ON THE CONSTRUCTION OF BOUNDARY LAYERS IN THE COMPRESSIBLE-INCOMPRESSIBLE LIMIT

NING JIANG AND NADER MASMOUDI

ABSTRACT. In this paper we give a more precise construction of the boundary layer used to prove the strong decay of the acoustic waves in the paper [4]. Indeed, the construction in [4] is not completely correct in the case when eigenvalues are not simple and some additional conditions should be imposed on the basis of eigenvectors chosen. There is an extra interesting algebraic structure based on a sequence of symmetric operators that should be used in this choice. This construction can be extended to the case of Robin boundary condition as well as to the full acoustic operator which includes the heat conductivity effect. Moreover, this paper allows us to extend the construction of the viscous boundary layer in the hydrodynamic limit of the Boltzmann equation used in our paper [7] to the general case.

1. Introduction

The paper [4] was devoted to the study of the so-called incompressible limit for solutions of the compressible isentropic Navier-Stokes equations in a bounded domain with the natural physical boundary conditions for a viscous fluid, namely the homogeneous Dirichlet boundary conditions. It was in some sense, the sequel of [9, 10] where various cases were considered, namely the cases of a periodic flow, of a flow in the whole space, or in a bounded domain with other boundary conditions than Dirichlet conditions (see also [3] where the dispersion in the whole space is proved).

A new striking phenomenon caused by the boundary conditions was observed in [4]. As is well known physically when looking at the incompressible limit, one expects that, as the Mach number ε goes to 0, fast acoustic waves are generated carrying the energy of the potential part of the flow (and a normalized part of the internal energy of the gas). For periodic flows (or for some particular boundary conditions), these waves subsist forever and their frequencies grow as ε goes to zero. Mathematically speaking, this means that the solutions of compressible (isentropic) Navier-Stokes equations may only converge *weakly* in L^2 to the solutions of incompressible Navier-Stokes equations – and they do as shown in [9]. However, in the case of a viscous flow in a bounded domain with the usual Dirichlet boundary condition, under a generic assumption on the domain, it was proved in [4] that the acoustic waves are instantaneously (asymptotically) damped, due to the formation of a thin boundary layer. This layer dissipates the energy carried by the waves and, from a mathematical viewpoint, it yields a *strong* convergence in L^2. For more recent results about the compressible-incompressible limit, we refer to [5, 8]. We also refer to [1, 12] for some review papers. The phenomenon of dissipation due to the boundary layer is also present for a related problem which presents, as is classical, striking analogies with the one studied here, namely the so-called "Ekman pumping" for rotating fluids (see for instance [6, 11, 2]).

The main ingredient in [4] was the construction of a boundary layer that was used as a test function very much like in the method of "oscillating test functions" of L. Tartar. However, the construction in [4] presents a small gap when the eigenvalues of the Laplacian with Neumann boundary condition are *not simple*. Here we would like to make this more precise. Furthermore, we consider the full acoustic and diffusion operators including the heat conductivity effect, which also gives some new phenomena different from the isentropic case. The construction we give here is also used to extend the construction of the so-called viscous boundary layer used in the hydrodynamic limit of the Boltzmann equation [7] to the general case.

We will use the same notations as in [4]. We first recall the definition of the viscous wave operator $\mathcal{A}_\varepsilon = \mathcal{A} + \varepsilon\mathcal{D}$ where $\mathcal{A}$ and $\mathcal{D}$ are acoustic and diffusion operators respectively, defined on $\mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)^d \times \mathcal{D}'(\Omega)$ by

$$\mathcal{A}U = \begin{pmatrix} \mathrm{div}\mathbf{u} \\ \nabla_x(\rho+\theta) \\ \gamma\mathrm{div}\mathbf{u} \end{pmatrix}, \quad \mathcal{D}U = \begin{pmatrix} 0 \\ \nu\Delta_x\mathbf{u} + (\xi+\nu)\nabla_x\mathrm{div}\mathbf{u} \\ \kappa\Delta_x\theta \end{pmatrix},$$

where $U = (\rho\,, \mathbf{u}\,, \theta)^T$ and we assume that $\gamma > 0$, $\nu > 0$ and $\xi + \nu > 0$.

Let $\{\lambda_{k,0}^2\}_{k\geq 1}$ be the nondecreasing sequence of eigenvalues and $\{\Psi_{k,0}\}_{k\geq 1}$ the orthonormal basis of $L^2(\Omega)$ functions with zero mean value of eigenvectors of the Laplace operator $-\Delta_{\mathrm{N}}$ with homogeneous Neumann boundary conditions:

$$-\Delta_x\Psi_{k,0} = \lambda_{k,0}^2\Psi_{k,0} \quad \text{in} \quad \Omega\,, \quad \frac{\partial\Psi_{k,0}}{\partial\mathrm{n}} = 0 \quad \text{on} \quad \partial\Omega\,. \tag{1.1}$$

The eigenvalues and eigenvectors of $\mathcal{A}$ read as follows

$$\phi_{k,0}^{\pm} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1+\gamma}}\Psi_{k,0}\,, \mathbf{u}_{k,0}^{\pm} = \pm\frac{\nabla_x\Psi_{k,0}}{i\lambda_{k,0}}\,, \frac{\gamma}{\sqrt{1+\gamma}}\Psi_{k,0}\right)^T \in \mathbb{C}^{2+d}\,, \tag{1.2}$$

$$\mathcal{A}\phi_{k,0}^{\pm} = i\sqrt{1+\gamma}\lambda_{k,0}^{\pm}\phi_{k,0}^{\pm} \quad \text{in} \quad \Omega\,, \quad \mathbf{u}_{k,0}^{\pm}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\,, \tag{1.3}$$

where $\lambda_{k,0}^{\pm} = \pm\lambda_{k,0}$.

In the paper [4], for the isentropic case, Desjardins, Grenier, Lions and the second author constructed the viscous boundary layers under two conditions: the first, a geometric condition on the domain Ω (the "assumption (H)" in [4]); the second, an orthogonality condition for multiple eigenvalues, namely that if $\lambda_{k,0} = \lambda_{l,0}$ and $k \neq l$, then

$$\int_{\partial\Omega} \nabla_x\Psi_{k,0}\cdot\nabla_x\Psi_{l,0}\mathrm{d}\sigma_x = 0\,. \tag{1.4}$$

In the present paper, we consider the more general non-isentropic case. The analogue of the orthogonality condition (1.4) for the isentropic case will be (3.16) for the non-isentropic case. Actually, this condition (both for isentropic and non-isentropic cases) is **not completely sufficient and we need some higher order orthogonality conditions.** This is the main concern of the present paper.

Let us explain more how the condition (1.4) should be enforced. Assume that λ^2 is an eigenvalue of (1.1) and denote by $\mathrm{H}_0 = \mathrm{H}_0(\lambda)$ the eigenspace associated to λ^2, i.e.

$$\mathrm{H}_0(\lambda) = \{\Psi \in \mathcal{D}(-\Delta_x) : -\Delta_x\Psi = \lambda^2\Psi \;\; \text{in} \;\; \Omega, \frac{\partial\Psi}{\partial\mathrm{n}} = 0 \;\; \text{on} \;\; \partial\Omega\} \tag{1.5}$$

where $\mathcal{D}(-\Delta_x) = H^2(\Omega) \cap \{\Psi : \frac{\partial\Psi}{\partial\mathrm{n}} = 0 \;\text{on}\; \partial\Omega\}$ denotes the domain of $-\Delta_x$ with Neumann boundary condition. On the finite dimensional space $\mathrm{H}_0(\lambda)$, we can define a quadratic form Q_1 (its associated bilinear form is also denoted by Q_1) and a symmetric operator $L_1 = L_1^\lambda$ by

$$Q_1(\Psi, \tilde{\Psi}) = \int_{\partial\Omega} \nabla_x\Psi\cdot\nabla_x\tilde{\Psi}\,\mathrm{d}\sigma_x = \int_\Omega L_1(\Psi)\tilde{\Psi}\,\mathrm{d}x\,. \tag{1.6}$$

Thus the condition (1.4) can be restated as

$$Q_1(\Psi_{k,0}, \Psi_{l,0}) = 0\,, \quad \text{if} \quad \Psi_{k,0}, \Psi_{l,0} \in \mathrm{H}_0(\lambda) \quad \text{and} \quad k \neq l\,. \tag{1.7}$$

This condition means that the eigenvectors $\Psi_{k,0}$ for $\lambda_{k,0} = \lambda$ are orthogonal for the symmetric operator L_1. Of course, since $L^2(\Omega)$ is the direct sum of the spaces $\mathrm{H}_0(\lambda)$ for different $\lambda's$, from the definition of L_1 on each eigenspace $\mathrm{H}_0(\lambda)$, we can define an operator L_1^λ on $L^2(\Omega)$ which leaves each eigenspace $\mathrm{H}_0(\lambda)$ invariant. But this is not necessary, so we will think of L_1^λ as acting on $\mathrm{H}_0(\lambda)$ for a fixed multiple eigenvalue λ.

The orthogonality condition (1.4) turns out to be enough for the construction of the boundary layer if the eigenvalues of L_1 are simple, namely, if

$$\lambda_{k,1} = \int_{\partial\Omega} |\nabla_x \Psi_{k,0}|^2 \,\mathrm{d}\sigma \neq \int_{\partial\Omega} |\nabla_x \Psi_{l,0}|^2 \,\mathrm{d}\sigma_x = \lambda_{l,1} \tag{1.8}$$

for all $l \neq k$ such that $\lambda_{l,0} = \lambda_{k,0} = \lambda$. [Here the expression of $\lambda_{k,1}$ is for the isentropic case. The analog for the general non-isentropic case is given by (3.13).] However, if λ_1 is an eigenvalue of L_1 with multiplicity greater than or equal to 2, more precisely, if there exists $l \neq k$ such that $\lambda_{l,0} = \lambda_{k,0} = \lambda$, and $\lambda_{l,1} = \lambda_{k,1} = \lambda_1$, then we need an extra orthogonality condition as following: Let $\mathrm{H}_1 = \mathrm{H}_1(\lambda_1)$ be defined by

$$\mathrm{H}_1(\lambda_1) = \{\Psi \in \mathrm{H}_0(\lambda) : L_1\Psi = \lambda_1\Psi\}\,. \tag{1.9}$$

On the finite dimensional space H_1, there exists a quadratic form Q_2 and a symmetric operator L_2 [see the definition below], the extra condition is

$$Q_2(\Psi_{k,0}, \Psi_{l,0}) = 0\,, \quad \text{if} \quad \Psi_{k,0}, \Psi_{l,0} \in \mathrm{H}_1(\lambda_1) \quad \text{and} \quad k \neq l\,. \tag{1.10}$$

This condition is enough if L_2 has only simple eigenvalue on the vector space H_1. However, if L_2 has eigenvalue with multiplicity greater than or equal to 2, we need additional condition on H_2, and so on.

