Homogenization and boundary layer

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This paper deals with the homogenization of elliptic systems with Dirichlet boundary condition, when the coefficients of both the system and the boundary data are ε-periodic. We show that, as $\varepsilon \to 0$, the solutions converge in L^2 with a power rate in ε, and identify the homogenized limit system. Due to a boundary layer phenomenon, this homogenized system depends in a non trivial way on the boundary. Our analysis answers a longstanding open problem, raised for instance in [7]. It extends substantially previous results obtained for polygonal domains with sides of rational slopes as well as our previous paper [12] where the case of irrational slopes was considered.

1 Introduction

This paper is about the homogenization of elliptic systems in divergence form

$$-\nabla \cdot (A(\cdot/\varepsilon) \nabla u)(x) = 0, \quad x \in \Omega, \tag{1.1}$$

set in a bounded domain Ω of $\mathbb{R}^d$, $d \geq 2$, with an oscillating Dirichlet data

$$u(x) = \varphi(x, x/\varepsilon), \quad x \in \partial\Omega. \tag{1.2}$$

As is customary, $\varepsilon > 0$ is a small parameter, and $A = A^{\alpha\beta}(y) \in M_N(\mathbb{R})$ is a family of functions of $y \in \mathbb{R}^d$, indexed by $1 \leq \alpha, \beta \leq d$, with values in the set of $N \times N$ matrices. Also, $u = u(x)$ and $\varphi = \varphi(x, y)$ take their values in $\mathbb{R}^N$. We remind, using Einstein convention for summation, that for each $1 \leq i \leq N$,

$$(\nabla \cdot A(\cdot/\varepsilon) \nabla u)_i(x) := \partial_{x_\alpha} \left[A^{\alpha\beta}_{ij}(\cdot/\varepsilon)\, \partial_{x_\beta} u_j \right](x).$$

In the sequel, Greek letters $\alpha, \beta, \ldots$ will range between 1 and d and Latin letters $i, j, k, \ldots$ will range between 1 and N. We make three hypotheses:

i) Ellipticity: For some $\lambda > 0$, for all family of vectors $\xi = \xi^\alpha_i \in \mathbb{R}^{Nd}$

$$\lambda \sum_\alpha \xi^\alpha \cdot \xi^\alpha \leq \sum_{\alpha,\beta,i,j} A^{\alpha,\beta}_{ij} \xi^\beta_j \xi^\alpha_i \leq \lambda^{-1} \sum_\alpha \xi^\alpha \cdot \xi^\alpha.$$

ii) Periodicity: $\forall y \in \mathbb{R}^d$, $\forall h \in \mathbb{Z}^d$, $\forall x \in \partial\Omega$, $A(y+h) = A(y)$, $\quad \varphi(x, y) = \varphi(x, y+h)$.

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iii) Smoothness: The functions A and φ, as well as the domain Ω are smooth. It is actually enough to assume that ϕ and Ω are in some H^s for s big enough, but we will not try to compute the optimal regularity.

We are interested in the limit $\varepsilon \to 0$, *i.e.* the homogenization of system (1.1)-(1.2).

Systems of type (1.1) are involved in various domains of material physics, notably in linear elasticity and in thermodynamics [24, 7, 1, 19]. In many cases they come with a right hand side f. Our analysis extends easily to that case. In the context of thermodynamics, $d = 2$ or 3, $N = 1$, u is the temperature, and $\sigma = A(\cdot/\varepsilon)\nabla u$ is the heat flux given by Fourier law. The parameter ε models heterogeneity, that is short-length variations of the material conducting properties. The boundary term φ in (1.2) corresponds to a prescribed temperature at the surface of the body. In the context of linear elasticity, $d = 2$ or 3, $N = d$, u is the unknown displacement, f is the external load and A is a fourth order tensor that models Hooke's law.

Note that other boundary conditions can be encountered, such as the Neumann condition

$$n(x) \cdot (A(x/\varepsilon)\nabla u)\,(x) \;=\; \varphi(x, x/\varepsilon), \quad x \in \partial\Omega, \tag{1.3}$$

where $n(x)$ is the normal vector. Still in thermodynamics, it corresponds to a given heat flux at the solid surface. One could also account for heat sources inside the body, by the addition of a source term in (1.1).

Elliptic systems with periodic coefficients are also a classical topic in the mathematical theory of homogenization. We refer to the well known book [7] for a good overview (see also the more recent books [16, 10, 9, 26]). As regards divergence form systems, two problems have been widely studied and are by now well understood:

1. *the non-oscillating Dirichlet problem*, that is (1.1)-(1.2) with $\varphi = \varphi(x)$.
2. *the oscillating Neumann problem*, that is (1.1)-(1.3) with a standard compatibility condition on φ.

Note that in both problems, the usual energy estimate provides a uniform bound on the solution u^ε in $H^1(\Omega)$.

For the non-oscillating Dirichlet problem, one shows that u^ε weakly converges in $H^1(\Omega)$ to the solution u^0 of the homogenized system

$$\begin{cases} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = 0, & x \in \Omega, \\ \qquad\qquad u^0(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{1.4}$$

The so-called homogenized matrix A^0 comes from the averaging of the microstructure. It involves the periodic solution $\chi = \chi^\gamma(y) \in M_N(\mathbb{R}),\ 1 \le \gamma \le d$, of the famous *cell problem*:

$$-\partial_{y_\alpha}\left[A^{\alpha\beta}(y)\,\partial_{y_\beta}\chi^\gamma(y)\right] \;=\; \partial_{y_\alpha}A^{\alpha\gamma}(y), \quad \int_{[0,1]^d} \chi^\gamma(y)\,dy = 0. \tag{1.5}$$

The homogenized matrix is then given by:

$$A^{0,\alpha\beta} \;=\; \int_{[0,1]^d} A^{\alpha\beta} \;+\; \int_{[0,1]^d} A^{\alpha\gamma}\partial_{y_\gamma}\chi^\beta.$$

One may even go further in the analysis, and obtain a two-scale expansion of u^ε. Denoting

$$u^1(x,y) \;:=\; -\chi^\alpha(y)\partial_{x_\alpha} u^0(x), \tag{1.6}$$

it is proved in [7] that

$$u^\varepsilon(x) \;=\; u^0(x) \;+\; \varepsilon u^1(x, x/\varepsilon) \;+\; O(\sqrt{\varepsilon}), \;\text{ in }\; H^1(\Omega). \tag{1.7}$$

Actually, an open problem in this area is to compute the next term in the expansion in the presence of a boundary. This will follow from the analysis of this paper (see below and section5).

For the oscillating Neumann problem, two cases must be distinguished. On one hand, if $\partial\Omega$ does not contain flat pieces, or if it contains finitely many flat pieces whose normal vectors do not belong to $\mathbb{R}\mathbb{Z}^n$, then

$$\varphi(\cdot, \cdot/\varepsilon) \to \overline{\varphi} := \int_{[0,1]^d} \varphi \;\text{ weakly in } L^2(\partial\Omega)$$

and u^ε converges weakly to the solution u^0 of

$$\begin{cases} -\nabla\cdot\left(A^0\nabla u^0\right)(x) = 0, & x\in\Omega,\\ n(x)\cdot\left(A^0\nabla u^0\right)(x) = \overline{\varphi}(x), & x\in\partial\Omega. \end{cases} \tag{1.8}$$

On the other hand, if $\partial\Omega$ does contain a flat piece whose normal vector belongs to $\mathbb{R}\,\mathbb{Q}^d$, then the family $\varphi(\cdot, \cdot/\varepsilon)$ may have a continuum of accumulation points as $\varepsilon \to 0$. Hence, u^ε may have a continuum of accumulation points in H^1 weak, corresponding to different Neumann boundary data. We refer to [7] for all details.

On the basis of these results, it seems natural to address the homogenization of (1.1)-(1.2) with an oscillating Dirichlet data. At first glance, this case looks similar to the aforementioned ones. *However, this homogenization problem turns out to be much different, and much more difficult.* Up to our knowledge, besides restrictive settings to be described later on, it has remained unsolved. There are two main sources of difficulties:

i) One has uniform L^p bounds on the solutions u^ε of (1.1)-(1.2), but no uniform H^1 bound *a priori.* This is due to the fact that

$$\|x \mapsto \varphi(x, x/\varepsilon)\|_{H^{1/2}(\partial\Omega)} = O(\varepsilon^{-1/2}), \;\text{ resp. } \|x \mapsto \varphi(x, x/\varepsilon)\|_{L^p(\partial\Omega)} = O(1),\; p > 1.$$

The usual energy inequality, resp. the estimates in article [5, page 8, Theorem 3] yield

$$\|u^\varepsilon\|_{H^1(\Omega)} = O(\varepsilon^{-1/2}), \;\text{ resp. } \|u^\varepsilon\|_{L^p(\Omega)} = O(1),\; p > 1.$$

This indicates that singularities of u^ε are *a priori* stronger than in the usual situations. It will be rigorously established in the core of the paper.

ii) Furthermore, one can not expect these stronger singularities to be periodic oscillations. Indeed, the oscillations of φ are at the boundary, along which they do not have any periodicity property. Hence, it is reasonable that u^ε should exhibit concentration near $\partial\Omega$, with no periodic character, as $\varepsilon \to 0$. This is a so-called *boundary layer phenomenon.* The key point is to describe this boundary layer, and its effect on the possible weak limits of u^ε. This causes strong mathematical difficulties, that have been recognized for long. Quoting [7, page xiii]:

Of particular importance is the analysis of the behavior of solutions near boundaries and, possibly, any associated boundary layers. Relatively little seems to be known about this problem.

We stress that there is also a boundary layer in the non-oscillating Dirichlet problem, although it has in this case a lower amplitude. More precisely, it is responsible for the $O(\sqrt{\varepsilon})$ loss in the error estimate (1.7). If either the L^2 norm, or the H^1 norm in a relatively compact subset $\omega \Subset \Omega$ is considered, one may avoid this loss as strong gradients near the boundary are filtered out. Following Allaire and Amar (see [2, Theorem 2.3]):

$$u^\varepsilon \;=\; u^0(x) \;+\; O(\varepsilon) \;\text{ in }\; L^2(\Omega), \quad u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) \;+\; O(\varepsilon) \;\text{ in }\; H^1(\omega). \tag{1.9}$$

Still following [2], another way to put the emphasis on the boundary layer is to introduce the solution $u_{bl}^{1,\varepsilon}(x)$ of

$$\left\{ \begin{aligned} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla u_{bl}^{1,\varepsilon} &= 0, \quad x \in \Omega \subset \mathbb{R}^d, \\ u_{bl}^{1,\varepsilon} &= -u^1(x, x/\varepsilon), \quad x \in \partial\Omega, \end{aligned} \right. \tag{1.10}$$

Then, one can show that

$$u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon u_{bl}^{1,\varepsilon}(x) \;+\; O(\varepsilon), \;\text{ in }\; H^1(\Omega). \tag{1.11}$$

or

$$u^\varepsilon(x) \;=\; u^0(x) + \varepsilon u^1(x, x/\varepsilon) + \varepsilon u_{bl}^{1,\varepsilon}(x) \;+\; O(\varepsilon^2), \;\text{ in }\; L^2(\Omega). \tag{1.12}$$

Note that system (1.10) is a special case of (1.1)-(1.2). Thus, the homogenization of the oscillating Dirichlet problem may give a refined description of the non-oscillating one. This is another motivation for its study. We refer to section 5 for the study of this case. Let us also mention that there are many homogenization problem that require the introduction of a boundary layer (for severl different reasons), we can mention [3, 15, 4, 20, 13].

Before stating our main result, let us present former works on this problem. Until recently, *they were all limited to convex polygons with rational normals.* This means that

$$\Omega \;:=\; \cap_{k=1}^K \left\{ x, \quad n^k \cdot x > c^k \right\}$$

is bounded by K hyperplanes,*whose unit normal vectors n^k belong to $\mathbb{R}\,\mathbb{Q}^d$.* Under this stringent assumption, the study of (1.1)-(1.2) can be carried out. In short, the key point is the addition of boundary layer correctors to the formal two-scale expansion:

$$u^\varepsilon(x) \;\sim\; u^0(x) \;+\; \varepsilon u^1(x, x/\varepsilon) \;+\; \sum_k v_{bl}^k\left(x, \frac{x}{\varepsilon}\right), \tag{1.13}$$

where $v_{bl}^k = v_{bl}^k(x, y) \in \mathbb{R}^n$ is defined for $x \in \Omega$, and y in the half-space

$$\Omega^{\varepsilon,k} \;=\; \left\{ y, \quad n^k \cdot y > c^k/\varepsilon \right\}.$$

These correctors satisfy

$$\left\{ \begin{aligned} &-\nabla_y \cdot A(y) \nabla_y v_{bl}^k = 0, \quad y \in \Omega^{\varepsilon,k}, \\ &v_{bl}^k = \varphi(x, y) - u_0(x), \quad y \in \partial\Omega^{\varepsilon,k}. \end{aligned} \right. \tag{1.14}$$

We refer to the papers by Moskow and Vogelius [18], and Allaire and Amar [2] for all details. These papers deal with the special case (1.10), but the results adapt to more general oscillating data. Note that x is only involved as a parameter in (1.14). Note also that the assumption $n^k \in \mathbb{R}\,\mathbb{Z}^d$ yields periodicity of the function $A(y)$ tangentially to the hyperplanes. This periodicity property is used in a crucial way in the aforementioned references. It allows straightforward resolution of the boundary layer systems (1.14). Moreover, thanks to a lemma of Tartar, one gets exponential convergence of $v^k_{bl}(x, y)$ to some $v^k_{bl,*}(x) = \varphi^k_*(x) - u^0(x)$, when y goes to infinity transversely to the k-th hyperplane. In order for the boundary layer correctors to vanish at infinity (and to be $o(1)$ in L^2), one must have $v^k_{bl,*} = 0$, which provides the boundary condition for u^0. Hence, u^0 should satisfy a system of the type

$$\begin{cases} -\nabla \cdot \left(A^0 \nabla u^0\right)(x) = 0, & x \in \Omega, \\ \qquad\qquad u^0(x) = \varphi_*(x), & x \in \partial\Omega. \end{cases} \tag{1.15}$$

where $\varphi_*(x) := \varphi^k_*(x)$ on the k-th side of Ω. Nevertheless, this picture is not completely correct. Indeed, there is still *a priori* a dependence of φ^k_* on ε, through the domain $\Omega^{\varepsilon,k}$. In fact, Moskow and Vogelius exhibit examples for which there is an infinity of accumulation points for the φ^k_*'s, as $\varepsilon \to 0$. Eventually, they show that the accumulation points of u^ε in L^2 are the solutions u^0 of systems like (1.15), in which the φ^k_*'s are replaced by their accumulation points. See [18] for rigorous statements and proofs. We stress that their analysis relies heavily on the peculiar shape of Ω, especially the rationality assumption.

