$. Indeed, we have

$$\begin{aligned}\|HR^{4,1}_q(v(t),f(t))\|_{L^p(L^2)}&\le C\sum_{|q-q'|\le 2}\|\nabla_gHS_{q'-1}f\|_{L^\infty(L^2)}\|\Delta_{q'}\nabla v(t)\|_{L^p}\\&\le C\sum_{|q-q'|\le 2}\|\Delta_{q'}\nabla v(t)\|_{L^p}\end{aligned}$$

where we used that $\|\nabla_gHS_{q'-1}f\|_{L^\infty(L^2)}\le C$. Hence, we conclude as for $R^1_q(v,f)$. Besides,

$$\begin{aligned}\|HR^{4,2}_q(f(t))\|_{L^p(L^2)}&\le C\sum_{|q-q'|\le 2}\|\nabla_gHS_{q'-1}f\|_{L^\infty(L^2)}\|\Delta_{q'}\nabla_gU\|_{L^p}\\&\le C\sum_{|q-q'|\le 2}\|\Delta_{q'}f(t)\|_{L^p}\end{aligned}$$

Hence, we conclude as for $R_q^1(v,f)$ and get

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR_q^{5,2}(f(t))\|_{L^p(L^2)} \le 2^{C\lambda\|\nabla v\|_{\tilde{L}^1_T(C^0)}} M_\lambda^\sigma(f). \tag{22}$$

We write $R_q^5(v,f) = R_q^{5,1}(v,f) + R_q^{5,2}(f)$. The estimate for $R_q^5(v,f)$ is similar to the one for $R_q^2(v,f)$ with the only difference that we have to use the regularity of ∇v. We have

$$[\Delta_q, T_{\partial_j v^i}]f = -\sum_{j=1}^d \sum_{q'} [S_{q'-1}\partial_j v^i, \Delta_q]\partial_j \Delta_{q'} f.$$

The terms of the above sum are equal to 0 except if $|q-q'| \le 2$. Moreover, by definition of the operators Δ_q, we have

$$[S_{q'-1}\partial_j v^i, \Delta_q]\Delta_{q'} f(x) = 2^{qd}\int_{\mathbb{R}^d} h(2^q(x-y))(S_{q'-1}\partial_j v^i(x) - S_{q'-1}\partial_j v^i(y))\Delta_{q'} f(y)dy.$$

So we infer that

$$\|HR_q^{5,1}(v,f)\|_{L^2(M)} \le 2^{-q}|\nabla^2 S_{q'-1}v| 2^{qd}\Big(\Big(2^q|\cdot| \times |h(2^q\cdot)|\Big) \star \|\nabla_g H\Delta_{q'} f\|_{L^2(M)}\Big)(x).$$

Hence,

$$\|HR_q^{5,1}(v,f)\|_{L^p(L^2(M))} \le 2^{-q}\|\nabla^2 S_{q'-1}v\|_{L^p}\|\nabla_g H\Delta_{q'} f\|_{L^\infty(L^2(M))}.$$

Then, we have, using Inequality (18),

$$\begin{aligned}2^{q\sigma-\Phi_{q,\lambda}(t)}&\|HR_q^{5,1}(v,f)\|_{L^p(L^2(M))}\\ &\le C\sum_{\substack{|q-q'|\le 2\\ q''\le q'-1}} 2^{(\sigma-1)(q-q'')-\Phi_{q,\lambda}(t)+\Phi_{q'',\lambda}(t)} M_\lambda^{\sigma+1}(v)\|\nabla_g H\Delta_{q'} f\|_{L^\infty(L^2)}.\end{aligned}$$

Hence,

$$2^{q\sigma-\Phi_{q,\lambda}(t)}\|HR_q^{5,1}(v,f)\|_{L^p(L^2(M))} \le C\sum_{q''\le q+1} 2^{-\beta(q-q'')} M_\lambda^{\sigma+1}(v)\|\nabla_g Hf\|_{L^\infty(L^2)}$$

and the sum is uniformly bounded since $\sigma - 1 + \lambda\|\nabla v\|_{\tilde{L}^1_T(C^0)} \le -\beta$. Then, we argue in a similar way for $\|HR_q^{5,2}(f)\|_{L^p(L^2(M))}$ and get

$$\|HR_q^{5,2}(f)\|_{L^p(L^2(M))} \le 2^{-q}\|\nabla S_{q'-1}f\|_{L^p(L^2(M))}\|\nabla_g H\Delta_{q'} f\|_{L^\infty(L^2(M))}.$$

and we conclude as above with $M_\lambda^{\sigma+1}(v)$ replaced by $M_\lambda^\sigma(f)$.

Finally, the estimate for $R_q^6(v,f)$ is exactly the same as the one for $R_q^3(v,f)$ since, we also have that $\|\nabla_g Hf\|_{L^\infty(L^2)} \le C$.

3 Global existence

Now, we turn to the proof of our main theorem. First, we notice that the local existence in with $v \in L^\infty_{loc}([0,T);W^{1,r}) \cap L^2_{loc}([0,T);W^{2,r})$ and $f \in L^\infty_{loc}([0,T);W^{1,r}(H^{-s}))$ can be easily deduced from standard arguments. Moreover, from regularity estimates for the heat equation, we have for all $0 < T_0 < T$, $v \in L^\infty_{loc}((T_0,T);W^{2-\varepsilon,r})$.