This process can be continued inductively to find the condition we have to impose on the eigenvectors of $-\Delta_x$. Indeed, we can construct recursively, on each eigenspace $\mathrm{H}_0(\lambda)$ of $-\Delta_x$, a sequence of symmetric operators $L_q, (q \in \mathbb{N})$ in the following way: Let $L_0 = -\Delta_x$, we define L_1 on each one of the eigenspace $\mathrm{H}_0(\lambda)$ of L_0 by (1.6) [In the non-isentropic case, (1.6) should be replaced by (3.19)]. Assume that the operators L_p were constructed for $p \leq q-1, q \geq 2$ in such a way that each operator L_p leaves invariant the eigenspaces of the operators $L_{p'}$ for $p' < p$. Now, to construct L_q, it is enough to construct L_q on each eigenspace $\mathrm{H}_1(\lambda_1) \cap \mathrm{H}_2(\lambda_2) \cap \cdots \cap \mathrm{H}_{q-1}(\lambda_{q-1})$, where $\lambda_1, \lambda_2, \cdots, \lambda_{q-1}$ are *multiple* eigenvalues of $L_1, L_2, \cdots, L_{q-1}$ respectively. This is done by constructing a quadratic form Q_q on each space $\mathrm{H}_1(\lambda_1) \cap \mathrm{H}_2(\lambda_2) \cap \cdots \cap \mathrm{H}_{q-1}(\lambda_{q-1})$ and defining L_q by

$$Q_q(\Psi, \tilde{\Psi}) = \int_\Omega L_q(\Psi)\tilde{\Psi} \,\mathrm{d}x\,, \quad \text{for all} \quad \Psi, \tilde{\Psi} \in \mathrm{H}_1(\lambda_1) \cap \mathrm{H}_2(\lambda_2) \cap \cdots \cap \mathrm{H}_{q-1}(\lambda_{q-1})\,. \tag{1.11}$$

The precise construction of the quadratic form Q_q on the space $\mathrm{H}_1(\lambda_1) \cap \mathrm{H}_2(\lambda_2) \cap \cdots \cap \mathrm{H}_{q-1}(\lambda_{q-1})$ will be done in the proof of Theorem 1.1. The theorem and its proof will be stated for the more general non-isentropic case, and it is easy to deduce the isentropic case. Once there is an eigenvalue λ_q such that $\dim \mathrm{H}_q(\lambda_q) = 1$, i.e. λ_q is a simple eigenvalue of L_q, no additional orthogonality conditions are needed and we can just take $L_{q'} = Id$ on $\mathrm{H}_1(\lambda_1) \cap \mathrm{H}_2(\lambda_2) \cap \cdots \cap \mathrm{H}_q(\lambda_q)$ for $q' \geq q+1$. (Note: for this case, the orthogonality condition (1.13) is reduced to $\int_\Omega \Psi_{k,0}\Psi_{l,0} \,\mathrm{d}x = 0$, which does not give any new condition).

Let $N \in \mathbb{N}$ be an integer. This is the integer that will appear in the order of the approximation in the next proposition. The eigenvectors $\Psi_{k,0}$ for $\lambda_{k,0} = \lambda$ should be chosen in such a way that they are orthogonal to all the operators L_n for $n \leq N+2$, which means that

$$Q_n(\Psi_{k,0}, \Psi_{l,0}) = \int_\Omega L_n(\Psi_{k,0})\Psi_{l,0} \,\mathrm{d}x = 0\,, \tag{1.12}$$

if $\Psi_{k,0}, \Psi_{l,0} \in \mathrm{H}_1(\lambda_1) \cap \mathrm{H}_2(\lambda_2) \cap \cdots \cap \mathrm{H}_{n-1}(\lambda_{n-1})$ and $k \neq l$. In fact, the condition (1.12) is equivalent to: for $1 \leq q \leq n$,

$$Q_q(\Psi_{k,0}, \Psi_{l,0}) = \int_\Omega L_q(\Psi_{k,0})\Psi_{l,0} \,\mathrm{d}x = 0\,, \quad \text{if} \quad l \neq k, \Psi_{k,0}, \Psi_{l,0} \in \mathrm{H}_0(\lambda)\,. \tag{1.13}$$

Now we state the main result of this paper, which is the same as the Proposition 2 in [4], except that we add the higher order orthogonality conditions. As we mentioned above, the main purpose of the present paper is to fill the gap in the proof of the Proposition 2 in [4] and to rewrite it in the non-isentropic setting.

Theorem 1.1. *Let Ω be a C^2 bounded domain of $\mathbb{R}^d$ and let $k \geq 1$, $N \geq 0$. Let the eigenvectors $\Psi_{k,0}$ of $-\Delta_x$ satisfy the orthogonality conditions (1.7) and (1.13). Then, there exists approximate eigenvalues $i\lambda^{\pm}_{k,\varepsilon,N}$ and eigenvectors $\phi^{\pm}_{k,\varepsilon,N} = (\rho^{\pm}_{k,\varepsilon,N}, \mathbf{u}^{\pm}_{k,\varepsilon,N}, \theta^{\pm}_{k,\varepsilon,N})^T$ of $\mathcal{A}_\varepsilon$ such that*

$$\mathcal{A}_\varepsilon \phi^{\pm}_{k,\varepsilon,N} = i\lambda^{\pm}_{k,\varepsilon,N}\phi^{\pm}_{k,\varepsilon,N} + R^{\pm}_{k,\varepsilon,N}\,, \tag{1.14}$$

with

$$i\lambda^{\pm}_{k,\varepsilon,N} = \pm i\lambda_{k,0} + i\lambda^{\pm}_{k,1}\sqrt{\varepsilon} + O(\varepsilon)\,, \tag{1.15}$$

and the real part of $i\lambda^{\pm}_{k,1}$, i.e. $\mathfrak{Re}(i\lambda^{\pm}_{k,1}) < 0$. Furthermore, for all $p \in [1,\infty]$, we have

$$\|R^{\pm}_{k,\varepsilon,N}\|_{L^p(\Omega)} \leq C_p(\sqrt{\varepsilon})^{N+\frac{1}{p}}\,. \tag{1.16}$$

Remarks: 1) In the isentropic case, to get that $\mathfrak{Re}(i\lambda^{\pm}_{k,1}) < 0$, we need an additional geometric condition on the domain Ω, the "assumption (H)" in [4]. For the non-isentropic case, this geometric condition is not needed because of the additional dissipative effect coming from heat conductivity, see the Remark after (3.14).

2) The proof gives a more precise expansion of the eigenvalues $i\lambda^{\pm}_{k,\varepsilon,N}$ and eigenvectors $\phi^{\pm}_{k,\varepsilon,N}$.

In Section 2, we collect some results which will be used in the proof of the Theorem 1.1. In section 3, we prove Theorem 1.1. The main idea is to construct a boundary layer similar to the one in [4]. For the convenience of the readers, we give a complete proof here.

2. Preliminaries

In this section, we collected some results needed in the construction of the boundary layers.

2.1. The operator $\mathcal{A} - i\lambda^{\pm}_{k,0}$.

First, for any $\phi, \tilde{\phi} \in L^2(\Omega, \mathbb{C}\times\mathbb{C}^d\times\mathbb{C})$, we introduce a scalar product

$$\langle \phi | \tilde{\phi}\rangle = \int_\Omega \left(\rho\tilde{\rho} + \mathbf{u}\cdot\tilde{\mathbf{u}} + \tfrac{1}{\gamma}\theta\tilde{\theta}\right)\,\mathrm{d}x\,, \tag{2.1}$$

for any $\phi = (\rho\,, \mathbf{u}\,, \theta)^T$ and $\tilde{\phi} = (\tilde{\rho}\,, \tilde{\mathbf{u}}\,, \tilde{\theta})^T$. Under this scalar product, the eigenvectors $\phi^{\pm}_{k,0}$ in (1.2) have norm 1.

Now, we consider the operator $\mathcal{A} - i\lambda^{\tau}_{k,0}$, where the acoustic mode $k \geq 1$ and $\tau = +$ or $-$. This operator, especially its pseudo inverse will be important in the construction of the boundary layer. The kernel and its orthogonal under the inner product (2.1) are

$$\begin{aligned}\mathrm{Null}(\mathcal{A} - i\lambda^{\tau}_{k,0}) &= \mathrm{Span}\{\phi^{\tau}_{l,0} : \lambda_{l,0} = \lambda_{k,0}\}\,,\\ \mathrm{Null}(\mathcal{A} - i\lambda^{\tau}_{k,0})^{\perp} &= \mathrm{Span}\{\phi^{+}_{l,0}\,, \phi^{-}_{l,0} : \lambda_{l,0} \neq \lambda_{k,0}\} \oplus \mathrm{Span}\{\phi^{-\tau}_{l,0} : \lambda_{l,0} = \lambda_{k,0}\}\\ &\quad \oplus \mathrm{Null}(\mathcal{A})\,.\end{aligned} \tag{2.2}$$

Next, we define a bounded pseudo inverse of $\mathcal{A} - i\lambda^{\tau}_{k,0}$

$$\big(\mathcal{A} - i\lambda^{\tau}_{k,0}\big)^{-1} : \mathrm{Null}\big(\mathcal{A} - i\lambda^{\tau}_{k,0}\big)^{\perp} \longrightarrow \mathrm{Null}\big(\mathcal{A} - i\lambda^{\tau}_{k,0}\big)^{\perp}\,,$$

by

$$\big(\mathcal{A} - i\lambda^{\tau}_{k,0}\big)^{-1}\phi^{\delta}_{l,0} = \tfrac{1}{i\lambda^{\delta}_{l,0} - i\lambda^{\tau}_{k,0}}\phi^{\delta}_{l,0}\,, \quad \text{for any} \quad \phi^{\delta}_{l,0} \quad \text{with} \quad \lambda^{\delta}_{l,0} \neq \lambda^{\tau}_{k,0}\,, \tag{2.3}$$

and

$$\big(\mathcal{A} - i\lambda^{\tau}_{k,0}\big)^{-1}(\rho, \mathbf{u}, -\rho)^T = \tfrac{1}{i\lambda^{\tau}_{k,0}}(\rho, \mathbf{u}, -\rho)^T\,, \tag{2.4}$$

for any $(\rho, \mathbf{u}, -\rho)^T \in \mathrm{Null}(\mathcal{A})$ and $\tau, \delta \in \{+,-\}$. It is obvious that this pseudo-inverse operator is a bounded operator.