A step towards more generality has been made in our recent paper [12], in which generic convex polygonal domains are considered. Indeed, we assume in [12] that *the normals* $n = n^k$ *satisfy the Diophantine condition:*

$$\text{For all } \ \xi \in \mathbb{Z}^d \setminus \{0\} \quad |P_{n^\perp}(\xi)| > \kappa\, |\xi|^{-l}, \quad \text{for some } \ \kappa,\ l > 0, \tag{1.16}$$

where $P_{n^\perp}$ is the projector orthogonally to n. Note that for dimension $d = 2$ this condition amounts to:

$$\text{For all } \ \xi \in \mathbb{Z}^d \setminus \{0\} \quad |n^\perp \cdot \xi| \ := \ | - n_2\xi_1 + n_1\xi_2| > \kappa\, |\xi|^{-l}, \quad \text{for some } \ \kappa,\ l > 0,$$

whereas for $d = 3$, it is equivalent to:

$$\text{For all } \ \xi \in \mathbb{Z}^d \setminus \{0\} \quad |n \times \xi| > \kappa\, |\xi|^{-l}, \quad \text{for some } \ \kappa,\ l > 0.$$

Condition (1.16) is generic in the sense that it holds for almost every $n \in S^{d-1}$, see section 2 for more details.

Under this Diophantine assumption, one can perform the homogenization of problem (1.1)-(1.2). *Stricto sensu*, only the case (1.10), $d = 2, 3$ is treated in [12], but our analysis extends straightforwardly to the general setting. Despite a loss of periodicity in the tangential variable, we manage to solve the boundary layer equations, and prove convergence of v^k_{bl} away from the boundary. The main idea is to work with quasi-periodic functions instead of periodic ones. Interestingly, and contrary to the "rational case", the field φ^k_* does not depend on ε. As a result, we establish convergence of the whole sequence u^ε to the single solution u^0 of (1.15). We stress that, even in this polygonal setting, the boundary data φ_* depends in a non trivial way on the boundary. In particular, it is not simply the average of φ with respect to y, contrary to what happens in the Neumann case.

Pondering on this previous study, we are able to treat in this paper the case of smooth domains. Our main result is the following

Theorem 1 (Homogenization in smooth domains)
Let Ω be a smooth bounded domain of $\mathbb{R}^d$, $d \geq 2$. We assume that it is uniformly convex (all the principal curvatures are bounded from below).
Let u^ε be the solution of system (1.1)-(1.2)*, under the ellipticity, periodicity and smoothness conditions i)-iii).*
There exists a boundary term φ_* (depending on φ, A and Ω), with $\varphi_* \in L^p(\partial\Omega)$ for all finite p, and a solution u^0 of (1.15)*, with $u^0 \in L^p(\Omega)$ for all finite p, such that:*

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \;\leq\; C_\alpha\, \varepsilon^\alpha, \qquad \textit{for all } 0 < \alpha < \frac{d-1}{3d+5}. \tag{1.17}$$

Let us make a few remarks on this theorem:

1. We only treat with full details the case where Ω is the disc. The general case of uniformly convex Ω follows from a much similar analysis, and is briefly discussed in section 4.

2. As regards (more) general domains, one can still carry out most of the analysis if there is no flat piece in the boundary which has a normal vector which belongs to $\mathbb{R}\,\mathbb{Q}^d$. In such a case, one can still prove a result similar to theorem 1 with a worse rate of convergence. This will be done in a forthcoming paper.

3. The value $\frac{d-1}{3d+5}$ in the theorem comes from the optimization of several small parameters involved and hence is not sharp. Finding the sharp rate seems a very interesting open problem.

4. The dependence of φ_* in x only happens through the normal $n(x)$ and through the function $\varphi(x,.)$, where x is fixed. More precisely, $\varphi_*(x) = \mathcal{A}[\varphi(x,.), A(.), n(x)]$ where $\mathcal{A}$ is a functional that will be constructed in the next section.

The outline of the paper is as follows. We investigate in section 2 the case where Ω is a half-space: $\Omega = \{x,\; x \cdot n > c\}$, under condition (1.16). We remind some results obtained in [12], and give some refined ones. In particular, we construct the functional $\mathcal{A}$. In section 3, we prove the theorem in the case where $d = 2$, Ω is the unit disk and φ factors into $\varphi(x,y) = v_0(y)\,\varphi_0(x)$ for some smooth $v_0 \in M_N(\mathbb{R})$ and $\varphi_0 \in \mathbb{R}^N$. Then, we indicate in section 4 how to extend the proof to general smooth, uniformly convex domains Ω and general boundary data φ. Finally, we give an application of our result to the study of the higher order approximation of (1.4).

2 The half-space problem

We consider here a half-space: $\Omega = \{x,\; x \cdot n > c\}$. We suppose that the unit inward normal n satisfies the small divisor assumption (1.16). This assumption is almost surely satisfied. More precisely, let $(d-1)l > 1$ and let $\mathcal{A}_\kappa$ be the set

$$\mathcal{A}_\kappa = \Big\{ n \in \mathbb{S}^{d-1}\;,\; \forall \xi \in \mathbb{Z}^d \setminus \{0\}\;, |P_{n^\perp}(\xi)| \geq \kappa\,|\xi|^{-l} \Big\}. \tag{2.1}$$

We claim that there exists a constant C such that $m(\mathcal{A}_\kappa^c) \le C\kappa^{d-1}$ where m denotes the Lebesgue measure on the sphere $\mathbb{S}^{d-1}$. Indeed,

$$\mathcal{A}_\kappa = \cap_{\xi\in\mathbb{Z}^d, |\xi|\neq 0} \left\{ n,\ \left|P_{n^\perp}\left(|\xi|^{-1}\xi\right)\right| \ge \kappa\, |\xi|^{-(l+1)} \right\}$$

from which

$$\mathcal{A}_\kappa^c = \cup_{\xi\in\mathbb{Z}^d, |\xi|\neq 0} \left\{ n,\ \left|P_{n^\perp}\left(|\xi|^{-1}\xi\right)\right| < \kappa\, |\xi|^{-(l+1)} \right\}.$$

Completing the unit vector $\xi_1 := |\xi|^{-1}\xi$ into an orthonormal basis $\xi_2, \ldots, \xi_d$, and writing $n = \sum_{i=1}^d n_i\, \xi_i$, one has

$$\left\{ n \in \mathbb{S}^{d-1},\ \left|P_{n^\perp}\left(|\xi|^{-1}\xi\right)\right| < \kappa\, |\xi|^{-(l+1)} \right\} = \left\{ n \in \mathbb{S}^{d-1},\ \left(\sum_{i=2}^d n_i^2\right)^{1/2} < \kappa\, |\xi|^{-(l+1)} \right\}$$

with a Lebesgue measure which is clearly less than $C\kappa^{d-1}\, |\xi|^{(1-d)(l+1)}$. Hence, we deduce that

$$m(\mathcal{A}_\kappa^c) \le C\kappa^{d-1} \sum_{\xi\in\mathbb{Z}^d, |\xi|\neq 0} |\xi|^{(1-d)(l+1)}. \tag{2.2}$$

This estimate will be used later on.

2.1 The boundary layer analysis

In the half-space case, we expect the solution u^ε of (1.1)-(1.2) to behave like

$$u^\varepsilon(x) \sim u^0(x) + \varepsilon u^1(x, x/\varepsilon) + v_{bl}(x, x/\varepsilon)$$

where u^1 was given in (1.6) and where $v_{bl} = v_{bl}^\varepsilon$ models the boundary layer. At a formal level, it satisfies

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y\, v_{bl}(x,y) = 0, & y.n > c/\varepsilon \\ v_{bl}(x,y) = \varphi(x,y) - u_0(x), & y.n = c/\varepsilon. \end{cases} \tag{2.3}$$

and should decay when y goes to infinity transversely to the boundary $y.n = c/\varepsilon$. Remark that x is not involved in the differential operators and that the ε dependence only comes from the domain, namely c/ε. This suggests to have a look at the problem

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y\, v(y) = 0, & y.n > a \\ v(y) = v_0(y), & y.n = a. \end{cases} \tag{2.4}$$

for a periodic and smooth $v_0 = v_0(y)$. We consider v_0 and v with values in $\mathbb{R}^N$, but of course all results can be extended to $M_{N,p}(\mathbb{R})$, treating the p columns of the matrices separately.

System (2.4) has been examined in the recent paper [12]. Loosely, we have shown:

1. Well-posedness of (2.4), in an appropriate space of *quasiperiodic* functions. Our well-posedness result holds for general normal vector n, with or without the Diophantine assumption. Moreover, it is valid for any $N \ge 1$. We stress that in the scalar case $N = 1$, simpler arguments based on the maximum principle would lead to well-posedness in L^∞.

2. Convergence of the solution v to some constant field v_* as y goes to infinity transversely to the boundary. This convergence result uses assumption (1.16).

We shall recall here a few elements of these two aspects of the boundary layer analysis. We shall then refine these elements, focusing on the dependence of v and v_* on a and n.

Well-posedness

Let M be an orthogonal matrix of $O(d)$ that maps the canonical vector $e_d = (0,...,0,1)$ to the normal vector n. The matrix M is not unique: it is only defined modulo an orthogonal matrix of $O(d-1)$. By the change of variable $y = Mz$, system (2.4) becomes

$$\begin{cases} -\nabla_z \cdot B(Mz)\nabla_z \mathbf{v}(z) = 0, & z_d > a, \\ \mathbf{v}(z) = v_0(Mz), & z_d = a, \end{cases} \tag{2.5}$$

where $\mathbf{v}(z) = v(Mz)$. Denoting $A^{\alpha\beta}_{ij}$, resp. $B^{\alpha\beta}_{ij}$, $1 \le i,j \le N$, the coefficients of $A^{\alpha\beta}$, resp. $B^{\alpha\beta}$, we get that

$$\forall i,j, \quad B_{ij} = (M^t)A_{ij}M$$

which is a product of matrices in $M_d(\mathbb{R})$. Indeed, from $y = Mz$, we get that $\nabla_z = M^t\nabla_y$ and $\nabla_y = M\nabla_z$. Hence for any vector e, $\mathrm{div}_y(e) = \mathrm{div}_z(M^te)$. We also denote $z = (z', z_d)$ the tangential and normal component of z.

Let now $N \in M_{d,d-1}(\mathbb{R})$ be defined by

$$Nz' \;=\; M(z',0).$$

The structure of (2.5) suggests to look for a solution of the type:

$$\mathbf{v}(z) \;=\; V\big(Nz', z_d\big), \quad V\big(\theta, t\big) \text{ 1-periodic in } \theta \in \mathbb{R}^d. \tag{2.6}$$

This means that we look for a $\mathbf{v}$ which is quasi-periodic in z'. We point out that if n is the multiple of a rational vector, as in former papers [18, 2], then one can choose N in such a way that all coefficients of (2.5) are periodic in z' (with an integer period, possibly greater than one). In such a case, one can look for a $\mathbf{v}$ periodic in z', which simplifies greatly the boundary layer analysis.

Accordingly to (2.6), we define

$$\mathcal{B}\big(\theta,t\big) = B\big(\theta + t\,n\big), \quad V_0\big(\theta,t\big) = v_0\big(\theta + t\,n\big)$$

This leads to the following system, for $\theta \in \mathbb{T}^d$, $t > a$:

$$\begin{cases} -\begin{pmatrix} N^t\nabla_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}(\theta,t)\begin{pmatrix} N^t\nabla_\theta \\ \partial_t \end{pmatrix} V(\theta,t) = 0, & t > a \\ V(\theta,t) = V_0(\theta,t), & t = a. \end{cases} \tag{2.7}$$

The well-posedness of system (2.7) is established in Proposition 2 of [12] which states that

Proposition 2 *There exists a unique smooth solution V of* (2.7) *such that*

$$\int_{\mathbb{T}^d}\int_a^{+\infty} \Big(|N^t\,\nabla_\theta\partial^\gamma_\theta V|^2 + |\partial^l_t\partial^\gamma_\theta V|^2\Big)\,dt\,d\theta \;<\; C$$

for $l \in \mathbb{N}$, $l \ge 1$, and $\gamma \in \mathbb{N}^d$ and where we denote $\partial^\gamma_\theta = \partial^{\gamma_1}_{\theta_1}...\partial^{\gamma_d}_{\theta_d}$. Here and in all the paper $\mathbb{N}$ denotes the set of integers including 0.

As a consequence, $\mathbf{v}(z) = V(Nz', z_d)$ is a smooth solution of (2.5).

Behaviour at infinity

At this stage, one still needs to understand the asymptotic behaviour of $V(\theta, t)$, as $t \to +\infty$. In the "periodic case", this follows from a lemma of Tartar (see [2]). In the wider quasi-periodic setting, *and together with the Diophantine assumption* (1.16), we have

Proposition 3 (see [12]) *There exists a constant vector $v_* \in \mathbb{R}^N$ such that*

$$\lim_{t\to+\infty} V = v_*.$$

Moreover,

$$\left| \partial_\theta^\alpha \partial_t^k (V - v_*)\right| \le C(1+t)^{-m},$$

for all $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}$, uniformly in θ.

In general and without any Diophantine assumption on n, we have

$$\|V(t)\|_{H^s(\mathbb{T}^d)} \le C + Ct^{1/2} \tag{2.8}$$

which can be obtain by writing $V(t) = V(0) + \int_0^T \partial_t V$ and then using the L^2 bound on $\partial_t V$.