We want to prove that we can extend the solution beyond the time T. It is enough to prove that $\nabla v \in L^\infty((0,T)\times\mathbb{R}^2)$.

The local existence result tells that, for any T_0 in $]0,T[$, the solution (v,f) of (1) belongs to the space $L^\infty_{loc}([T_0,T[;W^{2-\varepsilon,r}\times W^{1,r}(H^{-s}))$ for any $\varepsilon>0$. Sobolev type embeddings of Corollary 1.5 imply that

$$(v,\tau)\in L^\infty_{loc}\Big([T_0,T[;\widetilde{B}^{2-\varepsilon-2\left(\frac{1}{r}-\frac{1}{p}\right)}_{p,\infty}\times \widetilde{B}^{1-2\left(\frac{1}{r}-\frac{1}{p}\right)}_{p,\infty}\Big).$$

Choosing $\varepsilon<1-2/r$ and $p=\infty$ in the above assertion implies that $(v,\tau)\in L^\infty_{loc}(\widetilde{C}^{1+\sigma}\times \widetilde{C}^{\sigma}(H^{-s}))$ where $\sigma=1-\varepsilon-2/r>0$. So we can apply Theorem 1.7 and we can choose T_0 such that, with the notations of Theorem 2.1, we have

$$\|\nabla v\|_{\widetilde{L}^1_{[T_0,T]}(C^0)}\leq \frac{min(\sigma-\beta,1-\sigma-\beta)}{3\lambda}.$$

The deteriorating regularity estimate of Theorem 2.1 applied with σ and between T_0 and T tells exactly that f satisfies

$$M^\sigma_\lambda(f)\leq 3\|f\|_{C^\sigma(H^{-s})}+\frac{3C}{\lambda}M^{\sigma+1}_\lambda(v). \tag{23}$$

Now, we have to estimate ∇v. The two dimensional Navier-Stokes equation can be written as

$$\partial_t v-\nu\Delta v=P(v\cdot\nabla v)+PD\tau$$

where P denotes the Leray projector on the divergence free vector field. Exactly along the same lines as in the proof of Theorem2.1, we have

$$2^{q(\sigma+1)-\Phi_{q,\lambda}(t)}\|P(v\cdot\nabla v)-P(S_qv\cdot\nabla\Delta_q v)\|_{L^\infty}\leq CM^{\sigma+1}_\lambda(v)\bigg(\|S_q\nabla v(t)\|_{L^\infty}+\sum_{q'\geq q}2^{q-q'}\|\nabla\Delta_{q'}v(t)\|_{L^\infty}\bigg).$$

Moreover, it is obvious that

$$2^{q(\sigma-\frac12)-\Phi_{q,\lambda}(t)}\|P(S_qv\cdot\nabla\Delta_q v)\|_{L^\infty}\leq C\|v(t)\|_{H^{\frac12}}M^{\sigma+1}_\lambda(v).$$

So it turns out that

$$2^{q(\sigma+1)-\Phi_{q,\lambda}(t)}\|\Delta_qP(v\cdot\nabla v)\|_{L^\infty}\leq CM^{\sigma+1}_\lambda(v)\bigg(\|S_q\nabla v(t)\|_{L^\infty}+\sum_{q'\geq q}2^{(q-q')}\|\nabla v(t)\|_{L^\infty}+2^{\frac{3q}{2}}\|v(t)\|_{H^{\frac12}}\bigg). \tag{24}$$

Using well known estimates on the heat equation (see for instance [4]) and Inequalities (23) and (24) , we get that

$$M^{\sigma+1}_\lambda(v)\leq\|v_0\|_{C^{\sigma+1}}+\bigg(\frac{C}{\lambda}+2^{\frac{3q}{2}}F_q(T_0,T)\bigg)M^s_\lambda(v)+\frac{C}{\nu}M^\sigma_\lambda(\tau)$$

with

$$F_q(T_0,T)\stackrel{\text{def}}{=}\sup_{t\in[T_0,T]}\int_{T_0}^t e^{c\nu2^{2q}(t-t')}\|v(t')\|_{H^{\frac12}}dt'.$$

Hölder inequality implies immediately that

$$F_q(T_0,T)\leq\frac{C}{\nu^{\frac34}}2^{-\frac{3q}{2}}\|v\|_{L^4_{T_0,T]}(H^{\frac12})}.$$

Moreover, it is easy to see that

$$M_\lambda^\sigma(\tau) \leq M_\lambda^\sigma(f).$$

So, we infer that

$$M_\lambda^{\sigma+1}(v) \leq \|v_0\|_{C^{\sigma+1}} + \frac{3C}{\nu}\|\tau_0\|_{C^\sigma} + \left(\frac{C}{\lambda} + \frac{C}{\lambda\nu} + \frac{C}{\nu^{\frac{3}{4}}}\|v\|_{L^4_{T_0,T]}(H^{\frac{1}{2}})}\right) M_\lambda^{\sigma+1}(v)$$

Now it is enough to choose T_0 such that the quantity

$$\left(\frac{C}{\lambda} + \frac{C}{\lambda\nu} + \frac{C}{\nu^{\frac{3}{4}}}\|v\|_{L^4_{T_0,T]}(H^{\frac{1}{2}})}\right)$$

is small enough. Then as σ is greater than 0, the solution (v, τ) of the system (1) is such that $(\nabla v, \tau)$ belongs to $L^\infty([T_0, T] \times \mathbb{R}^2)$; this concludes the proof of Theorem 1.1.

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