The following lemma will be used frequently in this paper.

Lemma 2.1. *For each acoustic mode $k \geq 1$ and $\tau = +$ or $-$, let $\phi^\tau_{k,0}$ be defined in* (1.2), *and let ν^τ_k be a given number and f^τ_k and g^τ_k be given vectors. Then the following system for $\phi^\tau_k = (\rho^\tau_k\,, \mathbf{u}^\tau_k\,, \theta^\tau_k)^T$*

$$\begin{aligned}(\mathcal{A} - i\lambda^\tau_{k,0})\phi^\tau_k = i\nu^\tau_k \phi^\tau_{k,0} + f^\tau_k\,,\\ \mathbf{u}^\tau_k{\cdot}\mathrm{n} = g^\tau_k \quad on \quad \partial\Omega\end{aligned} \tag{2.5}$$

can be solved modulo Null$(\mathcal{A} - i\lambda^\tau_{k,0})$ under the following two conditions:

- *$i\nu^\tau_k$ satisfies*
$$i\nu^\tau_k = \int_{\partial\Omega} g^\tau_k \Psi_{k,0} \mathrm{d}\sigma_x - \langle f^\tau_k | \phi^\tau_{k,0}\rangle\,. \tag{2.6}$$
- *If $\lambda_{k,0}$ is an eigenvalue with multiplicity greater than 1, a compatibility condition is needed*
$$\int_{\partial\Omega} g^\tau_k \Psi_{l,0} \mathrm{d}\sigma_x = \langle f^\tau_k | \phi^\tau_{l,0}\rangle\,, \quad for \quad \lambda_{l,0} = \lambda_{k,0} \quad with \quad k \neq l\,. \tag{2.7}$$

Under these two conditions, the solutions to (2.5) *can be uniquely represented as*

$$\phi^\tau_k = P_0\phi^\tau_k + P_0^\perp \phi^\tau_k = \sum_{\lambda_{k,0}=\lambda_{l,0}} \langle \phi^\tau_k \,|\, \phi^\tau_{l,0}\rangle \phi^\tau_{l,0} + P_0^\perp \phi^\tau_k\,, \tag{2.8}$$

where $P_0^\perp \phi^\tau_k \in$ Null$\big(\mathcal{A} - i\lambda^\tau_{k,0}\big)^\perp$ is completely determined, and $P_0\phi^\tau_k$ is the orthogonal projection on Null$\big(\mathcal{A} - i\lambda^\tau_{k,0}\big)$ which is not determined.

Proof. For any $g^\tau_k \in H^{\frac{1}{2}}(\partial\Omega)$, there exists $\tilde{\mathbf{u}}^\tau_k \in H^1(\Omega\,;\mathbb{R}^\mathrm{D})$, such that $\gamma\tilde{\mathbf{u}}^\tau_k{\cdot}\mathrm{n} = g^\tau_k$, where γ is the usual trace operator from $H^1(\Omega\,;\mathbb{R}^\mathrm{D})$ to $H^{\frac{1}{2}}(\partial\Omega)$. We define

$$\tilde{\phi}^\tau_k = \phi^\tau_k - (0, \tilde{\mathbf{u}}^\tau_k, 0)^T\,. \tag{2.9}$$

Then $\tilde{\phi}^\tau_k$ has zero the normal velocity on the boundary $\partial\Omega$, thus is in the domain of $\mathcal{A}$. From (2.5), $\tilde{\phi}^\tau_k$ satisfies

$$(\mathcal{A} - i\lambda^\tau_{k,0})\tilde{\phi}^\tau_k = -(\mathcal{A} - i\lambda^\tau_{k,0})(0, \tilde{\mathbf{u}}^\tau_k, 0) + i\nu^\tau_k \phi^\tau_{k,0} + f^\tau_k\,. \tag{2.10}$$

The solvability of (2.10) is that the right-hand side must be in Null$\big(\mathcal{A} - i\lambda^\tau_{k,0}\big)^\perp$. Thus, the inner product of (2.10) with $\phi^\tau_{k,0}$ is zero, which gives (2.6), while the inner product with $\phi^\tau_{l,0}$ with $\lambda_{k,0} = \lambda_{l,0}, k \neq l$ gives (2.7). Under these conditions, by applying the pseudo-inverse operator $\big(\mathcal{A} - i\lambda^\tau_{k,0}\big)^{-1}$ defined in (2.3)-(2.4), we can uniquely solve $\tilde{\phi}^\tau_k$ in Null$\big(\mathcal{A} - i\lambda^\tau_{k,0}\big)^\perp$, denoted by $\bar{\phi}^\tau_k$. However, the projection of $\tilde{\phi}^\tau_k$ on Null$\big(\mathcal{A} - i\lambda^\tau_{k,0}\big)$ is *not* determined. In other words,

$$\tilde{\phi}^\tau_k = \bar{\phi}^\tau_k + \sum_{\lambda_{k,0}=\lambda_{l,0}} \langle \tilde{\phi}^\tau_k \,|\, \phi^\tau_{l,0}\rangle \phi^\tau_{l,0}\,.$$

Using (2.9), we get (2.8), where

$$P_0^\perp \phi^\tau_k = \bar{\phi}^\tau_k + (0, \tilde{\mathbf{u}}^\tau_k, 0)^T - \sum_{\lambda_{k,0}=\lambda_{l,0}} \langle (0, \tilde{\mathbf{u}}^\tau_k, 0)^T \,|\, \phi^\tau_{l,0}\rangle \phi^\tau_{l,0}\,.$$

In (2.8), the projection of ϕ^τ_k on Null$\big(\mathcal{A} - i\lambda^\tau_{k,0}\big)$, i.e. the first term in the right-hand side of (2.8), can *not* be uniquely determined. It is easy to see that the projection of ϕ^τ_k on Null$\big(\mathcal{A} - i\lambda^\tau_{k,0}\big)^\perp$, i.e. $P_0^\perp \phi^\tau_k$, is uniquely determined, although the lifting of the trace g^τ_k is not unique.

□

2.2. **Geometry of the boundary $\partial\Omega$.** Next, we collect some differential geometry properties related to the boundary $\partial\Omega$ which can be considered as a $(d-1)$ dimension compact Riemannian manifold with a metric induced from the standard Euclidean metric of $\mathbb{R}^d$. Let $T(\partial\Omega)$ and $T(\partial\Omega)^\perp$ denote the tangent and normal bundles of $\partial\Omega$ in $\mathbb{R}^d$ respectively.

There is a tubular neighborhood $U_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$ of $\partial\Omega$ such that the nearest point projection map is well defined and smooth. More precisely, we have the following lemma:

Lemma 2.2. *If $\partial\Omega$ is a compact C^k submanifold of dimension $d-1$ embedded in $\mathbb{R}^d$, then there is $\delta = \delta_{\partial\Omega} > 0$ and a map $\pi \in C^{k-1}(U_\delta\,; \partial\Omega)$ such that the following properties hold:*

(i): for all $x \in \Omega \subset \mathbb{R}^d$ with $\mathrm{dist}(x\,, \partial\Omega) < \delta$;

$$\pi(x) \in \partial\Omega\,, \quad x - \pi(x) \in T^\perp_{\pi(x)}(\partial\Omega)\,, \quad |x - \pi(x)| = \mathrm{dist}(x\,, \partial\Omega)\,, \textit{and}$$

$$|z - x| > \mathrm{dist}(x\,, \partial\Omega) \quad \textit{for any} \quad z \in \partial\Omega \setminus \{\pi(x)\}\,;$$

(ii):

$$\pi(x+z) \equiv x\,, \quad \textit{for} \quad x \in \partial\Omega\,, z \in T_x(\partial\Omega)^\perp\,, |z| < \delta\,,$$

(iii): Let $\mathrm{Hess}\pi_x$ denote the Hessian of π at x, then

$$\mathrm{Hess}\pi_x(V_1\,, V_2) = \mathrm{h}_x(V_1\,, V_2)\,, \quad \textit{for} \quad x \in \partial\Omega \quad V_1\,, V_2 \in T_x(\partial\Omega)\,,$$

where h_x is the second fundamental form of $\partial\Omega$ at x.

The proof of this lemma is classical, for example, see [13], where the lemma is proved for general submanifold.

The viscous boundary layer we will construct has significantly different behavior over the tangential and normal directions near the boundary. This inspire us to consider the following new coordinate system, which we call the curvilinear coordinate for the tubular neighborhood U_δ defined in Lemma 2.2. Because $\partial\Omega$ is a $(d-1)$ dimensional manifold, so locally $\pi(x)$ can be represented as

$$\pi(x) = (\pi^1(x)\,, \cdots\,, \pi^{d-1}(x))\,. \tag{2.11}$$

More precisely, the representation (2.11) could be understood in the following sense: we can introduce a new coordinate system $(\xi^1\,, \cdots\,, \xi^d)$ by a homeomorphism which is locally defined as $\xi : \xi(x) = (\xi'(x)\,, \xi^d(x))$ where $\xi' = (\xi^1\,, \cdots\,, \xi^{d-1})$, such that $\xi(\pi(x)) = (\xi', 0)$ and $\mathrm{d}(x) = \xi^d$, where $\mathrm{d}(x)$ is the distance function to the boundary $\partial\Omega$, i.e.

$$\mathrm{d}(x) = \mathrm{dist}(x\,, \partial\Omega) = |x - \pi(x)|\,. \tag{2.12}$$

For the simplicity of notation, we denote "$\xi'(x) = \pi(x)$" which is the meaning of (2.11).