Note that V and v_* depend *a priori* on n and a in (2.7). But, as n satisfies the small divisor assumption, it does not belong to $\mathbb{R}\mathbb{Q}^d$, which implies

Proposition 4 (see [12]) *The limit at infinity v_* does not depend on a.*

As mentioned in the introduction, this is in sharp contrast with the rational case.

2.2 Refined estimates

The results described above are not enough to be used within the context of smooth domains. Roughly, our idea to handle a smooth convex domain Ω is to see it as the intersection of the half spaces whose boundaries are the tangent hyperplanes to $\partial\Omega$. Using a good sequence of such half-spaces and the corresponding boundary layer correctors, one may hope to obtain in the limit a homogenized problem in Ω. However, this idea will require some uniform control of the correctors, with respect to the normal vectors n at $\partial\Omega$. This is the purpose of the present paragraph. We start with a uniform L^∞ bound:

Proposition 5 *For all $n \in \cup_{\kappa>0} A_\kappa$, the solution v of* (2.4) *given by $v(Mz) = V(Nz', z_d)$, where V solves (2.7), satisfies:*

$$\sup_y |\partial^\alpha v(y)| \le M_\alpha, \quad \forall \alpha \in \mathbb{N}^d$$

The constant M_α depends linearly on the $W^{s,\infty}$ norm of v_0 for some $s = s(\alpha)$ large enough. It depends neither on n nor on a. In particular, $v_* = v_*(n)$ is bounded uniformly in n.

Proof. First, let us remark that this result is trivial in the scalar case, thanks to the maximum principle. For the vector case, we proceed as follows. By classical elliptic regularity arguments, it is enough to handle the case $\alpha = 0$. We introduce the Green function $G_n(y, \tilde{y})$ i.e. the matrix function that satisfies

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y G_n(y,\tilde{y}) = \delta(y-\tilde{y})\, I_N, & y\cdot n > a \\ G_n(y,\tilde{y}) = 0, & y\cdot n = a, \end{cases}$$

where I_N denotes the identity over $\mathbb{R}^N$. We also introduce

$$P_n(y,\tilde{y}) := -n \cdot \left(A^t(\tilde{y})\nabla_{\tilde{y}} G_n(y,\tilde{y})\right) \; = \; -n_\alpha \, (A^t)^{\alpha\beta} \partial_{\tilde{y}_\beta} G_n(y,\tilde{y})$$

(where $(A^t)^{\alpha\beta}_{ij} \; = \; A^{\beta\alpha}_{ji}$) the corresponding Poisson kernel, defined for $\tilde{y}\cdot n = a$. From the work of Avellaneda and Lin [5], we can get various estimates on G_n and P_n. Actually, the estimates given in [5] relate to $G^\varepsilon = G^\varepsilon(x,\tilde{x})$ and $P^\varepsilon = P^\varepsilon(x,\tilde{x})$, which are the Green and Poisson kernels of the operator $\nabla \cdot A(\cdot/\varepsilon)\nabla$, considered in a smooth bounded domain with Dirichlet conditions. But their estimates extend straightforwardly to the half-space case, and then, using the scaling relation

$$G_n(y,\tilde{y}) \; = \; \varepsilon^{d-2} G^\varepsilon_n(\varepsilon y, \varepsilon\tilde{y}),$$

we can obtain estimates on G_n and P_n. More precisely, by the rescaling of [5, estimate (4.6)], we have

$$|G_n(y,\tilde{y})| \; \le \; \frac{C\,\delta(y)\,\delta(\tilde{y})}{|y-\tilde{y}|^d}, \quad \forall y \neq \tilde{y}, \quad \delta(y) \; := \; y\cdot n - a, \quad \delta(\tilde{y}) \; := \; \tilde{y}\cdot n - a. \tag{2.9}$$

In particular, this implies

$$|P_n(y,\tilde{y})| \; \le \; \frac{C\,\delta(y)}{|y-\tilde{y}|^d}, \quad \forall y \neq \tilde{y}. \tag{2.10}$$

Moreover, one can have a better decay for gradients of these quantities. Indeed, the bound in [5, estimate (4.6)] is

$$|G^\varepsilon_n(x,\tilde{x})| \; \le \; \frac{C\,\delta^\varepsilon(x)\,\delta^\varepsilon(\tilde{x})}{|x-\tilde{x}|^d}, \quad \forall x \neq \tilde{x}, \quad \delta^\varepsilon(x) \; := \; x\cdot n - \varepsilon a, \quad \delta^\varepsilon(\tilde{x}) \; := \; \tilde{x}\cdot n - \varepsilon a.$$

Denoting $r = \min(\delta^\varepsilon(x), |x-\tilde{x}|)$, we notice that $G^\varepsilon_n(\cdot,\tilde{x})$ satisfies the equation

$$\nabla \cdot A(\cdot/\varepsilon)\nabla G^\varepsilon_n(\cdot,\tilde{x}) = 0$$

over $\{x',\, |x'-x| < r/2\}$. Applying [5, Lemma 20], we get

$$\begin{aligned} |\nabla_x G^\varepsilon(x,\tilde{x})| \; &\le \; \frac{C}{r} \sup_{|x'-x|<r/2} |G^\varepsilon(x',\tilde{x})| \; \le \; \frac{C'}{r} \sup_{|x'-x|<r/2} \frac{\delta^\varepsilon(x')\delta^\varepsilon(\tilde{x})}{|x'-\tilde{x}|^d} && (2.11) \\ &\le \; \frac{C''}{r} \frac{\delta^\varepsilon(x)\,\delta^\varepsilon(\tilde{x})}{|x-\tilde{x}|^d} && (2.12) \end{aligned}$$

¿From there, we deduce that

$$|\nabla_y G_n(y,\tilde{y})| \; \le \; C\left(\frac{\delta(\tilde{y})}{|y-\tilde{y}|^d} \; + \; \frac{\delta(y)\,\delta(\tilde{y})}{|y-\tilde{y}|^{d+1}}\right), \quad \forall y \neq \tilde{y}. \tag{2.13}$$

Note that, inverting the roles of x and $\tilde{x}$ (and those of A and A^t), we get also

$$|\nabla_{\tilde{y}} G_n(y, \tilde{y})| \;\le\; C \left(\frac{\delta(y)}{|y-\tilde{y}|^d} \;+\; \frac{\delta(y)\,\delta(\tilde{y})}{|y-\tilde{y}|^{d+1}} \right), \quad \forall y \neq \tilde{y}. \tag{2.14}$$

Moreover, the inequality (2.13) implies in particular that

$$|\nabla_y P_n(y, \tilde{y})| \;\le\; C \left(\frac{1}{|y-\tilde{y}|^d} \;+\; \frac{\delta(y)}{|y-\tilde{y}|^{d+1}} \right). \tag{2.15}$$

Note that all the constants in the previous inequalities are independent of both n and a.

Thanks to (2.10), in order to show that v is bounded uniformly in n and a, it is enough to prove that v has the integral representation

$$v(y) \;=\; -\int_{\tilde{y}\cdot n = a} P_n(y, \tilde{y}) v_0(\tilde{y})\, d\tilde{y}. \tag{2.16}$$

Indeed, using the estimate on P_n in the integral leads to the fact that for all $y \cdot n > a$,

$$|v(y)| \;\le\; C\|v_0\|_{L^\infty} \int_{\mathbb{R}^{d-1}} \frac{|y\cdot n - a|}{(|\tilde{y}'| + |y\cdot n - a|)^d} d\tilde{y}' \;\le\; C\|v_0\|_{L^\infty}.$$

To prove (2.16), we introduce

$$w(y) \;:=\; -\int_{\{\tilde{y}\cdot n = a\}} P_n(y, \tilde{y})\, v_0(\tilde{y})\, d\tilde{y}.$$

Thanks to the bound (2.10), this matrix function is well-defined and uniformly bounded over $\{y \cdot n > a\}$. We claim that it satisfies (2.4). First, for any ψ^k smooth and compactly supported in $\{y\cdot n = a\}$, it is classical that

$$w^k(y) \;:=\; -\int_{\{\tilde{y}\cdot n = a\}} P_n(y, \tilde{y})\, \psi^k(\tilde{y}) v_0(\tilde{y})\, d\tilde{y},$$

satisfies

$$\begin{cases} -\nabla_y \cdot\, A(y)\nabla_y\, w^k = 0, & y.n > a \\ w^k(y) = -\psi^k(y) v_0(y), & y.n = a. \end{cases}$$

Then, we choose ψ^k such that $\psi^k = 1$ for $|y| \le k$, and let k go to infinity. On one hand, using (2.10) in the integral formula for w^k, we get that w^k converges locally uniformly to w over the closed half-plane $\{y \cdot n \ge a\}$. On the other hand, passing to the limit in the previous system, one obtains that w solves (2.4). Standard elliptic regularity arguments show that w is smooth. Moreover, as $w \in L^\infty$, all its derivatives are also bounded.

We now define $u := v - w$. It is a smooth function, with all derivatives bounded. It satisfies the homogeneous system

$$-\nabla_y \cdot\, A(y)\nabla_y\, u = 0, \quad y.n > a, \quad u = 0, \quad y.n = a.$$

We can prove that $u = 0$ by a duality argument. More precisely, let f be smooth and compactly supported in $\{y \cdot n > a\}$. As f is arbitrary, it is enough to show that $\int u \cdot f = 0$. Therefore, let us introduce U given by

$$U(\tilde{y}) \;=\; \int_{y\cdot n > a} G_n(y, \tilde{y})\, f(y) dy.$$

It satisfies

$$-\nabla_{\tilde{y}} \cdot\, A^t(\tilde{y}) \nabla_{\tilde{y}}\, U = f, \quad \tilde{y} \cdot n > a, \quad U = 0, \quad \tilde{y} \cdot n = a.$$

The idea is to write

$$\int_{y\cdot n > 0} u \cdot f \;=\; -\int_{\{y\cdot n > a\}} u \cdot \left(\nabla_y \cdot\, A^t(y) \nabla_y\, U\right) \tag{2.17}$$

$$\;=\; +\int_{\{y\cdot n > a\}} A(y)\nabla u \cdot \nabla_y\, U \;=\; 0, \tag{2.18}$$

where the last two equalities come from successive integration by parts. To make this reasoning rigorous, one must have some decay properties for the integrands. Precisely, it is enough to show that

$$I_1(R) \;:=\; \int_{\{y\cdot n > a,\, |y| = R\}} u \cdot \left(n \cdot\, A^t(y) \nabla_y\, U\right) \quad \text{and} \quad I_2(R) \;:=\; \int_{\{y\cdot n > a,\, |y| = R\}} A(y)\nabla u \cdot n\, U$$

go to zero as $R \to +\infty$. By the first part of proposition 3, we know that v is bounded. Moreover, by (2.10), w is also bounded, and so is u. Besides, by (2.14), we have that, for y far enough from the support of f, $|\nabla U(y)| \;\le\; C/|y|^d$. Combining these bounds yields: $I_1(R) \to 0$ as $R \to +\infty$. As regards I_2, we use the second part of proposition 3, which shows that $\delta(y)^m \nabla v(y)$ is bounded for all $m \in \mathbb{N}$. Moreover, using (2.15), we get that $\delta(y)\nabla w(y)$, and so $\delta(y)\nabla u(y)$, is bounded. Finally, by (2.9), we obtain that $|U(y)| \;\le\; C\delta(y)/|y|^d$. Hence, $I_2(R) \to 0$ as $R \to +\infty$. This concludes the proof of the proposition.

Besides this bound, we need some extra decay estimates on $V - v_*$ and their derivatives. For such estimates, the Diophantine assumption $n \in \mathcal{A}_\kappa$ plays a role, and the decay deteriorates as κ goes to zero. This is made quantitative in

Proposition 6 *The solution V of (2.7) satisfies: for all $\alpha \in \mathbb{N}^d,\ k \in \mathbb{N},\ m \in \mathbb{N}$,*

$$\left|\partial_\theta^\alpha \partial_t^k \left(V(\theta, t) - v_*\right)\right| \;\le\; \frac{C_{m,\alpha,k}}{\kappa} \Big(1 + \kappa(t-a)\Big)^{-m}, \qquad \textit{uniformly in } \theta. \tag{2.19}$$

The constant $C_{m,\alpha,k}$ depends linearly on the $W^{s,\infty}$ norm of v_0 for some $s = s(m, \alpha, k)$ large enough.

Proof. Throughout the sequel, $\kappa \le 1$. We leave to the reader to check *a posteriori* that there is no loss of generality in taking $a = 0$. To prove (2.19), it is enough to prove

$$|\partial_\theta^\alpha \partial_t^k \left(V(\theta, t) - v_*\right)| \;\le\; C_{m,\alpha,k}\, \kappa^{-1}, \qquad t \le 1, \tag{2.20}$$

$$|\partial_\theta^\alpha \partial_t^k \left(V(\theta, t) - v_*\right)| \;\le\; C_{m,\alpha,k}\, \kappa^{-1}\, (\kappa t)^{-m}\,, \qquad t \ge 1. \tag{2.21}$$

We remind the Sobolev bounds

$$\|N^t \nabla_\theta V(t)\|_{H^s}^2 + \|\partial_t V(t)\|_{H^s}^2 \;<\; C_s\,, \quad \forall s. \tag{2.22}$$

As $V = \int_0^t \partial_t V + V_0$, they yield a uniform bound on V and its derivatives for $t \leq 1$. Combined with the uniform bound on v_* coming from the previous proposition, it implies the first inequality (one can even take κ^0 instead of κ^{-1}).