It is easy to see that $\nabla_x \mathrm{d}$ is perpendicular to the level surface of the distance function d, i.e. the set $S^z = \{x \in \Omega : \mathrm{d}(x) = z\}$. In particular, on the boundary, $\nabla_x \mathrm{d}$ is perpendicular to $S_0 = \partial\Omega$. Without loss of generality, we can normalize the distance function so that $\nabla_x \mathrm{d}(x) = -\mathrm{n}(x)$ when $x \in \partial\Omega$. By the definition of the projection π, we have

$$\pi(x + t\nabla_x \mathrm{d}(x)) = \pi(x) \quad \text{for } t \text{ small}\,, \tag{2.13}$$

and consequently, $\nabla_x \pi^\alpha \cdot \nabla_x \mathrm{d} = 0$, for $\alpha = 1\,, \cdots\,, d-1$. In particular, for t small enough, $\nabla_x \pi^\alpha(x) \in T_x(\partial\Omega)$ when $x \in \partial\Omega$.

In the following section we prove the Proposition 1.1, one of the key idea is that the boundary layer terms have different length scales on the tangential and normal directions of each level set S^z, in particular the boundary $\partial\Omega = S^0$. So in order to solve the ansatz, we need to project the vector fields onto tangential and normal directions by inner product with $\nabla_x \pi^\alpha$ and $\nabla_x \mathrm{d}$.

Next, we calculate the induced Riemannian metric from $\mathbb{R}^d$ on the family of level set S^z. In a local coordinate system, these Riemannian metric can be represented as

$$g = g_{\alpha\beta} \mathrm{d}\pi^\alpha \otimes \mathrm{d}\pi^\beta\,,$$

where $g_{\alpha\beta} = \langle \frac{\partial}{\partial \pi^\alpha}, \frac{\partial}{\partial \pi^\beta} \rangle$. Noticing that $\frac{\partial}{\partial x^i} = \frac{\partial \pi^\alpha}{\partial x^i}\frac{\partial}{\partial \pi^\alpha}$, and $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = \delta_{ij}$, the metric $g_{\alpha\beta}$ can be determined by

$$g_{\alpha\beta}\frac{\partial \pi^\alpha}{\partial x^i}\frac{\partial \pi^\beta}{\partial x^i} = 1\,.$$

3. Proof of Theorem 1.1

In this section, we prove the Theorem 1.1. The main idea is to build approximate modes of $\mathcal{A}_\varepsilon$ in terms of $\phi^\pm_{k,0}$, we formally solve the equation

$$\mathcal{A}_\varepsilon \phi^\pm_{k,\varepsilon,N} = i\lambda^\pm_{k,\varepsilon,N}\phi^\pm_{k,\varepsilon,N} + R^\pm_{k,\varepsilon,N}\,, \tag{3.1}$$

where we make for $\phi^\pm_{k,\varepsilon,N}$ and $\lambda^\pm_{k,\varepsilon,N}$ the following ansatz.

$$\phi^\pm_{k,\varepsilon,N} = \sum_{i=0}^N \left(\sqrt{\varepsilon}^i\phi^{\pm,\mathrm{int}}_{k,i}(x) + \sqrt{\varepsilon}^i\phi^{\pm,\mathrm{b}}_{k,i}(\pi(x), \tfrac{\mathrm{d}(x)}{\sqrt{\varepsilon}})\chi(x)\right)\,, \tag{3.2}$$

and $\lambda^\pm_{k,\varepsilon,N} = \sum\limits_{i=0}^N \sqrt{\varepsilon}^i\lambda^\pm_{k,i}$, where $\phi^{\pm,\mathrm{int}}_{k,i} = (\rho^{\pm,\mathrm{int}}_{k,i}, \mathbf{u}^{\pm,\mathrm{int}}_{k,i}, \theta^{\pm,\mathrm{int}})^T$ and $\phi^{\pm,\mathrm{b}}_{k,i} = (\rho^{\pm,\mathrm{b}}_{k,i}, \mathbf{u}^{\pm,\mathrm{b}}_{k,i}, \theta^{\pm,\mathrm{b}})^T$ are smooth functions with boundary conditions $\mathbf{u}^{\pm,\mathrm{int}}_{k,i} + \mathbf{u}^{\pm,\mathrm{b}}_{k,i} = 0$ on $\partial\Omega$, $\phi^{\pm,\mathrm{b}}_{k,i}$ being rapidly decreasing to 0 in the ζ variable defined by $\zeta = \frac{\mathrm{d}(x)}{\sqrt{\varepsilon}}$. We also require that for $i \geq 1$, we have

$$\langle \phi^{\pm,\mathrm{int}}_{k,j} | \phi^{\pm,\mathrm{int}}_{k,0} \rangle = 0 \tag{3.3}$$

for all $j \geq 1$. Here the function $\mathrm{d}(x)$ is defined in (2.12).

In the ansatz (3.2), $\chi(y) \in C_0^\infty(\Omega)$ is a smooth cut-off function such that $\chi(y) = 1$ in a neighborhood of $\partial\Omega$ and $\chi(y) = 0$ if $\mathrm{d}(y) > \delta$ for some δ small enough. Here $\delta > 0$ is taken as in Lemma 2.2 so that π is uniquely determined in $\{0 < \mathrm{d}(y) < \delta\}$.

Straightforward calculations show that for $\phi^{\mathrm{b}} = (\rho^{\mathrm{b}}, \mathbf{u}^{\mathrm{b}}, \theta^{\mathrm{b}})^T$,

$$\mathcal{A}_\varepsilon\phi^{\mathrm{b}} = \tfrac{1}{\sqrt{\varepsilon}}\mathcal{A}^{\mathrm{d}}\phi^{\mathrm{b}} + \mathcal{A}^\pi\phi^{\mathrm{b}} + \mathcal{D}^{\mathrm{b}}\phi^{\mathrm{b}} + \sqrt{\varepsilon}\mathcal{F}^{\mathrm{b}}\phi^{\mathrm{b}} + \varepsilon\mathcal{G}^{\mathrm{b}}\phi^{\mathrm{b}}\,,$$

where

$$\mathcal{A}^{\mathrm{d}}\phi^{\mathrm{b}} = \begin{pmatrix} \partial_\zeta(\mathbf{u}^{\mathrm{b}}\cdot\nabla_x\mathrm{d}) \\ \partial_\zeta(\rho^{\mathrm{b}} + \theta^{\mathrm{b}})\nabla_x\mathrm{d} \\ \gamma\partial_\zeta(\mathbf{u}^{\mathrm{b}}\cdot\nabla_x\mathrm{d}) \end{pmatrix}, \quad \mathcal{A}^\pi\phi^{\mathrm{b}} = \begin{pmatrix} \partial_{\pi^\alpha}(\mathbf{u}^{\mathrm{b}}\cdot\nabla_x\pi^\alpha) \\ \partial_{\pi^\alpha}(\rho^{\mathrm{b}} + \theta^{\mathrm{b}})\nabla_x\pi^\alpha \\ \gamma\partial_{\pi^\alpha}(\mathbf{u}^{\mathrm{b}}\cdot\nabla_x\pi^\alpha) \end{pmatrix},$$

$$\mathcal{D}^{\mathrm{b}}\phi^{\mathrm{b}} = \begin{pmatrix} 0 \\ \nu|\nabla_x\mathrm{d}|^2\partial_\zeta^2\mathbf{u}^{\mathrm{b}} + (\xi+\nu)\partial_\zeta^2(\mathbf{u}^{\mathrm{b}}\cdot\nabla_x\mathrm{d})\nabla_x\mathrm{d} \\ \kappa|\nabla_x\mathrm{d}|^2\partial_\zeta^2\theta^{\mathrm{b}} \end{pmatrix},$$

$$\begin{aligned}\mathcal{F}^{\mathrm{b}}\phi^{\mathrm{b}} = &\begin{pmatrix} 0 \\ 2\nu\partial^2_{\zeta\pi^\alpha}\mathbf{u}^{\mathrm{b}}(\nabla_x\mathrm{d}\cdot\nabla_x\pi^\alpha) + (\xi+\nu)[(\partial^2_{\zeta\pi^\alpha}\mathbf{u}^{\mathrm{b}}\cdot\nabla_x\pi^\alpha)\nabla_x\mathrm{d} + (\partial^2_{\zeta\pi^\alpha}\mathbf{u}^{\mathrm{b}}\cdot\nabla_x\mathrm{d})\nabla_x\pi^\alpha] \\ 2\kappa\partial^2_{\zeta\pi^\alpha}\theta^{\mathrm{b}}(\nabla_x\mathrm{d}\cdot\nabla_x\pi^\alpha) \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ \nu\partial_\zeta\mathbf{u}^{\mathrm{b}}\Delta_x\mathrm{d} + (\xi+\nu)\partial_\zeta\mathbf{u}^{\mathrm{b}}\cdot\nabla_x^2\mathrm{d} \\ \kappa\partial_\zeta\theta^{\mathrm{b}}\Delta_x\mathrm{d} \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}^{\mathrm{b}}\phi^{\mathrm{b}} = &\begin{pmatrix} 0 \\ \nu[\partial_{\pi^\alpha}\mathbf{u}^{\mathrm{b}}\Delta_x\pi^\alpha + \partial^2_{\pi^\alpha\pi^\beta}\mathbf{u}^{\mathrm{b}}(\nabla_x\pi^\alpha\cdot\nabla_x\pi^\beta)] \\ \kappa[\partial_{\pi^\alpha}\theta^{\mathrm{b}}\Delta_x\pi^\alpha + \partial^2_{\pi^\alpha\pi^\beta}\theta^{\mathrm{b}}(\nabla_x\pi^\alpha\cdot\nabla_x\pi^\beta)] \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ (\xi+\nu)[\partial_{\pi^\alpha}\mathbf{u}^{\mathrm{b}}\cdot\nabla_x^2\pi^\alpha + (\partial^2_{\pi^\alpha\pi^\beta}\mathbf{u}^{\mathrm{b}}\cdot\nabla_x\pi^\alpha)\nabla_x\pi^\beta] \\ 0 \end{pmatrix}.\end{aligned}$$