To obtain the second inequality, that is the decay of $V - v_*$ as $t \to +\infty$, we go along the lines of [12], but keep track of the dependence on κ. If $n \in \mathcal{A}_\kappa$, then

$$\int_{\mathbb{T}^d} |N^t \nabla_\theta \tilde{W}|^2 \;\geq\; c\kappa^2 \, \|\tilde{W}\|^2_{H^{-l}(\mathbb{T}^d)} \tag{2.23}$$

for smooth enough $\tilde{W} = \tilde{W}(\theta)$ with zero average. Hence, the previous Sobolev bounds yield

$$\int_a^{+\infty} \kappa^2 \|\tilde{V}\|^2_{H^s(\mathbb{T}^d)} + \|\partial_t^k V\|^2_{H^s(\mathbb{T}^d)} \;\leq\; C(F, s, k) \;<\; +\infty, \tag{2.24}$$

for all $k \geq 1$ where we decompose

$$V(\theta, t) = \tilde{V}(\theta, t) \;+\; \bar{V}(t), \qquad \int_{\mathbb{T}^d} \tilde{V}\, d\theta = 0.$$

Proceeding exactly as in [12], we introduce

$$f(T) \;:=\; \int_{\mathbb{T}^d} \int_T^{+\infty} \Big(|N^t \nabla_\theta \, V|^2 + |\partial_t \, V|^2 \Big) \, dt \, d\theta,$$

and

$$W \;:=\; V - \int_{\mathbb{T}^d} V(\theta, T) d\theta.$$

After multiplication of (2.7) by W, integration from T to infinity, and a few manipulations, one ends up with

$$f(T) \;\leq\; C(-f'(T))^{1/2} \left(\int_{\mathbb{T}^d} |\tilde{V}(\theta, T)|^2 \, d\theta \right)^{1/2}. \tag{2.25}$$

To estimate $\int_{\mathbb{T}^d} |\tilde{V}(\theta, T)|^2 \, d\theta$, we use that

$$\left(\int_{\mathbb{T}^d} |\tilde{V}|^2 \, d\theta \right)^{1/2} \leq\; C \left(\|\tilde{V}\|_{H^{-l}(\mathbb{T}^d)} \right)^{1/p} \; \left(\|\tilde{V}\|_{H^{l/(p-1)}(\mathbb{T}^d)} \right)^{1-1/p}, \tag{2.26}$$

By (2.23), the first factor at the r.h.s. of (2.26) is controlled by $\left(\frac{-f'(T)}{\kappa^2}\right)^{1/2p}$. For the second factor, we use a simple interpolation inequality:

Lemma 7 *If $h \in H^1(\mathbb{R})$, then we have $\|h\|_\infty \leq C \|h\|_{L^2}^{1/2} \|h'\|_{L^2}^{1/2}$.*

Proof of the lemma. We write for each $t \in \mathbb{R}$ and $r > 0$,

$$|h(t)| = |h(t-r) + \int_{t-r}^t h'(s) ds| \leq |h(t-r)| + r^{1/2} \left(\int_{t-r}^t h'(s)^2 ds \right)^{1/2}.$$

Integrating in r between 0 and $R > 0$, we get $R\,|h(t)| \;\leq\; R^{1/2}\, \|h\|_{L^2} \;+\; R^{3/2}\, \|h'\|_{L^2}$. The result follows by optimizing in R.

Thanks to this lemma and the uniform Sobolev bounds on $\kappa\tilde{V}$ and $\partial_t V$, the second factor in the r.h.s. is controlled by $C/\kappa^{1/2-1/(2p)}$. Finally, (2.25) leads to

$$f(T) \,\leq\, C_p \left(-\frac{f'(T)}{\kappa}\right)^{\frac{p+1}{2p}} \tag{2.27}$$

and hence $f(T) \,\leq\, C_m(\kappa T)^{-m}$ for each $m > 1$, where $m = \frac{p+1}{p-1}$.

As regards higher order derivatives, it can be shown that the function

$$f_s(T) \,:=\, \sum_{|\alpha|+k\leq s} f_{\alpha,k}(T) \,=\, \sum_{|\alpha|+k\leq s} \int_{\mathbb{T}^d}\int_T^{+\infty} \Big(|N^t\,\nabla_\theta\,\partial_\theta^\alpha\,\partial_t^k\,V|^2 + |\partial_t\,\partial_\theta^\alpha\,\partial_t^k V|^2\Big)\,dt\,d\theta.$$

satisfies the same bound: $f_s(T) \,\leq\, C_{s,m}\,(\kappa T)^{-m}$. This is proved through induction on s. Let α, k such that $|\alpha| + k = s$. Applying $\partial_\theta^\alpha\,\partial_t^k$ to (2.7) leads to the equation

$$-\begin{pmatrix} N^t\,\nabla_\theta \\ \partial_t \end{pmatrix}\cdot\,\mathcal{B}(\theta,t)\begin{pmatrix} N^t\,\nabla_\theta \\ \partial_t \end{pmatrix}\partial_\theta^\alpha\,\partial_t^k V = \begin{pmatrix} N^t\,\nabla_\theta \\ \partial_t \end{pmatrix}\cdot G_{\alpha,k}, \tag{2.28}$$

where

$$|G_{\alpha,k}| \,\leq\, C_{\alpha,k} \sum_{|\beta|+l\,\leq\,s-1} \Big|\big(N^t\,\nabla_\theta,\partial_t\big)\partial_\theta^\beta\,\partial_t^l V\Big|.$$

If we multiply the equation (2.28) by

$$W_{\alpha,k} := \,\partial_\theta^\alpha\,\partial_t^k V - \int_{\mathbb{T}^d}\partial_\theta^\alpha\,\partial_t^k V(\theta,T)d\theta,$$

and integrate by parts, we get

$$\begin{aligned}
f_{\alpha,k}(T) \,&\leq\, C\left((-f'_{\alpha,k}(T))^{1/2} \,+\, \|G_{\alpha,k}(\cdot,T)\|_{L^2(\mathbb{T}^d)}\right)\,\|\partial_\theta^\alpha\,\partial_t^k\tilde{V}(\cdot,T)\|_{L^2(\mathbb{T}^d)} \\
&\quad + \|G_{\alpha,k}\|_{L^2(\mathbb{T}^d\times\{t>T\})}\,f_{\alpha,k}(T)^{1/2} \\
&\leq\, C_{\alpha,k}\left((-f'_s(T))^{1/2}\|\partial_\theta^\alpha\,\partial_t^k\tilde{V}(\theta,T)\|_{L^2(\mathbb{T}^d)} \,+\, \|G_{\alpha,k}\|^2_{L^2(\mathbb{T}^d\times\{t>T\})}\right) \\
&\leq\, C_{\alpha,k}\left((-f'_s(T))^{1/2}\|\partial_\theta^\alpha\,\partial_t^k\tilde{V}(\theta,T)\|_{L^2(\mathbb{T}^d)} \,+\, C_{s-1,m}(\kappa T)^{-m}\right)
\end{aligned}$$

using the induction assumption. Along the previous lines, we end up with

$$f_s(T) = \sum_{|\alpha|+k\leq s} f_{\alpha,k}(T) \,\leq\, C_{s,p}\left(-\frac{f'_s(T)}{\kappa}\right)^{\frac{p+1}{2p}} \,+\, C_{s-1,m}\,(\kappa T)^{-m}.$$

which gives the desired bound.

Using these bounds and the Sobolev embeddings (see also lemma 7) we get that

$$\begin{aligned}
\Big|\,\partial_\theta^\alpha\,\tilde{V}(\theta,t)\Big| \,&\leq C_{\alpha,m}\frac{1}{\sqrt{\kappa}}(\kappa t)^{-m} \\
\Big|\,\partial_\theta^\alpha\,\partial_t^{k+1}V(\theta,t)\Big| \,&\leq C_{\alpha,k,m}(\kappa t)^{-m}
\end{aligned} \tag{2.29}$$

for all $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$, $k \in \mathbb{N}$, uniformly in θ. As regards $\bar{V}(t)$, we use that

$$|\bar{V}(t+h) - \bar{V}(t)| \le \int_t^{t+h} \left| \frac{d}{dt} \bar{V} \right| \le C_m \int_t^{t+h} (1+\kappa s)^{-m-1} \, ds \le C \frac{1}{\kappa} (\kappa t)^{-m}.$$

This implies

$$|\bar{V}(t) - v_*| \le C \frac{1}{\kappa} (\kappa t)^{-m}. \tag{2.30}$$

The estimates (2.29), (2.30) imply (2.21). Moreover, it is clear from the proof that the constant $C_{m,\alpha,k}$ in (2.20)-(2.21) is independent of v_0 for $\|v_0\|_{W^{s,\infty}} \le 1$, $s = s(m,\alpha,k)$ large enough. As the map $v_0 \mapsto V$ is linear, this shows that the constant $C_{m,\alpha,k}$ can be chosen linear in $\|v_0\|_{W^{s,\infty}}$, s large enough. This concludes the proof of the proposition.

Thanks to the previous propositions, we have at hand refined estimates on v and $v - v_*$. Such estimates will be crucial in our homogenization proof for smooth domains. Indeed, our proof will rely on the construction of accurate expansions of u^ε, in which correctors like v will appear as leading terms. Still, for the next terms of the expansion, other boundary layer correctors will be needed. They will satisfy the same type of equations as v, but with additional source terms. Therefore, we need to extend the estimates of the previous propositions to this slightly larger setting.

Instead of (2.7), we consider the system

$$\left\{ \begin{aligned} -\begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \mathcal{B}(\theta,t) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} U(\theta,t) = F(\theta,t), \quad t > a \\ U(\theta,t) = 0, \quad t = a, \end{aligned} \right. \tag{2.31}$$

set on $\mathbb{T}^d \times \{t > a\}$. We assume that the source term $F = F(\theta,t)$ is smooth and in the Schwartz class with respect to t. As explained in paper [12], the well-posedness and asymptotic properties of (2.7) extend to the system (2.31). In particular, there is a unique smooth solution $U = U(\theta,t)$, with the Sobolev bounds

$$\begin{aligned} &\|N^t \nabla_\theta U\|^2_{H^s(\mathbb{T}^d \times \{t>a\})} + \|\partial_t U\|^2_{H^s(\mathbb{T}^d \times \{t>a\})} \\ &\qquad < C \left(\|(t-a)\, F\|^2_{H^s(\mathbb{T}^d \times \{t>a\})} + \|F\|^2_{H^s(\mathbb{T}^d \times \{t>a\})} \right). \end{aligned} \tag{2.32}$$

Moreover, there is a constant u_* such that $U - u_*$ is in the Schwartz class with respect to t.

Like the resolution of (2.7) provides a solution to (2.4), the resolution of (2.31) provides a solution to

$$\left\{ \begin{aligned} -\nabla_y \cdot A(y) \nabla_y u(y) = f(y), \quad y.n > a \\ u(y) = 0, \quad y.n = a. \end{aligned} \right. \tag{2.33}$$

As before, u and U, resp. f and F are connected through

$$u(y = Mz) = \mathbf{u}(z) = U(Nz', z_d), \quad f(y = Mz) = \mathbf{f}(z) = F(Nz', z_d).$$

We want to derive some bounds on u and $U - u_*$, in terms of f and F. We state

Proposition 8 *Let $\mu \geq 0$, $\nu \geq 1$, $m_0 \geq 4$. Assume that for all $m \geq m_0$, $\alpha \in \mathbb{N}^d$ and for all $k \in \mathbb{N}$,*

$$\begin{aligned} \left|\partial_y^\alpha f(y)\right| &\leq C_\alpha^f\, \kappa^{-\mu} && \text{uniformly in } y, \\ \left|\partial_\theta^\alpha \partial_t^k F(\theta,t)\right| &\leq C_{m,\alpha,k}^F \left(\kappa^\nu(1+t-a)\right)^{-m} && \text{uniformly in } \theta. \end{aligned} \tag{2.34}$$

Then, for all $\delta > 0$, there exists $m_1 = m_1(m_0,\mu,\nu,\delta)$ such that: for all $m \geq m_1$, $\alpha \in \mathbb{N}^d$ and for all $k \in \mathbb{N}$

$$\begin{aligned} \left|\partial_y^\alpha u(y)\right| &\leq C_{\alpha,\delta}^u\, \kappa^{-\mu}\, \kappa^{-2\nu} && \text{uniformly in } y, \\ \left|\partial_\theta^\alpha \partial_t^k (U-u_*)(\theta,t)\right| &\leq C_{m,\alpha,k,\delta}^U \left(\kappa^{\nu+\delta}\,(1+t-a)\right)^{-m} && \text{uniformly in } \theta. \end{aligned} \tag{2.35}$$

Proof. As for proposition 6, we can assume $a = 0$. We can restrict to the case $\mu = 0$ as well. Indeed, suppose that the result holds in such a case, and take $\mu > 0$. If (2.34) is satisfied, it implies trivially that

$$\left|\partial_y^\alpha f(y)\right| \leq C_\alpha^f\, \kappa^{-\mu}, \quad \left|\partial_\theta^\alpha \partial_t^k F(\theta,t)\right| \leq C_{m,\alpha,k}^F\, \kappa^{-\mu} \left(\kappa^\nu(1+t)\right)^{-m}.$$

By linearity of the equations, and using the result with $\mu = 0$, and $\delta/2$ instead of δ, we get (for $m \geq m_1(m_0,0,\nu,\delta/2)$)

$$\left|\partial_y^\alpha u(y)\right| \leq C_{\alpha,\delta/2}^u\, \kappa^{-\mu}\, \kappa^{-2\nu}, \quad \left|\partial_\theta^\alpha \partial_t^k (U-u_*)(\theta,t)\right| \leq C_{m,\alpha,k,\delta/2}^U\, \kappa^{-\mu} \left(\kappa^{\nu+\delta/2}\,(1+t)\right)^{-m}.$$

The last inequality reads

$$\left|\partial_\theta^\alpha \partial_t^k (U-u_*)(\theta,t)\right| \leq C_{m,\alpha,k,\delta/2}^U \left(\kappa^{\nu+\delta/2+\mu/m}\,(1+t)\right)^{-m} \leq C_{m,\alpha,k,\delta/2}^U \left(\kappa^{\nu+\delta}\,(1+t)\right)^{-m}$$

for $m \geq \max(m_1, 2\mu/\delta)$, which proves our claim. From now on, $\mu = 0$.

We start with the inequality on u. As u satisfies an elliptic system, it is enough to treat the case $\alpha = 0$: standard regularity arguments provide the bound for higher derivatives. By a combination of the two inequalities in (2.34), we have

$$\left|f(y)\right| \leq \frac{C_m}{1+(\kappa^\nu t)^m}, \ \text{ uniformly in } y',$$

for $m \geq m_0$. We use here the notation $y = y' + t\,n$, with $y' \cdot n = 0$ and $t \geq 0$. We rescale system (2.33), introducing $\tilde{y} = \kappa^\nu y$, $\tilde{u}(\tilde{y}) = u(y)$ and so on. Dropping the tildes, we get

$$\left\{ \begin{aligned} -\nabla_y \cdot\, A(\cdot/\kappa^\nu)\nabla_y\, u(y) &= \frac{1}{\kappa^{2\nu}} f(y), \quad y.n > 0 \\ u(y) &= 0, \quad y.n = 0. \end{aligned} \right. \tag{2.36}$$

where this new f satisfies in particular (for some C depending on m_0)

$$\left|f(y)\right| \leq \frac{C}{1+t^4}. \tag{2.37}$$

We must show that $\kappa^{2\nu} u$ is uniformly bounded. We use temporarily the notation ε instead of κ^ν. Let $G^\varepsilon = G^\varepsilon(y_1, y_2)$ the Green function associated to the operator $-\nabla_y \cdot A(\cdot/\varepsilon)\nabla_y$ in the domain $y \cdot n > 0$. Then,

$$\varepsilon^2 u(y_1) \;=\; \int_{y_2 \cdot n > 0} G^\varepsilon(y_1, y_2)\, f(y_2)\, dy_2.$$

This representation formula can be established similarly to what we did for (2.16).

As for proposition 5, we then rely on estimates proved by Avellaneda and Lin. Namely, we have the following inequalities:

$$|G^\varepsilon(y_1, y_2)| \;\le\; \frac{C_d}{|y_1 - y_2|^{d-2}} \;\text{ for }\; d > 2, \quad |G^\varepsilon(y_1, y_2)| \;\le\; C_d\,(|\ln|y_1 - y_2|| + 1) \;\text{ for }\; d = 2, \tag{2.38}$$

cf [5, Theorem 13]), and

$$|G^\varepsilon(y_1, y_2)| \;\le\; C_d \frac{t_1\, t_2}{|y_1 - y_2|^d} \;\text{ for }\; d \ge 2, \quad t_1 \;:=\; y_1 \cdot n, \quad t_2 \;:=\; y_2 \cdot n, \tag{2.39}$$

cf [5, estimate (4.6)]. As mentioned earlier, the inequalities in [5] are given in the case of a bounded domain, but they extend straightforwardly to the half-space case.

To apply these estimates to the Green formula, we decompose the integral into

$$\int_{y_2 \cdot n > 0} G^\varepsilon(y_1, y_2)\, f(y_2)\, dy_2 \;=\; \int_{\substack{|y_1 - y_2| < 1, \\ y_2 \cdot n > 0}} G^\varepsilon(y_1, y_2)\, f(y_2)\, dy_2 \;+\; \int_{\substack{|y_1 - y_2| > 1, \\ y_2 . n > 0}} G^\varepsilon(y_1, y_2)\, f(y_2)\, dy_2$$

Then, combining (2.37) and (2.38) yields a uniform (in ε and y) bound for the first term. As regards the second term, we write

$$\left| \int_{\substack{|y_1 - y_2| > 1, \\ y_2 \cdot n > 0}} G^\varepsilon(y_1, y_2)\, f(y_2)\, dy_2 \right| \;\le\; C \int_{y_2 \cdot n > 0} \frac{t_1\, t_2}{|y_1 - y_2|^d} \frac{1}{1 + t_2^4} dy_2 \tag{2.40}$$

$$\le\; C' \int_{\mathbb{R}_+} \frac{t_1\, t_2}{|t_1 - t_2| + 1} \frac{1}{1 + t_2^4} dt_2. \tag{2.41}$$

The last integral comes from integration with respect to the tangential variable. It is an easy verification that this last integral is bounded uniformly in ε and y. For the sake of brevity, we leave it to the reader. This concludes the proof of the first inequality in (2.35).

The estimate of $U - u_*$ is established much like the estimate of $V - v_*$ in proposition 6, and we will only sketch the proof. As for V, the estimate for $t \le 1$ comes from the global Sobolev estimate (2.32) and the uniform bound on u (so on u_*). Note that, using the bound for F in (2.32), we obtain

$$\|N^t \nabla_\theta U\|^2_{H^s} + \|\partial_t U\|^2_{H^s} \;<\; C_s\, \kappa^{-2 m_0 \nu} \quad \forall s \tag{2.42}$$

which corresponds to a fixed loss in κ.

For $t \ge 1$, we introduce again the functions

$$f_s(T) \;:=\; \sum_{|\alpha| + k \le s} f_{\alpha,k}(T) \;:=\; \sum_{|\alpha| + k \le s} \int_{\mathbb{T}^d} \int_T^{+\infty} \Big(|N^t\, \nabla_\theta\, \partial_\theta^\alpha\, \partial_t^k\, U|^2 + |\partial_t\, \partial_\theta^\alpha\, \partial_t^k V|^2 \Big)\, dt\, d\theta.$$

We obtain, along the same lines as before: for all $p > 1$,

$$
\begin{aligned}
f_0(T) \;\le\; C_p \left(\|N^t \nabla_\theta U\|_{H^{l/(p-1)}} + \|\partial_t U\|_{H^{l/(p-1)}} \right)^{1-\frac{1}{p}} \left(- \frac{f_0'(T)}{\kappa} \right)^{\frac{p+1}{2p}} \\
+ \left| \int_{\mathbb{T}^d} \int_T^{+\infty} F(\theta,t) \left(U(\theta,t) - \int_{\mathbb{T}^d} U(\cdot, T) \right) d\theta dt \right|. \quad (2.43)
\end{aligned}
$$

An integration by parts provides

$$
\begin{aligned}
\int_{\mathbb{T}^d} \int_T^{+\infty} F(\theta,t) \left(U(\theta,t) - \int_{\mathbb{T}^d} U(\cdot,T) \right) d\theta dt = - \int_{\mathbb{T}^d} \int_T^{+\infty} \mathcal{F}(\theta,t) \partial_t U(\theta,t) d\theta dt \\
- \int_{\mathbb{T}^d} \mathcal{F}(\theta,T) \left(U(\theta,T) - \int_{\mathbb{T}^d} U(\cdot,T) \right) d\theta
\end{aligned}
$$

where $\mathcal{F}(\theta,t) = -\int_t^{+\infty} F(\theta,s) ds$. It follows that

$$
\begin{aligned}
& \left| \int_{\mathbb{T}^d} \int_T^{+\infty} F(\theta,t) \left(U(\theta,t) - \int_{\mathbb{T}^d} U(\cdot,T) \right) d\theta dt \right| \\
& \quad \le \; \|\mathcal{F}\|_{L^2(\mathbb{T}^d \times \{t > T\})} \|\partial_t U\|_{L^2(\mathbb{T}^d \times \{t>T\})} \; + \; \|\mathcal{F}(\cdot,T)\|_{H^l(\mathbb{T}^d)} \, \kappa^{-1} \|N^t \nabla_\theta U(\cdot,T)\|_{L^2(\mathbb{T}^d)}
\end{aligned}
$$

using (2.23) for the last term. From there, combining (2.34), (2.42) and (2.43), we obtain easily that

$$
f_0(T) \;\le\; C_{m,p}\, \kappa^{-M} \left(\left(- \frac{f_0'(T)}{\kappa} \right)^{\frac{p+1}{2p}} \; + \; (\kappa^\nu \, T)^{-m} \right)
$$

for all $m \ge m_0$ and some fixed M depending only on m_0. Now, given some $\delta > 0$, if $m \ge \max(m_0, M/\delta)$ and $\frac{p+1}{p-1} \ge \max(m_0, M/\delta)$, then

$$
f_0(T) \;\le\; C_{m,p} \left(\left(- \frac{f_0'(T)}{\kappa^{\nu+\delta}} \right)^{\frac{p+1}{2p}} \; + \; \left(\kappa^{\nu+\delta} \, T \right)^{-m} \right).
$$

Hence $f_0(T) \le C_m (\kappa^{\nu+\delta}\, T)^{-m}$ for m large enough. Similar bounds hold for f_s, $s \ge 1$ which are obtained, as before, recursively. For the sake of brevity, we leave the details to the reader. This concludes the proof of the proposition.

We quote that, by the linearity of the map $(F, f) \mapsto (U, u)$, one can be more specific about the constants $C^u_{\alpha,\delta}$ and $C^U_{m,\alpha,k,\delta}$ in (2.35): one has

$$
C^u_{\alpha,\delta} + C^U_{m,\alpha,k,\delta} \;\le\; \mathcal{C}_{m,\alpha,k,\delta} \sum_{(m',\alpha',k') \in I_{m,\alpha,k}} \left(C^f_{\alpha'} \; + \; C^F_{m',\alpha',k'} \right) \tag{2.44}
$$

where $\mathcal{C}_{m,\alpha,k} > 0$ does not depend on f, F, and $I_{m,\alpha,k}$ is a finite subset of indices also independent of f, F.

Corollary 1 *The function $n \mapsto v_*(n)$ is Lipschitz on $\mathcal{A}_\kappa$, with a Lipschitz constant which is $O(\kappa^{-2})$ as κ goes to zero.*

Proof of the corollary. Let n_1 and n_2 be in $\mathcal{A}_\kappa$. We wish to show that

$$|v_*(n_1) - v_*(n_2)| \;\le\; \frac{C}{\kappa^2}\,|n_1 - n_2|.$$

For $i = 1, 2$, we introduce the solution V_{n_i} of (see previous section for notations):

$$\begin{cases} -\begin{pmatrix} N_i^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\; B(\theta + t n_i) \begin{pmatrix} N_i^t \nabla_\theta \\ \partial_t \end{pmatrix} V_{n_i} = 0, \quad t > 0, \\ \hfill V_{n_i}(\theta, t = 0) = \chi^\gamma(\theta), \quad t = 0. \end{cases} \tag{2.45}$$

We set $V := V_{n_1} - V_{n_2}$, $N := N_1 - N_2$, and so on. We have

$$\begin{cases} -\begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\; B(\theta + t n_1) \begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} V = F, \quad t > 0, \\ \hfill V(\theta, t = 0) = 0, \quad t = 0. \end{cases} \tag{2.46}$$

where

$$\begin{aligned} F \;&=\; \left(-\begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\; B(\theta + t n_1) \begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} \;+\; \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\; B(\theta + t n_2) \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} \right) V_{n_2} \\ &=\; -\begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\; B(\theta + t n_1) \begin{pmatrix} N^t \nabla_\theta \\ \partial_t \end{pmatrix} V_{n_2} \\ &\quad +\; \left(\begin{pmatrix} -N^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\; B(\theta + t n_1) \;+\; \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} \cdot\; (B(\theta + t n_2) - B(\theta + t n_1)) \right) \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} V_{n_2}. \end{aligned}$$

We also introduce the corresponding

$$v_{n_i}(y = M_1 z) \;=\; \mathbf{v_{n_i}}(z) \;=\; V_{n_i}(N_1 z', z_d), \quad i = 1, 2,$$

$v(y)$, $f(y)$. By the estimates of propositions 5 and 6, one has the following bounds:

$$\begin{aligned} \left|\partial_y^\alpha f(y)\right| \;&\le\; C_{m,\alpha} |n_1 - n_2| && \text{uniformly in } y, \\ \left|\partial_\theta^\alpha \partial_t^k F(\theta, t)\right| \;&\le\; C_{m,\alpha,k,\delta} |n_1 - n_2| \left(\kappa^{1+\delta}(1 + t - a)\right)^{-m} && \text{uniformly in } \theta \end{aligned}$$

for all $\delta > 0$ and m such that $\delta m > 1$. Applying our last proposition (see also (2.44)), we get that

$$|v(y)| \;=\; |v_{n_1}(y) - v_{n_2}(y)| \;\le\; \frac{C_\delta}{\kappa^{2+2\delta}}\,|n_1 - n_2|$$

uniformly in y, for all $\delta > 0$. Actually, one can improve a little this inequality and take $\delta = 0$. Indeed, the source term F can be split into

$$F \;=\; F' \;+\; F'' \;:=\; -\begin{pmatrix} N_1^t \nabla_\theta \\ \partial_t \end{pmatrix} G \;+\; L(n_1 - n_2, \theta, t, \partial_\theta, \partial_t) \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} V_{n_2}$$

where G satisfies:

$$|\partial_\theta^\alpha \partial_t^k G| \;\le\; C_{m,\alpha,k,\delta}\, |n_1 - n_2| \left(\kappa^{1+\delta} t\right)^{-m}, \quad \forall \delta > 0, \ \forall t > a = 0,$$

where $L(n, \theta, t, \partial_\theta, \partial_t)$ is a first-order smooth matricial operator, whereas

$$\left|\partial_\theta^\alpha \partial_t^k \begin{pmatrix} N_2^t \nabla_\theta \\ \partial_t \end{pmatrix} V_{n_2}\right| \;\le\; C_{m,\alpha,k}\; (\kappa t)^{-m}.$$

We insist that this last inequality involves only κ: one evaluates $(N_2^t\nabla_\theta, \partial_t)V_{n_2}$, so that additional estimates of type (2.29)-(2.30) (responsible for an additional loss in κ) are not needed.

This special form of the source term F allows to refine the estimate on $v = v_{n_1} - v_{n_2}$. One can write $v = v' + v''$, with

$$\begin{cases} -\nabla_y \cdot A(\cdot)\nabla_y v'(y) = f'(y) = \nabla \cdot g(y), & y.n > 0 \\ v'(y) = 0, & y.n = 0. \end{cases}$$

and

$$\begin{cases} -\nabla_y \cdot A(\cdot)\nabla_y v''(y) = f''(y), & y.n > 0 \\ v''(y) = 0, & y.n = 0. \end{cases}$$

We then proceed as in the proof of Proposition 8: we have the representation formula:

$$\begin{aligned} \kappa^{1+\delta} v'(\kappa^{1+\delta} y_1) &= -\int_{y_2\cdot n>0} \nabla_{y_2} G^{\kappa^{1+\delta}}(y_1, y_2)\, g(\kappa^{1+\delta} y_2) dy_2 \\ \kappa^2 v''(\kappa y_1) &= -\int_{y_2\cdot n>0} G^{\kappa}(y_1, y_2)\, f''(\kappa y_2) dy_2, \end{aligned}$$

where $G^\varepsilon(y_1, y_2)$ is as before the Green function associated to the operator $-\nabla_y \cdot A(\cdot/\varepsilon)\nabla_y$ in the domain $y \cdot n > 0$. Moreover, the source terms satisfy

$$|g(\kappa^{1+\delta} y_2)| \,+\, |f''(\kappa y_2)| \,\le\, \frac{C_{\delta,m}|n_1 - n_2|}{t_2^m}.$$

Like in the proof of Proposition (8), we can combine these estimates with the bounds established earlier on the Green function, to obtain

$$|v'(y)| \,\le\, \frac{C_\delta\, |n_1 - n_2|}{\kappa^{1+\delta}}, \quad |v''(y)| \,\le\, \frac{C\, |n_1 - n_2|}{\kappa^2}.$$

The result follows letting y go to infinity transversely to the boundary. This concludes the proof of the corollary.