We will use the following notations for the simplicity: $\mathcal{F}^{\rm b}\phi^{\rm b} = (0, \mathcal{F}^{\mathbf{u}}, \mathcal{F}^{\theta})^T$ and $\mathcal{G}^{\rm b}\phi^{\rm b} = (0, \mathcal{G}^{\mathbf{u}}, \mathcal{G}^{\theta})^T$, and furthermore,

$$
\begin{aligned}
\mathcal{F}^{\mathbf{u}}\cdot \nabla_x\pi &= \mathcal{F}^{\mathbf{u}}_{\pi\pi}(\mathbf{u}^{\rm b}\cdot\nabla_x\pi) + \mathcal{F}^{\mathbf{u}}_{\pi d}(\mathbf{u}^{\rm b}\cdot\nabla_x{\rm d})\,,\\
\mathcal{F}^{\mathbf{u}}\cdot \nabla_x{\rm d} &= \mathcal{F}^{\mathbf{u}}_{d\pi}(\mathbf{u}^{\rm b}\cdot\nabla_x\pi) + \mathcal{F}^{\mathbf{u}}_{dd}(\mathbf{u}^{\rm b}\cdot\nabla_x{\rm d})\,,\\
\mathcal{G}^{\mathbf{u}}\cdot \nabla_x\pi &= \mathcal{G}^{\mathbf{u}}_{\pi\pi}(\mathbf{u}^{\rm b}\cdot\nabla_x\pi) + \mathcal{G}^{\mathbf{u}}_{\pi d}(\mathbf{u}^{\rm b}\cdot\nabla_x{\rm d})\,,\\
\mathcal{G}^{\mathbf{u}}\cdot \nabla_x{\rm d} &= \mathcal{G}^{\mathbf{u}}_{d\pi}(\mathbf{u}^{\rm b}\cdot\nabla_x\pi) + \mathcal{G}^{\mathbf{u}}_{dd}(\mathbf{u}^{\rm b}\cdot\nabla_x{\rm d})\,,
\end{aligned}
$$

where $\mathcal{F}^{\mathbf{u}}_{\pi\pi}, \mathcal{F}^{\mathbf{u}}_{\pi d}, \mathcal{F}^{\mathbf{u}}_{d\pi}, \mathcal{F}^{\mathbf{u}}_{dd}, \mathcal{G}^{\mathbf{u}}_{\pi\pi}, \mathcal{G}^{\mathbf{u}}_{\pi d}, \mathcal{G}^{\mathbf{u}}_{d\pi}, \mathcal{G}^{\mathbf{u}}_{dd}$ are linear functions. For the θ components, we use the similar notations.

Now we start to solve the equation (3.1) inductively.

Step 0 : First, the order $\sqrt{\varepsilon}^{-1}$ in the boundary layer gives $\mathcal{A}^{\rm d}\phi^{\rm b}_{k,0} = 0$ which implies that

$$
\mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x{\rm d} = 0 \quad\text{and}\quad \rho^{\rm b}_{k,0} + \theta^{\rm b}_{k,0} = 0\,.
$$

In particular, $\mathbf{u}^{\rm b}_{k,0}\cdot{\rm n} = 0$ and consequently $\mathbf{u}^{\rm int}_{k,0}\cdot{\rm n} = 0$ on the boundary $\partial\Omega$.

The order $\sqrt{\varepsilon}^0$ of the interior part in (3.1) implies that

$$
\mathcal{A}\phi^{\pm,\rm int}_{k,0} = i\lambda^{\pm}_{k,0}\phi^{\pm,\rm int}_{k,0}\,.
$$

Comparing with (1.2) and (1.3), we have $\phi^{\pm,\rm int}_{k,0} = \phi^{\pm}_{k,0}$ and $\lambda^{\pm}_{k,0} = \pm i\sqrt{1+\gamma}\lambda_{k,0}$.

Remark: From now on, for the simplicity of representation, we only calculate for "+" case, the calculation for the "-" case is the same.

3.1. Order $\sqrt{\varepsilon}^0$ of the boundary layer.

The order $\sqrt{\varepsilon}^0$ in the boundary layer gives

$$
-\mathcal{A}^{\rm d}\phi^{\rm b}_{k,1} = (\mathcal{A}^{\pi} + \mathcal{D}^{\rm b} - i\lambda_{k,0})\phi^{\rm b}_{k,0},
$$

i.e.

$$
\begin{aligned}
-\begin{pmatrix} \partial_\zeta(\mathbf{u}^{\rm b}_{k,1}\cdot\nabla_x{\rm d}) \\ \partial_\zeta(\rho^{\rm b}_{k,1}+\theta^{\rm b}_{k,1})\nabla_x{\rm d} \\ \gamma\partial_\zeta(\mathbf{u}^{\rm b}_{k,1}\cdot\nabla_x{\rm d}) \end{pmatrix} &= \begin{pmatrix} \partial_{\pi^\alpha}(\mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x\pi^\alpha) \\ \partial_{\pi^\alpha}(\rho^{\rm b}_{k,0}+\theta^{\rm b}_{k,0})\nabla_x\pi^\alpha \\ \gamma\partial_{\pi^\alpha}(\mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x\pi^\alpha) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ \nu|\nabla_x{\rm d}|^2\partial^2_{\zeta\zeta}\mathbf{u}^{\rm b}_{k,0} + (\xi+\nu)\partial^2_{\zeta\zeta}(\mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x{\rm d})\nabla_x{\rm d} \\ \kappa|\nabla_x{\rm d}|^2\partial^2_{\zeta\zeta}\theta^{\rm b}_{k,0} \end{pmatrix} - i\lambda_{k,0}\begin{pmatrix} -\theta^{\rm b}_{k,0} \\ \mathbf{u}^{\rm b}_{k,0} \\ \theta^{\rm b}_{k,0} \end{pmatrix}.
\end{aligned} \tag{3.4}
$$

The process to solve this system will be illustrated in the following steps, which are the foundation to solve the more complicated ODE for $\phi^{\rm b}_{k,j}$. Before we solve the system (3.4), we recall that we already know the values of $\mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x{\rm d}$ and $\rho^{\rm b}_{k,0}+\theta^{\rm b}_{k,0}$ as well as the boundary data for $\mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x\pi$ and $\theta^{\rm b}_{k,0}$.

Step 1 is to solve $\rho^{\rm b}_{k,1}+\theta^{\rm b}_{k,1}$. This is achieved by taking the inner product of the second equation of (3.4) with $\nabla_x{\rm d}$. This gives $\partial_\zeta(\rho^{\rm b}_{k,1}+\theta^{\rm b}_{k,1}) = 0$ thus $\rho^{\rm b}_{k,1}+\theta^{\rm b}_{k,1} = 0$, noting that $\nabla_x\pi^\alpha\cdot\nabla_x{\rm d} = 0$ and $\mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x{\rm d} = 0$.

Step 2 is to solve the tangential part of $\mathbf{u}^{\rm b}_{k,0}$, i.e. $\mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x\pi$. By taking the inner product of the second equation of (3.4) with $\nabla_x\pi$, we get that $f^{\rm b} = \mathbf{u}^{\rm b}_{k,0}\cdot\nabla_x\pi$ satisfies the ODE

$$
\begin{aligned}
\mathcal{L}f^{\rm b} &= 0\,,\\
f^{\rm b}(\zeta=0) &= -\mathbf{u}^{\rm int}_{k,0}(x)\cdot\nabla_x\pi\,,\\
f^{\rm b}(\zeta\to\infty) &= 0\,,
\end{aligned} \tag{3.5}
$$

where the operator $\mathcal{L} = \nu|\nabla_x \mathrm{d}|^2\partial^2_{\zeta\zeta} - i\lambda_{k,0}$. The boundary condition second line of (3.5) should be understood as: when $\zeta = \frac{\mathrm{d}(x)}{\sqrt{\varepsilon}} = 0$, $x \in \partial\Omega$. The solution of (3.5) is given by

$$\mathbf{u}^{\mathrm{b}}_{k,0}(\pi(x),\zeta)\cdot\nabla_x\pi = \widetilde{Z}^{\mathrm{b,u}}_0(\zeta,\phi^{\mathrm{int}}_{k,0}) = -(\mathbf{u}^{\mathrm{int}}_{k,0}\cdot\nabla_x\pi)\exp\left(-\tfrac{1+i}{2}\sqrt{\tfrac{2\lambda_{k,0}}{\nu}}\tfrac{1}{|\nabla_x\mathrm{d}|}\zeta\right). \tag{3.6}$$

Note that on the right-hand side of (3.6), $\mathbf{u}^{\mathrm{int}}_{k,0}\cdot\nabla_x\pi$ in taken on the boundary $\partial\Omega$, i.e. $\zeta = 0$. Furthermore, $\widetilde{Z}^{\mathrm{b,u}}_0(\zeta,\cdot)$ is a linear function.

Step 3 is to solve $\theta^{\mathrm{b}}_{k,0}$ whose equation is obtained by subtracting γ times the first equation from the third equation of (3.4). Hence, $f^{\mathrm{b}} = \theta^{\mathrm{b}}_{k,0}$ satisfies the ODE

$$\begin{aligned}\mathcal{L}_\gamma f^{\mathrm{b}} &= 0\,,\\ f^{\mathrm{b}}(\zeta = 0) &= -\theta^{\mathrm{int}}_{k,0}\,,\\ f^{\mathrm{b}}(\zeta\to\infty) &= 0\,,\end{aligned} \tag{3.7}$$

where $\mathcal{L}_\gamma = \kappa|\nabla_x\mathrm{d}|^2\partial^2_{\zeta\zeta} - i(1+\gamma)\lambda_{k,0}$. The solution of (3.7) is given by

$$\theta^{\mathrm{b}}_{k,0}(\pi(x),\zeta) = \widetilde{Z}^{\mathrm{b},\theta}_0(\zeta,\phi^{\mathrm{int}}_{k,0}) = -\theta^{\mathrm{int}}_{k,0}\exp\left(-\tfrac{1+i}{2}\sqrt{\tfrac{2(1+\gamma)\lambda_{k,0}}{\kappa}}\tfrac{1}{|\nabla_x\mathrm{d}|}\zeta\right). \tag{3.8}$$

Again, on the right-hand side of (3.8), $\theta^{\mathrm{int}}_{k,0}$ is taken on the boundary $\partial\Omega$, i.e. $\zeta = 0$, and $\widetilde{Z}^{\mathrm{b},\theta}_0(\zeta,\cdot)$ is also a linear function.