3 The disk

We turn in this section to the core of the paper, that is the homogenization of system (1.1)-(1.2) for smooth domains Ω. To get rid of confusing technicalities, we will first consider the case of a unit disk:

$$d = 2, \quad \Omega \,=\, \{x,\ |x| < 1\},$$

with boundary data φ that factors into

$$\varphi(x, y) \,=\, v_0(y)\, \varphi_0(x)$$

for some smooth v_0 on $\mathbb{T}^d$ with values in $M_N(\mathbb{R})$ and some smooth φ_0 on $\partial\Omega$ with values in $\mathbb{R}^N$. The extension to the general framework of theorem 1 will be discussed in section 4. Let us stress that this extension, although a bit heavy to write down, contains no mathematical difficulties. Thus, all ideas are already contained in the simplified configuration studied here.

For all $x \in \mathbb{S}^1$, we denote by $n(x) = -x$ the unit inward normal vector. If $x \in \cup_{\kappa>0}\mathcal{A}_\kappa$, $n(x)$ satisfies the small divisor assumption (1.16). Thus, we can use the results of section 2: the boundary layer system (2.4) with $n = n(x)$ and with boundary data $v_0 \in M_N(\mathbb{R})$ has a solution $v = v(y) \in M_N(\mathbb{R})$ that converges (transversely to the boundary) to some $v_* = v_*(n) \in M_N(\mathbb{R})$. We set:

$$\varphi_*(x) \;:=\; v_*(n(x))\,\varphi_0(x).$$

From the beginning of section 2, we know that $\cup_{\kappa>0}\mathcal{A}_\kappa$ has full measure, so that φ_* is defined almost everywhere on the circle. Moreover, $\varphi_* \in L^\infty(\mathbb{S}^1;\, \mathbb{R}^N)$: corollary 1 implies its measurability and proposition 5 yields a uniform bound.

Theorem 1 is a direct consequence of

Proposition 9 *Let u^0 be the solution of system* (1.15)*, with the boundary data φ_* defined above. Then,*

$$\|u^\varepsilon - u^0\|_{L^2} \;=\; O(\varepsilon^\alpha),$$

as ε goes to zero, for all $\alpha < 1/11$.

We quote that, because φ_* has L^∞ regularity, the limit field u^0 is in $L^p(\Omega)$ for all $1 \le p \le \infty$.

We can also prove (using the next interpolation argument) that u^0 belongs to $W^{s_p,p}(\Omega)$ for some $s_p > 0$ for all $1 < p < \infty$. But this will not be used in the convergence proof.

The rest of the section is devoted to the proof of this proposition. We first split the problem in two: we write $u^\varepsilon = u^\varepsilon_{reg} + u^\varepsilon_{bl}$, with

$$\begin{cases} -\nabla \cdot A\left(\dfrac{x}{\varepsilon}\right)\nabla u^\varepsilon_{reg} = 0, & x \in \Omega \subset \mathbb{R}^d, \\ \qquad\qquad u^\varepsilon_{reg} = \varphi_*, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\nabla \cdot A\left(\dfrac{x}{\varepsilon}\right)\nabla u^\varepsilon_{bl} = 0, & x \in \Omega \subset \mathbb{R}^d, \\ u^\varepsilon_{bl} = \varphi(x, x/\varepsilon) - \varphi_*, & x \in \partial\Omega. \end{cases}$$

We will bound $\|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)}$ and $\|u^\varepsilon_{bl}\|_{L^2(\Omega)}$ separately. We stress that the difficult part is the bound on u^ε_{bl}. It is where the boundary layer analysis is involved, notably the sets $\mathcal{A}_\kappa$. The treatment of u^ε_{reg} enters the classical framework discussed in the introduction, and is essentially contained in previous studies.

Nevertheless, there is a little technical difficulty for this problem, namely the lack of regularity of φ_*. Indeed, the classical estimates on $u^\varepsilon_{reg} - u^0$ rely on expansions that require differentiating u^0. As u^0 is only in L^∞, we will need some regularizing sequences, indexed by another parameter δ. The choice of these sequences will be specified in the next subsection. Remark that we have now three small parameters: ε, κ, and δ. Special attention will be paid to the way our estimates depend on them. The rate $\varepsilon^{1/11}$ will follow from optimizing in κ, δ and α which will be defined later.

3.1 Classical approximation

We derive here estimates on $u^\varepsilon_{reg} - u^0$. We take care of the smoothness problem as follows. By corollary 1, φ_* is Lipschitz over A_κ, with Lipschitz constant less than C/κ^2. By standard results (see [8], $\varphi_*|_{\mathcal{A}_\kappa}$ admits a Lipschitz extension, uniformly bounded in κ (because φ_* is), and with the same Lipschitz constant.

Let us call this extension φ^κ_*. With obvious notations, we associate to this boundary data the fields $u^{\varepsilon,\kappa}_{reg}$ and $u^{0,\kappa}$.

Now, we notice that

$$\varphi^\kappa_* - \varphi_* \; = \; \varphi^\kappa_* \mathbf{1}_{\mathcal{A}^c_\kappa} \; - \; \varphi_* \mathbf{1}_{\mathcal{A}^c_\kappa}.$$

So, from the estimate (2.2), we have

$$\|\varphi^\kappa_* - \varphi_*\|_{L^2(\partial\Omega)} \; \le \; C\,\kappa^{1/2}.$$

Thus, using the results of [5], we get

$$\|u^{\varepsilon,\kappa}_{reg} - u^\varepsilon_{reg}\|_{L^2(\Omega)} \; \le \; C\,\kappa^{1/2}, \quad \|u^{0,\kappa} - u^0\|_{L^2(\Omega)} \; \le \; C\,\kappa^{1/2}.$$

It remains to estimate $u^{\varepsilon,\kappa}_{reg} - u^{0,\kappa}$. We introduce a sequence of smooth fields $\varphi^{\kappa,\rho}_*$ such that $\varphi^{\kappa,\rho}_* \; \rightarrow \; \varphi^\kappa_*$ in $L^2(\partial\Omega)$, as $\rho \rightarrow 0$. More precisely, we chose it in such a way that

$$\|\varphi^{\kappa,\rho}_* - \varphi^\kappa_*\|_{L^2(\partial\Omega)} \; \le \; C\,\|\nabla\varphi^\kappa_*\|_{L^\infty}\,\rho \; \le \; C'\,\frac{\rho}{\kappa^2}, \quad \|\varphi^{\kappa,\rho}_*\|_{H^s} \; \le \; C_s\,\rho^{1-s}\|\nabla\varphi^\kappa_*\|_{L^\infty} \; \le \; C'_s\,\frac{\rho^{1-s}}{\kappa^2},$$

for all $s \ge 0$. For instance, one can use a partition of unity to come down to local charts, and in each chart, use a convolution by an approximation of unity with support in $(-\rho,\rho)$. Now, as $\varphi^{\kappa,\rho}_*$ is smooth, we can apply the standard homogenization results. We have ([2]):

$$\|u^{\varepsilon,\kappa,\rho}_{reg} - u^{0,\kappa,\rho}\|_{L^2(\Omega)} \; \le \; C\,\|u^{0,\kappa,\rho}\|_{H^2(\Omega)}\,\varepsilon \; \le \; C'\,\|\varphi^{\kappa,\rho}_*\|_{H^{3/2}(\partial\Omega)}\,\varepsilon \; \le \; C''\,\frac{\varepsilon}{\rho^{1/2}\kappa^2}.$$

Moreover, using again the results of Avellaneda and Lin:

$$\|u^{\varepsilon,\kappa,\rho}_{reg} - u^{\varepsilon,\kappa}_{reg}\|_{L^2(\Omega)} \; \le \; C\,\frac{\rho}{\kappa^2}, \quad \|u^{0,\kappa,\rho} - u^{0,\kappa}\|_{L^2(\Omega)} \; \le \; C\,\frac{\rho}{\kappa^2}$$

Gathering all previous bounds, we end up with

$$\|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)} \; \le \; C\,\left(\kappa^{1/2} \; + \; \frac{\varepsilon}{\rho^{1/2}\kappa^2} \; + \; \frac{\rho}{\kappa^2}\right) \tag{3.1}$$

3.2 Boundary layer approximation

We shall construct in this paragraph an approximation of u^ε_{bl}, of boundary layer type. To construct the boundary layer, we will divide the circle into small arcs, each of length ε^α, with $1 > \alpha > 0$ to be determined, and we will approximate each arc by a segment so as to use the half-space analysis.

We first parametrize the boundary of $\partial\Omega$ by $\theta \rightarrow e^{i\theta}$ with $\theta \in [0,2\pi]$. We divide $[0,2\pi]$ into $Q = [\frac{1}{\varepsilon^\alpha}]$ small intervals, namely

$$[0,2\pi] = \bigcup_{q=1}^{Q} I_q, \quad I_q = [2\pi\frac{q-1}{Q}, 2\pi\frac{q}{Q}].$$

We also denote $\tilde{I}_q$ the interval which has the same center as I_q and half the size: $\tilde{I}_q = [2\pi\frac{q-3/4}{Q}, 2\pi\frac{q-1/4}{Q}]$. We also denote θ_q the center of I_q, namely $\theta_q = 2\pi\frac{q-1/2}{Q}$.

Let $\Psi = \Psi(\xi)$ be a smooth function with compact support satisfying:

i) $\Psi = 1$ for $|\xi| < \pi/2$.

ii) $\Psi = 0$ for $|\xi| > 2\pi$.

iii) $\sum_{i=1}^{Q} \Psi(Q(\theta - \theta_q)) = 1$.

It induces a partition of unity in the vicinity of the circle: for $x = (r\cos\theta, r\sin\theta)$ in an ε^α neighborhood of the circle,

$$1 = \sum_{i=1}^{Q} \phi_q(x) := \sum_{i=1}^{Q} \Psi\left(Q(\theta-\theta_q)\right)\, \Psi(Q(r-1)).$$

Clearly, we can write $\phi_q(x) = \psi\left(\frac{x-x_q}{\varepsilon^\alpha}\right)$ where $x_q = e^{i\theta_q}$ and *all derivatives of* ψ *are uniformly bounded*. We now divide the set $\{1,\dots,Q\}$ into two sets

$$\mathbb{Q}^g = \{q,\, 1 \le q \le Q,\, \tilde{I}_q \cap \mathcal{A}_\kappa \ne \emptyset\}. \tag{3.2}$$

$$\mathbb{Q}^b = \{q,\, 1 \le q \le Q,\, \tilde{I}_q \cap \mathcal{A}_\kappa = \emptyset\}. \tag{3.3}$$

It is clear that the cardinality of $\mathbb{Q}^b$ is bounded by $\frac{C\kappa}{\varepsilon^\alpha}$ for some constant C. We write

$$u^\varepsilon_{bl} = u^{\varepsilon,g} + u^{\varepsilon,b} := \sum_{q\in\mathbb{Q}^g} u^\varepsilon_q + \sum_{q\in\mathbb{Q}^b} u^\varepsilon_q$$

where u^ε_q satisfies

$$\begin{cases} -\nabla\cdot A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon_q = 0, & x\in\Omega,\\ u^\varepsilon_q = (\varphi(x,x/\varepsilon) - \varphi_*(x))\,\phi_q(x), & x\in\partial\Omega.\end{cases} \tag{3.4}$$

The boundary data for u^ε_q is localized in a small arc around x_q.

For $u^{\varepsilon,b}$, we use [5] and the bound on the cardinality of $\mathbb{Q}^b$ to get

$$\|u^{\varepsilon,b}\|_{L^2(\Omega)} \le \|u^{\varepsilon,b}\|_{L^2(\partial\Omega)} \le C\kappa^{1/2}. \tag{3.5}$$

It remains to handle $u^{\varepsilon,g}$, that is u^ε_q for $q\in\mathbb{Q}^g$. First, we pick for such q some n_q with $-n_q \in \tilde{I}_q$. Then, we give the following ansatz:

$$u^{\varepsilon,app}_q = \sum_{\substack{k,l\ge 0\\ k(1-\alpha)+l\le K_0}} \varepsilon^{k(1-\alpha)+l} v^{k,l}_q\left(\frac{x}{\varepsilon}, \frac{x-x_q}{\varepsilon^\alpha}, x\right). \tag{3.6}$$

For each k,l, the boundary layer corrector $v^{k,l}_q$ will be a function of (y,Y,x), with compact support in Y, and decaying fast to zero as y goes to infinity along n_q. The constant K_0 will be fixed in due course. Actually, to be more precise the boundary profile $v^{k,l}_q$ also depends on ε through the boundary condition (see for instance that the boundary data is taken at the hyperplane $y.n_q = -1/\varepsilon$ in (3.7)). However, the bounds will be uniform in ε and we elected not to keep an ε in the notation $v_q^{k,l}$.

Let us detail the construction of the first correctors, that is for $k + l \leq 1$. The higher order terms are handled similarly. Remember that

$$\varphi_*(x) \;:=\; v_*(n(x))\,\varphi_0(x) \qquad a.e.$$

We take $v_q^{0,0}$ to satisfy

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y\, v_q^{0,0}(y,Y,x) = 0, & y.n_q > -1/\varepsilon \\ v^{0,0}(y,Y,x) = \Big(\varphi(x,y) - v_*(n_q)\,\varphi_0(x)\Big)\,\psi(Y), & y.n_q = -1/\varepsilon. \end{cases} \tag{3.7}$$

Of course, the idea is that $v_q^{0,0}(x/\varepsilon, x/\varepsilon^\alpha, x)$ should cancel the trace of u_q^ε at the boundary. This is still not exactly so: first, to be able to construct the corrector, we replace the circle $|\varepsilon y| = |x| = 1$ by the flat line $y.n_q = -\frac{1}{\varepsilon}$ (recall that n_q points inward). Second, we replace $v_*(n(x))$ by $v_*(n_q)$. However, we will show in the next subsection that these approximations result in small errors, and do not affect the homogenization.