Step 4 is to solve $\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\nabla_x\mathrm{d}$. From the first equation of (3.4) (or equivalently the third equation), we have

$$-\partial_\zeta(\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\nabla_x\mathrm{d}) = \mathrm{div}_\pi(\mathbf{u}^{\mathrm{b}}_{k,0}\cdot\nabla_x\pi) + i\lambda_{k,0}\theta^{\mathrm{b}}_{k,0}\,.$$

By integrating from ζ to ∞, we get

$$\begin{aligned}\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\nabla_x\mathrm{d} &= \widetilde{Z}^{\mathrm{b}}_1(\zeta,\phi^{\mathrm{int}}_{k,0})\\ &= -\tfrac{1}{c_\nu}\mathrm{div}_\pi(\mathbf{u}^{\mathrm{int}}_{k,0}\cdot\nabla_x\pi)\exp(-c_\nu\zeta) - \tfrac{1}{c_\kappa}i\lambda_{k,0}\theta^{\mathrm{int}}_{k,0}\exp(-c_\kappa\zeta)\,,\end{aligned} \tag{3.9}$$

where $c_\nu = \frac{1+i}{2}\sqrt{\frac{2\lambda_{k,0}}{\nu}}\frac{1}{|\nabla_x\mathrm{d}|}$ and $c_\kappa = \frac{1+i}{2}\sqrt{\frac{2(1+\gamma)\lambda_{k,0}}{\kappa}}\frac{1}{|\nabla_x\mathrm{d}|}$. In particular, setting $\zeta = 0$ in (3.9), we have that on the boundary $\partial\Omega$,

$$-\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\mathrm{n} = Z^{\mathrm{b}}_1(\phi^{\mathrm{int}}_{k,0}) = \widetilde{Z}^{\mathrm{b}}_1(0,\phi^{\mathrm{int}}_{k,0}) = -\tfrac{1}{c_\nu}\mathrm{div}_\pi(\mathbf{u}^{\mathrm{int}}_{k,0}\cdot\nabla_x\pi) - \tfrac{1}{c_\kappa}i\lambda_{k,0}\theta^{\mathrm{int}}_{k,0}\,. \tag{3.10}$$

Note that $Z^{\mathrm{b}}_1(\cdot)$ is a linear function. Now we summarize that by solving the ODE system (3.4), we determine

- Step 1 : $\rho^{\mathrm{b}}_{k,1} + \theta^{\mathrm{b}}_{k,1}$;
- Steps 2, 3 : $\mathbf{u}^{\mathrm{b}}_{k,0}\cdot\nabla_x\pi$ and $\theta^{\mathrm{b}}_{k,0}$;
- Step 4 : $\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\nabla_x\mathrm{d}$ and hence $\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\mathrm{n}$. when we take $\zeta = 0$.

Similarly, in the next round, before solving $\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\nabla_x\pi$ and $\theta^{\mathrm{b}}_{k,1}$, we need their boundary values. In other words, the terms in ansatz (3.2) are determined by solving the ODEs from the boundary layers and operator equations from the interior *alternatively.*

3.2. **Order $\sqrt{\varepsilon}$ in the interior.** **Step 5:** The order $\sqrt{\varepsilon}$ in the interior part of (3.1) yields:

$$\begin{aligned}(\mathcal{A} - i\lambda_{k,0})\phi^{\mathrm{int}}_{k,1} &= i\lambda_{k,1}\phi^{\mathrm{int}}_{k,0}\,,\\ \mathbf{u}^{\mathrm{int}}_{k,1}\cdot\mathrm{n} = -\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\mathrm{n} &= Z^{\mathrm{b}}_1(\phi^{\mathrm{int}}_{k,0})\,.\end{aligned} \tag{3.11}$$

Applying Lemma 2.1 to the system (3.11), from the formula (2.6), $i\lambda_{k,1}$ can be solved as

$$i\lambda_{k,1} = \int_{\partial\Omega}(\mathbf{u}^{\mathrm{int}}_{k,1}\cdot\mathrm{n})\Psi_{k,0}\mathrm{d}\sigma_x = -\int_{\partial\Omega}(\mathbf{u}^{\mathrm{b}}_{k,1}\cdot\mathrm{n})\Psi_{k,0}\mathrm{d}\sigma_x\,. \tag{3.12}$$

From (3.10) and the fact that $\nabla_x \Psi_{k,0} = g^{\gamma\beta}\frac{\partial \Psi_{k,0}}{\partial \pi^\beta}\frac{\partial}{\partial \pi^\gamma}$, and $\nabla_x \pi^\alpha = g^{\alpha\delta}\frac{\partial}{\partial \pi^\delta}$, we have

$$\begin{aligned}\int_{\partial\Omega} \partial_{\pi^\alpha}(\nabla_x \Psi_{k,0}\cdot\nabla_x \pi^\alpha)\Psi_{k,0}\,\mathrm{d}\sigma_x &= -\int_{\partial\Omega} g_{\gamma\delta}g^{\alpha\delta}g^{\beta\gamma}\frac{\partial \Psi_{k,0}}{\partial \pi^\alpha}\frac{\partial \Psi_{k,0}}{\partial \pi^\beta}\,\mathrm{d}\sigma_x \\ &= -\int_{\partial\Omega} |\nabla_\pi \Psi_{k,0}|^2 \mathrm{d}\sigma_x\,,\end{aligned}$$

where ∇_π is the gradient on the tangential direction of the boundary $\partial\Omega$. Thus the formula (3.12) reads

$$\begin{aligned} i\lambda^\pm_{k,1} &= \int_{\partial\Omega} Z_1^{\mathrm{b}}(\phi^{\mathrm{int}}_{k,0})\Psi_{k,0}\mathrm{d}\sigma_x \\ &= -\tfrac{1+i}{2}\tfrac{1}{\sqrt{\lambda^3_{k,0}}}\int_{\partial\Omega}\Big(\sqrt{\nu}|\nabla_x\Psi_{k,0}|^2 + \tfrac{\gamma}{1+\gamma}\sqrt{\kappa}\lambda^2_{k,0}|\Psi_{k,0}|^2\Big)\,\mathrm{d}\sigma_x\,. \end{aligned} \tag{3.13}$$

An important property of (3.13) is that the real part of $i\lambda^\pm_{k,1}$ is strictly negative:

$$\mathfrak{Re}(i\lambda^\pm_{k,1}) < 0\,. \tag{3.14}$$

Remark: If only the isentropic case was treated as in [4], there would be no second term in the integrand of (3.13). We will only have $\mathfrak{Re}(i\lambda^\pm_{k,1}) \leq 0$. As mentioned in [4], the strict negativity is related the famous Schiffer's conjecture. Interestingly, in the present paper, because of the additional dissipation from heat conductivity, the geometric condition related the Schiffer's conjecture as in [4] is not needed.

If the multiplicity of $\lambda = \lambda_{k,0}$ as the eigenvalue of $L_0 = \Delta_x$ is greater than 1, then from (2.7), the following compatibility condition must be satisfied:

$$\int_{\partial\Omega} Z_1^{\mathrm{b}}(\phi^{\mathrm{int}}_{k,0})\Psi_{l,0}\mathrm{d}\sigma_x = 0\,,\quad \text{for}\quad \lambda_{l,0}=\lambda_{k,0}\quad\text{with}\quad k\neq l\,, \tag{3.15}$$

which reads as

$$-\tfrac{1+i}{2}\tfrac{1}{\sqrt{\lambda^3_{k,0}}}\int_{\partial\Omega}\Big(\sqrt{\nu}\nabla_x\Psi_{k,0}\cdot\nabla_x\Psi_{l,0} + \tfrac{\gamma}{1+\gamma}\sqrt{\kappa}\lambda^2_{k,0}\Psi_{k,0}\Psi_{l,0}\Big)\,\mathrm{d}\sigma_x = 0\,, \tag{3.16}$$

for $\lambda_{l,0} = \lambda_{k,0}$ and $k \neq l$. For the isentropic case (i.e. $\kappa = 0$), it is the condition (10) in [4]. We can define the quadratic form Q_1 and the symmetric operator L_1 on $\mathrm{H}_0(\lambda)$ as

$$Q_1(\Psi_{k,0},\Psi_{l.0}) = \int_{\partial\Omega} Z_1^{\mathrm{b}}(\phi^{\mathrm{int}}_{k,0})\Psi_{l,0}\mathrm{d}\sigma_x\,, \tag{3.17}$$

and

$$L_1\Psi_{k,0} = i\lambda_{k,1}\Psi_{k,0}\,, \tag{3.18}$$

which satisfies that

$$Q_1(\Psi_{k,0},\Psi_{l,0}) = \int_\Omega L_1(\Psi_{k,0})\Psi_{l,0}\,, \tag{3.19}$$

and the orthogonality condition (3.15) is

$$Q_1(\Psi_{k,0},\Psi_{l,0}) = 0\,,\quad \text{if}\quad \Psi_{k,0},\Psi_{l,0}\in \mathrm{H}_0(\lambda)\quad\text{and}\quad k\neq l\,. \tag{3.20}$$

From Lemma 2.1, under these conditions, the solution to (3.11) is

$$\phi^{\mathrm{int}}_{k,1} = P_0^\perp\phi^{\mathrm{int}}_{k,1} + P_0\phi^{\mathrm{int}}_{k,1}\,, \tag{3.21}$$

When $i\lambda_{k,0}$ is a simple eigenvalue of Δ_x, then $P_0\phi^{\mathrm{int}}_{k,1} \in \mathrm{Null}(\mathcal{A})$ vanishes. For this case, $\phi^{\mathrm{int}}_{k,1} = P_0^\perp\phi^{\mathrm{int}}_{k,1} \in \mathrm{Null}(\mathcal{A})^\perp$ is completely determined. When $i\lambda_{k,0}$ is not simple, $P_0\phi^{\mathrm{int}}_{k,1}$ is to be determined.