Note that $v_q^{0,0}$ has separate variables, in the sense that it reads

$$v_q^{0,0}(y,Y,x) \;=\; w_q^{0,0}(y)\,\varphi_0(x)\,\psi(Y) \tag{3.8}$$

where $w_q^{0,0} \in M_N(\mathbb{R})$ satisfies (2.4) with $n = n_q$, $a = -1/\varepsilon$, and boundary data $v_0 - v_*(n_q)$. By definition of v_*, it goes to zero as y goes to infinity along n_q.

The $v_q^{1,0}$ term is chosen as a solution of (we drop the lower-script q for easier reading):

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y\, v^{1,0}(y,Y,x) = \nabla_y \cdot A(y)\nabla_Y\, v^{0,0}(y,Y,x) & \\ \qquad\qquad + \nabla_Y \cdot A(y)\nabla_y\, v^{0,0}(y,Y,x), & y.n_q > -\dfrac{1}{\varepsilon}, \\ v^{1,0}(y,Y,x) = v_{bd}^{1,0}(Y,x), & y.n_q = -\dfrac{1}{\varepsilon}, \end{cases} \tag{3.9}$$

for some good boundary data $v_{bd}^{1,0}(Y,x)$ (independent of y). Roughly, this corrector takes care of the source terms of amplitude $O(\varepsilon^{-\alpha-1})$ generated by $v^{0,0}$, while the boundary data $v_{bd}^{1,0}$ ensures that it decays at infinity. As before, we can factorize these fields, through

$$\begin{aligned} v^{1,0}(y,Y,x) \;&=\; \sum_{\alpha'=1}^{d} w_{\alpha'}^{1,0}(y)\,\varphi_0(x)\,\partial_{\alpha'}\psi(Y) \\ v_{bd}^{1,0}(Y,x) \;&=\; \sum_{\alpha'=1}^{d} w_{bd,\alpha'}^{1,0}\,\varphi_0(x)\,\partial_{\alpha'}\psi(Y), \end{aligned}$$

where $w_{\alpha'}^{1,0}$ solves

$$\begin{cases} -\nabla_y \cdot A(y)\nabla_y\, w_{\alpha'}^{1,0} = \nabla_{y_{\beta'}} \cdot (A^{\beta'\alpha'}(y) w^{0,0}) + A_{\alpha'\beta'}(y)\nabla_{y_{\beta'}} w^{0,0} & y.n_q > -\dfrac{1}{\varepsilon} \\ w_{\alpha'}^{1,0}(y) = w_{bd,\alpha'}^{1,0}, & y.n_q = -\dfrac{1}{\varepsilon}. \end{cases} \tag{3.10}$$

Note that, up to considering a lift of the boundary data, this system is of the type (2.33). Note also that the source term decays fast as $y \cdot n_q$ goes to infinity. As we have already discussed, for any constant boundary data $w^{1,0}_{bd,\alpha'}$, this problem admits a solution that converges to a constant field as $y \cdot n_q$ goes to infinity. We chose precisely $w^{1,0}_{bd,\alpha'}$ so that this constant at infinity is zero. Of course, this gives rise to another error term, to be controlled in the next subsection.

The construction of $v^{0,1}$ follows the same lines. Thus, $v^{0,1}$ satisfies

$$\left\{\begin{aligned} -\nabla_y \cdot A(y)\nabla_y v^{0,1}(y,Y,x) &= \nabla_y \cdot A(y)\nabla_x v^{0,0}(y,Y,x) & & \\ &\quad + \nabla_x \cdot A(y)\nabla_y v^{0,0}, & y.n_q &> -\frac{1}{\varepsilon}, \\ v^{0,1} &= v^{0,1}_{bd}(Y,x), & y.n_q &= -\frac{1}{\varepsilon}. \end{aligned}\right. \tag{3.11}$$

so as to cancel the $O(\varepsilon^{-1})$ remainder terms due to $v^{0,0}$. Again, one can separate variables:

$$v^{0,1}(y,Y,x) \;=\; \sum_{\alpha'=1}^{d} w^{0,1}_{\alpha'}(y)\partial_{x_{\alpha'}}\varphi_0(x)\psi(Y)$$

where $w^{0,1}_{\alpha'} \;=\; w^{1,0}_{\alpha'}(y)$ solves the classical boundary layer system, with a rapidly decaying source term. The higher order profiles are built recursively, following this scheme. They satisfy the same type of equations, with source terms coming from the lower order profiles. More precisely, $v^{k,l}$ solves

$$\left\{\begin{aligned} & -\nabla_y \cdot A(y)\nabla_y v^{k,l}(y,Y,x) = \nabla_y \cdot A(y)\nabla_x v^{k,l-1}(y,Y,x) \\ & \quad + \nabla_x \cdot A(y)\nabla_y v^{k,l-1} + \nabla_y \cdot A(y)\nabla_Y v^{k-1,l}(y,Y,x)+ \\ & \quad + \nabla_Y \cdot A(y)\nabla_y v^{k-1,l}(y,Y,x) \\ & \quad + \nabla_Y \cdot A(y)\nabla_Y v^{k-2,l}(y,Y,x) + \nabla_x \cdot A(y)\nabla_x v^{k,l-2}(y,Y,x) \\ & \quad + \nabla_x \cdot A(y)\nabla_Y v^{k-1,l-1}(y,Y,x) + \nabla_Y \cdot A(y)\nabla_x v^{k-1,l-1}(y,Y,x) \quad y.n_q > -\frac{1}{\varepsilon}, \\ & v^{k,l} = v^{k,l}_{bd}(Y,x), \qquad\qquad y.n_q = -\frac{1}{\varepsilon}. \end{aligned}\right. \tag{3.12}$$

Note that the bounds on the $v^{k,l}$ and $v^{k,l}_{bd}$ for $k+l \geq 1$ depend on κ. More precisely, at each step of the construction, a little more than a power κ^2 is lost: uniformly in $q \in \mathbb{Q}^g$,

$$\forall \delta > 0,\ \forall k,l,\ \forall s, \quad \|\nabla^s_{y,Y,x} v^{k,l}\|_{L^\infty} \;+\; \|\nabla^s_{Y,x} v^{k,l}_{bd}\|_{L^\infty} \;\leq\; \frac{C_{\delta,k,l,s}}{\kappa^{(2+\delta)(k+l)}} \tag{3.13}$$

These inequalities are a simple consequence of propositions 5, 6 and 8. For $k+l=0$, it follows straightforwardly from Proposition 5. For $k+l=1$, we notice that $v^{k,l}(y,x,Y) - v^{k,l}_{bd}(x,Y)$ satisfies the equations in (2.33), with a zero boundary data and a source term $f^{k,l}$ that depends on $v^{0,0}$. More precisely, this system is derived from an enlarged system of type (2.31) with a source term $F^{k,l}$ depending on $V^{0,0}$. From the estimates of propositions 5 and

6, we obtain

$$\begin{aligned}
\left|\partial_y^\alpha f^{k,l}(y)\right| \;&\leq\; C_\alpha, && \text{uniformly in } y,\\
\left|\partial_\theta^\alpha \partial_t^\beta F^{k,l}(\theta,t)\right| \;&\leq\; C_{m,\alpha,\beta}\,\kappa^{-1}\Big(\kappa(1+t-a)\Big)^{-m} \\
&\leq\; C_{m,\alpha,\beta}\,\Big(\kappa^{1+\delta}(1+t-a)\Big)^{-m} && \text{uniformly in } \theta,
\end{aligned}$$

for any given $\delta > 0$, as soon as $m\delta \geq 1$. Then, Proposition 8 yields the good bounds on $v^{k,l}$, for $k+l=1$. Applying recursively Proposition 8, one can obtain (3.13) for all k,l. We leave the details to the reader.

3.3 Last error estimates and conclusion

To conclude the homogenization proof, we still need: *i)* to estimate in L^2 the approximate boundary layer

$$u^{\varepsilon,g,app} \;=\; \sum_{q\in\mathbb{Q}^g} u_q^{\varepsilon,app},$$

where $u_q^{\varepsilon,app}$ has the expansion (3.6) *ii)* to compare it in L^2 to

$$u^{\varepsilon,g} \;:=\; \sum_{q\in\mathbb{Q}^g} u_q^\varepsilon$$

where u_q^ε satisfies (3.4).

i) Note that for all q, the support of $u_q^{\varepsilon,app}$ has size $O(\varepsilon^\alpha)$ along the boundary and $O(1)$ transversely to the boundary. Moreover, when $|q-q'|\geq 2$, the supports of $u_q^{\varepsilon,app}$ and $u_{q'}^{\varepsilon,app}$ are disjoint. From there, we infer

$$\|u^{\varepsilon,g,app}\|^2_{L^2(\Omega)} \;\leq\; 2\sum_{q\in\mathbb{Q}^g}\|u_q^{\varepsilon,app}\|^2_{L^2(\Omega)}$$

and

$$\|u_q^{\varepsilon,app}\|_{L^2(\Omega)} \;\leq\; \left\|v_q^{0,0}\left(\cdot,\frac{\cdot}{\varepsilon^\alpha},\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} \;+\; C\,\varepsilon^{\alpha/2}\sum_{\substack{k+l\geq 1\\ k(1-\alpha)+l\leq K_0}} \varepsilon^{k(1-\alpha)+l}\|v_q^{k,l}\|_{L^\infty(\Omega)}.$$

Combining this last inequality with (3.13), we get

$$\|u_q^{\varepsilon,app}\|_{L^2(\Omega)} \;\leq\; \left\|v_q^{0,0}\left(\cdot,\frac{\cdot}{\varepsilon^\alpha},\frac{\cdot}{\varepsilon}\right)\right\|_{L^2(\Omega)} \;+\; C(k_0,\delta)\,\varepsilon^{\alpha/2}\sum_{\substack{k+l\geq 1\\ k(1-\alpha)+l\leq K_0}} \frac{\varepsilon^{k(1-\alpha)+l}}{\kappa^{(2+\delta)(k+l)}}.$$

By construction, $v_q^{0,0}$ goes fast to zero as $t = y\cdot n_q \to +\infty$. Using the notations of (3.8)

$$\left\|v_q^{0,0}\left(\cdot,\frac{\cdot}{\varepsilon^\alpha},\frac{\cdot}{\varepsilon}\right)\right\|^2_{L^2(\Omega)} \;\leq\; C\varepsilon^{\alpha+1}\int_{\mathbb{R}_+}\left|w_q^{0,0}(y'+(t+1/\varepsilon)\right|^2\,dt \;\leq\; C\frac{\varepsilon^{\alpha+1}}{\kappa^3},$$

where the last inequality follows from Proposition 6. By summing over q, we get

$$\|u^{\varepsilon,g,app}\|_{L^2(\Omega)} \;\leq\; C(k_0,\delta)\left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} + \frac{\varepsilon^{1/2}}{\kappa^{3/2}}\right) \;\leq\; C'(k_0,\delta)\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}}$$

as soon as $\alpha > 1/2$ and δ small enough, a condition that will be satisfied eventually.

ii) The difference $e^\varepsilon = u^{\varepsilon,g} - u^{\varepsilon,g,app}$ solves

$$\left\{ \begin{aligned} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla e^\varepsilon &= r^\varepsilon, \quad x \in \Omega, \\ e^\varepsilon &= \phi^\varepsilon, \quad x \in \partial\Omega. \end{aligned} \right. \tag{3.14}$$

We now comment on the errors r^ε and ϕ^ε.

The source term r^ε comes from the fact that $u_q^{\varepsilon,app}$ does not satisfy exactly the first equation of (3.4). Indeed, the expansion (1.7) has been cut at $k(1-\alpha)+l = K_0$. Crudely, we get $\|r^\varepsilon\|_{L^2} = O(\varepsilon^{K_0-2})$. Furthermore, the estimate (3.13) allows to specify the dependence with respect to κ. Introducing k_0 such that $K_0 = k_0(1-\alpha)$, we get

$$\|r^\varepsilon\|_{L^2(\Omega)} \;\le\; C(\delta, K_0) \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}}\right)^{k_0} \varepsilon^{-2}, \quad \forall \delta > 0. \tag{3.15}$$

For this inequality, we use that $\varepsilon^{1-\alpha}/\kappa^{2+\delta} < 1$, a condition that will be ensured by our choice of parameters.

The boundary term ϕ^ε comes from several approximations:

1. In the boundary data for $v_q^{0,0}$, we have written $v_*(n_q)$ instead of $v_*(n(x))$. In other words, we have replaced u_q^ε by the solution $\tilde{u}_q^\varepsilon$ of

$$\left\{ \begin{aligned} -\nabla \cdot A\left(\frac{x}{\varepsilon}\right) \nabla \tilde{u}_q^\varepsilon &= 0, \quad x \in \Omega, \\ \tilde{u}_q^\varepsilon = \left(\varphi(x, x/\varepsilon) - v_*(n_q)\varphi_0(x)\right) \phi_q(x)&, \quad x \in \partial\Omega. \end{aligned} \right.$$

   Note that the boundary data for both u_q^ε and $\tilde{u}_q^\varepsilon$ are non-zero only for θ in a vicinity of I_q. Due to the Lipschitz character of v_*, *cf* corollary 1, we deduce

$$\| \sum_{q=1}^{Q} \left(u_q^\varepsilon - \tilde{u}_q^\varepsilon\right) \|_{L^2(\partial\Omega)} \;\le\; \frac{C}{\kappa^2} \varepsilon^\alpha.$$

2. To be able to solve the boundary layer systems for $v_q^{k,l}$, $q \in \mathbb{Q}^g$, we have considered the flat line $y.n_q = -1/\varepsilon$, instead of the original circle $|y| = 1/\varepsilon$. Moreover, to force the decay to zero, we have added the inhomogeneous Dirichlet data $v_{q,bd}^{k,l}$, $k + l \ge 1$. All of this results in non zero boundary terms at the circle. Note that the q-th term is supported in a $O(\varepsilon^\alpha)$ neighborhood of x_q, which is at distance at most $O(\varepsilon^{2\alpha})$ from the flat line. Its amplitude is therefore bounded by

$$\begin{aligned} &\sum_{\substack{k+l\ge 1 \\ k(1-\alpha)+l\le K_0}} \varepsilon^{k(1-\alpha)+l} \,|v_{q,bd}^{k,l}| \;+\; \varepsilon^{2\alpha-1} \sum_{k(1-\alpha)+l\le K_0} \varepsilon^{k(1-\alpha)+l} \,\|\nabla v_q^{k,l}(y)\|_{L^\infty} \\ &\le\; C(\delta, K_0) \Big(\sum_{\substack{k+l\ge 1 \\ k(1-\alpha)+l\le K_0}} \frac{\varepsilon^{k(1-\alpha)+l}}{\kappa^{(2+\delta)(k+l)}} \;+\; \varepsilon^{2\alpha-1} \sum_{k(1-\alpha)+l\le K_0} \frac{\varepsilon^{k(1-\alpha)+l}}{\kappa^{(2+\delta)(k+l)}} \Big), \quad \forall \delta > 0, \\ &\le\; C(\delta, K_0) \Big(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \Big), \quad \forall \delta > 0. \end{aligned}$$

For the last inequality, we use that $\varepsilon^{1-\alpha}/\kappa^2 < 1$, a condition that will be ensured by our choice of parameters.

Gathering these bounds, we end up with

$$\|\phi^\varepsilon\|_{L^2(\partial\Omega)} \;\le\; C(\delta, K_0)\left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1}\right), \quad \forall \delta > 0. \tag{3.16}$$

Using the estimates (3.15), (3.16), and those of Avellaneda and Lin [5], we end up with

$$\|e^\varepsilon\|_{L^2(\Omega)} \;\le\; C(\delta, K_0)\left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \;+\; \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}}\right)^{k_0} \varepsilon^{-2}\right), \quad \forall \delta > 0. \tag{3.17}$$

Eventually, we have the following inequalities:

$$\left\{\begin{aligned} \|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)} \;&\le\; C\left(\kappa^{1/2} \;+\; \frac{\varepsilon}{\rho^{1/2}\kappa^2} \;+\; \frac{\rho}{\kappa^2}\right), \\ \|u^\varepsilon_{bl}\|_{L^2(\Omega)} \;&\le\; C\kappa^{1/2} \;+\; C(\delta, K_0)\left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \;+\; \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}}\right)^{k_0} \varepsilon^{-2}\right) \end{aligned}\right. \tag{3.18}$$

for arbitrary $\delta > 0$ and $K_0 \in \mathbb{N}$. To obtain the appropriate rate of convergence, it remains to optimize these inequalities with respect to the parameters κ, α and ρ.

First, for any given values of ε and κ, the r.h.s. of the upper inequality is minimized when $\varepsilon/(\rho^{1/2}\kappa^2) \;\sim\; \rho/\kappa^2$. This yields $\rho \sim \varepsilon^{2/3}$. With this choice,

$$\|u^\varepsilon - u^0\|_{L^2} \;\le\; C(\delta, K_0)\left(\kappa^{1/2} + \frac{\varepsilon^{2/3}}{\kappa^2} + \frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \;+\; \varepsilon^{2\alpha-1} \;+\; \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}}\right)^{k_0} \varepsilon^{-2}\right).$$

Note that the r.h.s must vanish when $\varepsilon \to 0$, which implies that $2\alpha - 1 > 0$. In turn, this implies that the second term in the sum can be neglected compared to the third one. Now, for any given value of ε, the quantity $\kappa^{1/2} + \frac{\varepsilon^{1-\alpha}}{\kappa^2} + \varepsilon^{2\alpha-1}$ is minimized when all three terms are of the same size. This yields $\alpha = 6/11$, and $\kappa \sim \varepsilon^{2/11}$.

With this scaling, we get

$$\|u^\varepsilon - u^0\|_{L^2} \;\le\; C(\delta, K_0)\left(\varepsilon^{(1-2\delta)/11} \;+\; \varepsilon^{k_0(1-2\delta)/11-2}\right),$$

for all δ and K_0. Then, for any $\delta \in (0, 1/2)$, we take K_0 large enough so that $(k_0 - 1)(1 - 2\delta)/11 > 2$. Hence,

$$\|u^\varepsilon - u^0\|_{L^2} \;\le\; C(\delta)\, \varepsilon^{1/11-2\delta},$$

which concludes the proof.

4 Extension to the general setting

We still need to explain how to extend our result to more general Ω, and to the case where ϕ is not factored. We shall follow the analysis and notations of section 3, and point out the arguments that need to be modified.

4.1 Uniformly convex domains

We assume that Ω is a smooth bounded open subset of $\mathbb{R}^d$, uniformly convex. We denote by m the measure on $\partial\Omega$. By our assumptions, the mapping

$$n : \partial\Omega \mapsto \mathbb{S}^{d-1}, \quad x \mapsto n(x)$$

is a diffeomorphism. This implies that for all $\kappa > 0$ the set

$$\mathcal{B}_\kappa \; := \; \{x \in \partial\Omega, \quad n(x) \in \mathcal{A}_\kappa\}$$

satisfies

$$m(\mathcal{B}_\kappa^c) \le C\kappa^{d-1}. \tag{4.1}$$

In particular, the set $\cup_{\kappa>0}\mathcal{B}_\kappa$ has full measure in $\partial\Omega$. For x in this set, we can define

$$\varphi_*(x) \; := v_*(n(x))\, \varphi_0(x)$$

which belongs to $L^\infty(\partial\Omega)$. As in section 3, we then introduce u^ε_{reg}, u_0 and u^ε_{bl}. In order to prove Theorem 1, we need to control: *i)* $\|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)}$ and *ii)* $\|u^\varepsilon_{bl}\|_{L^2(\partial\Omega)}$.

i) The analysis carried for the disk still works for our domains Ω, replacing $\mathcal{A}_\kappa$ by $\mathcal{B}_\kappa$. The only change is the κ^{d-1} in the measure estimate (4.1). Therefore, we end up with

$$\|u^\varepsilon_{reg} - u^0\|_{L^2(\Omega)} \; \le \; C \, \left(\kappa^{(d-1)/2} \; + \; \frac{\varepsilon}{\rho^{1/2}\kappa^2} \; + \; \frac{\rho}{\kappa^2} \right) \tag{4.2}$$

ii) The analysis is again almost unchanged. Let $\alpha > 0$. As $\partial\Omega$ is diffeomorphic to the sphere $\mathbb{S}^{d-1}$, it is easy to build a partition of unity $(\varphi_q)_{q\in\mathbb{Q}}$ in a vicinity of $\partial\Omega$, with cardinality $O(\varepsilon^{(1-d)\alpha})$, such that $\varphi_q|_{\partial\Omega}$ is supported in a set of measure $O(\varepsilon^{(d-1)\alpha})$. One can again distinguish between a bad set of indices $\mathbb{Q}^b$ and a good set $\mathbb{Q}^g$, and split u^ε_{bl} accordingly. All estimates remain the same, except for (3.5), in which the $\kappa^{1/2}$ term is replaced by $\kappa^{(d-1)/2}$, because of (4.1). Eventually, one obtains

$$\|u^\varepsilon_{bl}\|_{L^2(\Omega)} \; \le \; C\kappa^{(d-1)/2} \; + \; C(\delta, k_0) \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \; + \; \varepsilon^{2\alpha-1} \; + \; \left(\frac{\varepsilon^{1-\alpha}}{\kappa^{2+\delta}} \right)^{k_0} \; \varepsilon^{-2} \right) \tag{4.3}$$

Putting together (4.2)-(4.3) and optimizing yields the theorem.

4.2 General boundary data

So far, we have considered factored data, meaning

$$\varphi(x, y) \; = \; v(y)\, \varphi_0(y)$$

for some smooth periodic $v \in M_N(\mathbb{R})$ and some smooth $\varphi_0 \in \mathbb{R}^N$. We have established in such a case that

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \; \le \; C_{\alpha,\varphi}\, \varepsilon^\alpha, \quad \forall \alpha < \frac{d-1}{3d+5}.$$

Actually, the constant $C_{\alpha,\varphi}$ can be further specified. A closer look at the proof shows that

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} \; \le \; C_\alpha \, \|\varphi\|_{H^s(\partial\Omega\times\mathbb{T}^d)}\, \varepsilon^\alpha, \quad \forall \alpha < \frac{d-1}{3d+5}, \tag{4.4}$$

for some s large enough. More precisely, $s = s(\alpha)$ depends on $\frac{d-1}{3d+5} - \alpha$.

This refined estimate (4.4) allows to go from factored to non-factored data. Indeed, let $\varphi = \varphi(x, y) \in C^\infty(\partial\Omega \times \mathbb{T}^d)$. By expanding φ as a Fourier sum, we can write

$$\varphi(x, y) \; = \; \sum_{k \in \mathbb{Z}^d} \varphi^k(x, y) \; = \; \sum_{k \in \mathbb{Z}^d} e^{2\pi i\, k \cdot y} \varphi_0^k(x).$$

For each $k \in \mathbb{Z}^d$, the data φ^k is factored, so that we can apply the analysis of section 3. In particular, we can define a homogenized boundary data φ^k_*. We can then consider the solution $u^{\varepsilon,k}$ of (1.1)-(1.2) with boundary data φ^k, *resp.* the solution $u^{0,k}$ of (1.15) with boundary data φ^k_*. By estimate (4.4):

$$\|u^{\varepsilon,k} - u^{0,k}\|_{L^2(\Omega)} \; \le \; C_\alpha \, \|\varphi^k\|_{H^s(\Omega \times \mathbb{T}^d)} \, \varepsilon^\alpha, \quad \forall \alpha < \frac{d-1}{3d+5},$$

for s large enough (independent of k). As φ is smooth and periodic with respect to y, the k-th Fourier coefficient φ_0^k decays in $H^s(\partial\Omega)$ faster than any negative power of k. This leads to

$$\|\varphi^k\|_{H^s(\partial\Omega \times \mathbb{T}^d)} \; \le \; C_{s,N} \, |k|^{-N}, \qquad \forall \, k, n.$$

Combining the last two bounds yields the convergence of $u^\varepsilon = \sum_k u^{\varepsilon,k}$ to the solution $u^0 = \sum_k u^{0,k}$ of (1.15) with boundary data $\varphi_* = \sum_k \varphi^k_*$.

5 Next order approximation

As a byproduct of our main Theorem 1, we can tackle another related homogenization problem. Namely, we can build high order expansions for the non-oscillating Dirichlet problem

$$\begin{cases} -\nabla \cdot \left(A\left(\cdot/\varepsilon\right) \nabla u\right)(x) = 0, & x \in \Omega, \\ \qquad\qquad u(x) = \varphi(x), & x \in \partial\Omega. \end{cases} \tag{5.1}$$

where φ depends only on x. We have already mentioned this problem in the introduction: one has

$$u^\varepsilon(x) \; = \; u^0(x) + \varepsilon \chi(x/\varepsilon) \nabla u^0(x) + \varepsilon u^{1,\varepsilon}_{bl}(x) \; + \; r^\varepsilon(x),$$

where $r^\varepsilon = O(\varepsilon)$ in $H^1(\Omega)$, $r^\varepsilon = O(\varepsilon^2)$ in $L^2(\Omega)$. The fields u^0 and χ are defined through (1.4) and (1.5), whereas the boundary layer corrector $u^{1,\varepsilon}_{bl}$ satisfies (1.10). This is a special case of system (1.1)-(1.2), where the boundary data φ is factored into

$$\varphi(x, y) \; := \; -\chi(y) \nabla u^0(x).$$

We can associate to φ the homogenized boundary data φ_* and by Theorem 1, we get:

$$\|u^{1,\varepsilon}_{bl} - \overline{u}\|_{L^2(\Omega)} = O(\varepsilon^\alpha), \quad \forall \, \alpha < \frac{d-1}{3d+5}.$$

where $\overline{u}$ is the solution of (1.15). If we set:

$$u^1(x, y) \; := \; \chi(y) \nabla u^0(x) \; + \; \overline{u}(x).$$

we obtain

Theorem 10 *The solution u^ε of* (5.1) *admits the asymptotic expansion*

$$u^\varepsilon \;=\; u^0 \;+\; \varepsilon\, u^1(x, x/\varepsilon) \;+\; O(\varepsilon^{1+\alpha}) \qquad \textit{in } L^2(\Omega), \quad \forall\, \alpha < \frac{d-1}{3d+5}.$$

Thus, we improve the first estimate in (1.9). From this improved L^2 estimate, one can have some improved H^1 estimate in any relatively compact subset $\omega \Subset \Omega$. Namely, one can introduce the family of 1-periodic matrices

$$\Upsilon^{\alpha\beta} = \Upsilon^{\alpha\beta}(y) \in M_n(\mathbb{R}),\ \alpha, \beta = 1, ..., d,$$

satisfying

$$-\nabla_y \cdot A\, \nabla_y \Upsilon^{\alpha\beta} \;=\; B^{\alpha\beta} - \int_y B^{\alpha\beta}, \qquad \int_y \Upsilon^{\alpha\beta} = 0, \tag{5.2}$$

where

$$B^{\alpha\beta} \;:=\; A^{\alpha\beta} - A^{\alpha\gamma}\frac{\partial \chi^\beta}{\partial y_\gamma} - \frac{\partial}{\partial y_\gamma}\left(A^{\gamma\alpha}\chi^\beta\right).$$

Then, one can define

$$u^2(x, y) \;:=\; \Upsilon^{\alpha,\beta}\frac{\partial^2 u^0}{\partial x_\alpha \partial x_\beta} \;-\; \chi^\alpha \partial_\alpha \bar{u}. \tag{5.3}$$

Proceeding exactly as in [2], one is led to the following asymptotic expansion:

$$u^\varepsilon(x) \;=\; u^0(x) \;+\; \varepsilon u^1(x, x/\varepsilon) \;+\; \varepsilon^2 u^2(x, x/\varepsilon) \;+\; O(\varepsilon^{1+\alpha}) \quad \text{in } H^1(\omega), \quad \forall\, \alpha < \frac{d-1}{3d+5}.$$

Details are left to the reader.

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