Note that the system (3.11) is linear and the boundary data $Z_1^{\mathrm{b}}(\phi_{k,0}^{\mathrm{int}})$ is also linear in $\phi_{k,0}^{\mathrm{int}}$. So $P_0^{\perp}\phi_{k,1}^{\mathrm{int}}$ also linearly depends on $P_0^{\perp}\phi_{k,1}^{\mathrm{int}}$. Thus we denote

$$P_0^{\perp}\phi_{k,1}^{\mathrm{int}} = Z_1^{\mathrm{int}}(\phi_{k,0}^{\mathrm{int}})\,, \tag{3.22}$$

where $Z_1^{\mathrm{int}}(\cdot)$ is a linear function. Furthermore,

$$P_0\phi_{k,1}^{\mathrm{int}} = \sum_{l\neq k,\lambda_{l,0}=\lambda_{k,0}} a_{kl,1}\phi_{l,0}\,,$$

where $a_{kl,1} = \langle \phi_{k,1}^{\mathrm{int}}|\phi_{l,0}\rangle$ will be determined later. Here, we used that $a_{kk,1} = 0$. Note that the boundary data for $\mathbf{u}_{k,1}^{\mathrm{b}}\cdot\nabla_x\pi$ and $\theta_{k,1}^{\mathrm{b}}$ are determined modulo $P_0\phi_{k,1}^{\mathrm{int}}$. Later on, we will use the notation:

$$a_{kl,j} = \langle \phi_{k,j}^{\mathrm{int}}|\phi_{l,0}\rangle\,.$$

3.3. **Order $\sqrt{\varepsilon}$ of the boundary layer.** The order $\sqrt{\varepsilon}$ in the boundary layer is

$$-\mathcal{A}^{\mathrm{d}}\phi_{k,2}^{\mathrm{b}} = (\mathcal{A}^{\pi} + \mathcal{D}^{\mathrm{b}} - i\lambda_{k,0})\phi_{k,1}^{\mathrm{b}} + (\mathcal{F}^{\mathrm{b}} - i\lambda_{k,1})\phi_{k,0}^{\mathrm{b}}\,. \tag{3.23}$$

Step 1: As before, step 1 is to find the ODE satisfied by $\rho_{k,2}^{\mathrm{b}} + \theta_{k,2}^{\mathrm{b}}$ which is

$$-\partial_\zeta(\rho_{k,2}^{\mathrm{b}} + \theta_{k,2}^{\mathrm{b}})|\nabla_x \mathrm{d}|^2 = \big((\xi+2\nu)|\nabla_x\mathrm{d}|^2\partial_{\zeta\zeta}^2 - i\lambda_{k,0}\big)\,(\mathbf{u}_{k,1}^{\mathrm{b}}\cdot\nabla_x\mathrm{d}) + \mathcal{F}_{d\pi}^{\mathbf{u}}(\mathbf{u}_{k,0}^{\mathrm{b}}\cdot\nabla_x\pi)\,.$$

The right-hand side of the above equation is a linear operator on $\phi_{k,0}^{\mathrm{int}}$, (noticing the notations in (3.6) and (3.9)). Integrating from ζ to ∞ gives

$$\rho_{k,2}^{\mathrm{b}} + \theta_{k,2}^{\mathrm{b}} = Y_2^{\mathrm{b}}(\zeta,\phi_{k,0}^{\mathrm{int}})\,, \tag{3.24}$$

where $Y_2^{\mathrm{b}}(\zeta,\cdot)$ is a linear function. Note that $\rho_{k,j}^{\mathrm{b}} + \theta_{k,j}^{\mathrm{b}} = Y_j^{\mathrm{b}} = 0$ for $j = 0, 1$.

Step 2 is to solve $\mathbf{u}_{k,1}^{\mathrm{b}}\cdot\nabla_x\pi$. Using the same method which derives the ODE (3.5), i.e. taking the inner product of the second equation of (3.23) with $\nabla_x\pi$, we get that $f^{\mathrm{b}} = \mathbf{u}_{k,1}^{\mathrm{b}}\cdot\nabla_x\pi$ satisfies the ODE

$$\begin{aligned} \mathcal{L}(f^{\mathrm{b}}) &= (i\lambda_{k,1} - \mathcal{F}_{\pi\pi}^{\mathbf{u}})(\mathbf{u}_{k,0}^{\mathrm{b}}\cdot\nabla_x\pi)\,,\\ f^{\mathrm{b}}(\zeta=0) &= -\mathbf{u}_{k,1}^{\mathrm{int}}\cdot\nabla_x\pi\\ &= -(P_0\phi_{k,1}^{\mathrm{int}})_{\mathbf{u}}\cdot\nabla_x\pi - \big(Z_1^{\mathrm{int}}(\phi_{k,0}^{\mathrm{int}})\big)_{\mathbf{u}}\cdot\nabla_x\pi\,,\\ f^{\mathrm{b}}(\zeta\to\infty) &= 0\,, \end{aligned} \tag{3.25}$$

where $(\cdot)_{\mathbf{u}}$ denotes the $\mathbf{u}$ component. Again, for the second equation in (3.25), the terms on the right-hand side are taken values on the boundary $\partial\Omega$. From the linearity of the above ODE, we can decompose $\mathbf{u}_{k,1}^{\mathrm{b}}\cdot\nabla_x\pi = f_1^{\mathrm{b}} + f_2^{\mathrm{b}}$, where f_1^{b} is the solution of the ODE

$$\begin{aligned} \mathcal{L}f_1^{\mathrm{b}} &= 0\,,\\ f_1^{\mathrm{b}}(\zeta=0) &= -(P_0\phi_{k,1}^{\mathrm{int}})_{\mathbf{u}}\cdot\nabla_x\pi = -\sum_{l\neq k,\lambda_{l,0}=\lambda_{k,0}} a_{kl,1}\mathbf{u}_{l,0}^{\mathrm{int}}\cdot\nabla_x\pi\,,\\ f_1^{\mathrm{b}}(\zeta\to\infty) &= 0\,, \end{aligned} \tag{3.26}$$

Note that this ODE is the same as (3.5) with the linear combination of initial data, and $\widetilde{Z}_0^{\mathrm{b,u}}(\zeta,\cdot)$ is linear, so the solution is represented as $f_1^{\mathrm{b}} = \widetilde{Z}_0^{\mathrm{b,u}}(\zeta, P_0\phi_{k,1}^{\mathrm{int}})$. Besides, f_2^{b} satisfies the ODE

$$\begin{aligned} \mathcal{L}f_2^{\mathrm{b}} &= (i\lambda_{k,1} - \mathcal{F}_{\pi\pi}^{\mathbf{u}})\Big(\Big(\widetilde{Z}_0^{\mathrm{b,u}}(\phi_{k,0}^{\mathrm{int}})\Big)_{\mathbf{u}}\Big)\,,\\ f_2^{\mathrm{b}}(\zeta=0) &= -\big(Z_1^{\mathrm{int}}(\phi_{k,0}^{\mathrm{int}})\big)_{\mathbf{u}}\cdot\nabla_x\pi\,,\\ f_2^{\mathrm{b}}(\zeta\to\infty) &= 0\,, \end{aligned} \tag{3.27}$$

whose solution is represented as $f_2^{\mathrm{b}} = \widetilde{Z}_1^{\mathrm{b,u}}(\zeta, \phi_{k,0}^{\mathrm{int}})$, where $\widetilde{Z}_1^{\mathrm{b,u}}(\zeta, \cdot)$ is a linear function. Thus,

$$\mathbf{u}_{k,1}^{\mathrm{b}} \cdot \nabla_x \pi = \widetilde{Z}_0^{\mathrm{b,u}}(\zeta, P_0 \phi_{k,1}^{\mathrm{int}}) + \widetilde{Z}_1^{\mathrm{b,u}}(\zeta, \phi_{k,0}^{\mathrm{int}}) .$$

Step 3 is to solve $\theta_{k,1}^{\mathrm{b}}$. The method is the same as in deriving (3.7), i.e. subtracting γ times the first equation from the third equation of (3.23). Hence, $\theta_{k,1}^{\mathrm{b}}$ satisfies the ODE

$$\begin{aligned} \mathcal{L}_\gamma f^{\mathrm{b}} &= i\lambda_{k,1}(1+\gamma)\theta_{k,0}^{\mathrm{b}} - \mathcal{F}^\theta(\theta_{k,0}^{\mathrm{b}}) , \\ f^{\mathrm{b}}(\zeta = 0) &= -P_0(\phi_{k,1}^{\mathrm{int}})_\theta - \left(Z_1^{\mathrm{int}}(\phi_{k,0}^{\mathrm{int}})\right)_\theta , \\ f^{\mathrm{b}}(\zeta \to \infty) &= 0 , \end{aligned} \tag{3.28}$$

where $(\cdot)_\theta$ denotes the θ component. Using the same argument as in the last step, we can represent the solution of (3.28) as $\theta_{k,1}^{\mathrm{b}} = \widetilde{Z}_0^{\mathrm{b},\theta}(\zeta, P_0\phi_{k,1}^{\mathrm{int}}) + \widetilde{Z}_1^{\mathrm{b},\theta}(\zeta, \theta_{k,0}^{\mathrm{int}})$.

Step 4 is to get $\mathbf{u}_{k,2}^{\mathrm{b}} \cdot \nabla_x \mathrm{d} = \widetilde{Z}_1^{\mathrm{b}}(\zeta, P_0\phi_{k,1}^{\mathrm{int}})) + \widetilde{Z}_2^{\mathrm{b}}(\zeta, \phi_{k,0}^{\mathrm{int}})$. Then setting $\zeta = 0$, we have that on the boundary $\partial\Omega$

$$-\mathbf{u}_{k,2}^{\mathrm{b}} \cdot \mathrm{n} = Z_1^{\mathrm{b}}(P_0 \phi_{k,1}^{\mathrm{int}}) + Z_2^{\mathrm{b}}(\phi_{k,0}^{\mathrm{int}}) . \tag{3.29}$$

3.4. **Order ε of the interior.** The order ε of the interior part in (3.1) reads

$$\begin{aligned} (\mathcal{A} - i\lambda_{k,0})\phi_{k,2}^{\mathrm{int}} &= i\lambda_{k,1}\phi_{k,1}^{\mathrm{int}} + (i\lambda_{k,2} - \mathcal{D})\phi_{k,0}^{\mathrm{int}} , \\ \mathbf{u}_{k,2}^{\mathrm{int}} \cdot \mathrm{n} &= Z_1^{\mathrm{b}}(P_0\phi_{k,1}^{\mathrm{int}}) + Z_2^{\mathrm{b}}(\phi_{k,0}^{\mathrm{int}}) . \end{aligned} \tag{3.30}$$

Here we use again the relation $\mathbf{u}_{k,2}^{\mathrm{int}} = -\mathbf{u}_{k,2}^{\mathrm{b}}$ on $\partial\Omega$ and (3.29). Taking the inner product of (3.30) with $\phi_{k,0}^{\mathrm{int}}$ gives the first solvability condition

$$i\lambda_{k,2} = \int_{\partial\Omega} Z_1^{\mathrm{b}}(P_0\phi_{k,1}^{\mathrm{int}})\Psi_{k,0}\mathrm{d}\sigma_x + \int_{\partial\Omega} Z_2^{\mathrm{b}}(\phi_{k,0}^{\mathrm{int}})\Psi_{k,0}\mathrm{d}\sigma_x + \langle \mathcal{D}\phi_{k,0}^{\mathrm{int}} | \phi_{k,0}^{\mathrm{int}} \rangle . \tag{3.31}$$

Because of the linearity of Z_1^{b} and the orthogonality condition (3.20), the first term in the above equation vanishes, so $i\lambda_{k,2}$ is completely determined. To solve $\phi_{k,2}^{\mathrm{int}}$ from (3.30), we consider three cases:

Case 1: $\lambda_{k,0}$ is a simple eigenvalue of $L_0 = \Delta_x$. For this case, no orthogonality condition is needed.

Case 2: $i\lambda_{k,1}$ is a simple eigenvalue of L_1. Taking the inner product of (3.30) with $\phi_{l,0}^{\mathrm{int}}$ gives

$$a_{kl,1} = \frac{1}{i\lambda_{k,1} - i\lambda_{l,1}} \int_{\partial\Omega} Z_2^{\mathrm{b}}(\phi_{k,0}^{\mathrm{int}})\Psi_{l,0}\mathrm{d}\sigma_x . \tag{3.32}$$

Here, we used that $\langle \mathcal{D}\phi_{k,0}^{\mathrm{int}} | \phi_{l,0}^{\mathrm{int}} \rangle = 0$ for $k \neq l$. Thus $P_0\phi_{k,1}^{\mathrm{int}}$ is completely determined. No additional conditions on $\mathrm{H}_0(\lambda)$ rather than (3.20) is needed.

Case 3: $i\lambda_{k,1}$ is an eigenvalue of L_1 with multiplicity greater than or equal to 2. For this case, we need more orthogonality conditions on

$$\mathrm{H}_1(\lambda_1) = \{\Psi \in \mathrm{H}_0 : L_1\Psi = i\lambda_{k,1}\Psi\} .$$

These orthogonality conditions come from the equation (3.30). Taking the inner product with $\phi_{l,0}^{\mathrm{int}}$ for $l \neq k, \lambda_{l,0} = \lambda_{k,0}$ and $\lambda_{l,1} = \lambda_{k,1}$, we get

$$\int_{\partial\Omega} Z_2^{\mathrm{b}}(\phi_{k,0}^{\mathrm{int}})\Psi_{l,0}\mathrm{d}\sigma_x = 0 . \tag{3.33}$$

We can define the quadratic form Q_2 and the symmetric operator L_2 on $\mathrm{H}_1(\lambda_1)$ as

$$Q_2(\Psi_{k,0}, \Psi_{l.0}) = \int_{\partial\Omega} Z_2^{\mathrm{b}}(\phi_{k,0}^{\mathrm{int}})\Psi_{l,0}\mathrm{d}\sigma_x + \langle \mathcal{D}\phi_{k,0}^{\mathrm{int}} | \phi_{l,0}^{\mathrm{int}} \rangle . \tag{3.34}$$

and

$$L_2\Psi_{k,0} = i\lambda_{k,2}\Psi_{k,0}\,, \tag{3.35}$$

which satisfy that

$$Q_2(\Psi_{k,0},\Psi_{l,0}) = \int_\Omega L_2(\Psi_{k,0})\Psi_{l,0}\mathrm{d}x\,,$$

and the orthogonality condition (3.33) is

$$Q_2(\Psi_{k,0},\Psi_{l,0}) = 0\,,\quad \text{if}\quad \Psi_{k,0},\Psi_{l,0}\in \mathrm{H}_1(\lambda_1)\quad\text{and}\quad k\neq l\,. \tag{3.36}$$

Under these conditions, to represent the solution of the equation (3.30) in $\mathrm{Null}(\mathcal{A}-i\lambda_{k,0})^\perp$, we decompose the equation (3.30) into two parts: one is the linear combination of the equation (3.11), the other only includes known terms. More precisely, decompose $\phi^{\mathrm{int}}_{k,2} = \phi^1+\phi^2$, where ϕ^1 satisfies the equation

$$\begin{aligned}(\mathcal{A}-i\lambda_{k,0})\phi^1 &= i\lambda_{k,1}P_0\phi^{\mathrm{int}}_{k,1}\,,\\ \mathbf{u}^1\cdot\mathrm{n} &= Z^{\mathrm{b}}_1(P_0\phi^{\mathrm{int}}_{k,1})\,,\end{aligned} \tag{3.37}$$

whose solution in $\mathrm{Null}(\mathcal{A}-i\lambda_{k,0})^\perp$ is $Z^{\mathrm{int}}_1(P_0\phi^{\mathrm{int}}_{k,1})$, since the equation (3.37) is just a linear combination of equation (3.11). Besides ϕ^2 satisfies the equation

$$\begin{aligned}(\mathcal{A}-i\lambda_{k,0})\phi^2 &= (i\lambda_{k,2}-\mathcal{D})\phi^{\mathrm{int}}_{k,0} + i\lambda_{k,1}Z^{\mathrm{int}}_1(\phi^{\mathrm{int}}_{k,0})\,,\\ \mathbf{u}^2\cdot\mathrm{n} &= Z^{\mathrm{b}}_2(\phi^{\mathrm{int}}_{k,0})\,,\end{aligned} \tag{3.38}$$

whose solution in $\mathrm{Null}(\mathcal{A}-i\lambda_{k,0})^\perp$ is *completely* determined, and is denoted by $Z^{\mathrm{int}}_2(\phi^{\mathrm{int}}_{k,0})$. In summary, the solution of the operator equation (3.30) is

$$\phi^{\mathrm{int}}_{k,2} = P_0\phi^{\mathrm{int}}_{k,2} + Z^{\mathrm{int}}_1(P_0\phi^{\mathrm{int}}_{k,1}) + Z^{\mathrm{int}}_2(\phi^{\mathrm{int}}_{k,0})\,. \tag{3.39}$$

For **case 1**, everything is fully solved. For **case 2**, only $P_0\phi^{\mathrm{int}}_{k,2}$ is undetermined. While for **case 3**, $P_0\phi^{\mathrm{int}}_{k,1} = (P_1+P_1^\perp)(\phi^{\mathrm{int}}_{k,1})$ where P_1 is the orthogonal projection on $\mathrm{H}_1(\lambda_{k,1})$ i.e.

$$P_1\phi^{\mathrm{int}}_{k,1} = \sum_{l\neq k,\lambda_{l,0}=\lambda_{k,0},\lambda_{l,1}=\lambda_{k,1}} a_{kl,1}\phi_{l,0}.$$

while $P_1^\perp$ is the orthogonal projection on $\mathrm{H}_1^\perp(\lambda_{k,1})$, i.e.

$$P_1^\perp\phi^{\mathrm{int}}_{k,1} = \sum_{l\neq k,\lambda_{l,0}=\lambda_{k,0},\lambda_{l,1}\neq\lambda_{k,1}} a_{kl,1}\phi_{l,0}\,.$$

At this stage, $P_1^\perp\phi^{\mathrm{int}}_{k,1}$ is completely determined by (3.32) and $P_1\phi^{\mathrm{int}}_{k,1}$ will be determined later.

3.5. **Induction hypothesis.** For $j\geq 3$, we assume that we used the information from the boundary layer till the order $\sqrt{\varepsilon}^{j-2}$ and in the interior till the order $\sqrt{\varepsilon}^{j-1}$. Before we solve the ansatz for the order $\sqrt{\varepsilon}^{j-1}$ in the boundary layer and the order $\sqrt{\varepsilon}^{j}$ in the interior, we write down and induction hypothesis that summarizes what we were able to construct till now. We write this in the following 7 statements that we need to check for the order j. Here, $Y^{\mathrm{b}}_h(\zeta,\cdot), \widetilde{Z}^{\mathrm{b,u}}_h(\zeta,\cdot), \widetilde{Z}^{\mathrm{b},\theta}_h(\zeta,\cdot), \widetilde{Z}^{\mathrm{b}}_h(\zeta,\cdot), Z^{\mathrm{b}}_h(\cdot)$ and $Z^{\mathrm{int}}_h(\cdot)$ are linear functions defined in a way similar to (3.24), (3.6), (3.8), (3.9), (3.10) and (3.22) respectively.

($\mathbf{P^1_{j-1}}$) : For $2\leq i\leq j-1$, $\rho^{\mathrm{b}}_{k,i}+\theta^{\mathrm{b}}_{k,i} = \sum_{h=2}^{i} Y^{\mathrm{b}}_h(\zeta, P_0\phi^{\mathrm{int}}_{k,i-h})$; For $i=0,1$, $\rho^{\mathrm{b}}_{k,i}+\theta^{\mathrm{b}}_{k,i}=0$;

($\mathbf{P^2_{j-1}}$) : For $0\leq i\leq j-2$, $\mathbf{u}^{\mathrm{b}}_{k,i}\cdot\nabla_x\pi = \sum_{h=0}^{i}\widetilde{Z}^{\mathrm{b,u}}_h(\zeta,P_0\phi^{\mathrm{int}}_{k,i-h})$;

($\mathbf{P^3_{j-1}}$) : For $0\leq i\leq j-2$, $\theta^{\mathrm{b}}_{k,i} = \sum_{h=0}^{i}\widetilde{Z}^{\mathrm{b},\theta}_h(\zeta,P_0\phi^{\mathrm{int}}_{k,i-h})$;

($\mathbf{P^4_{j-1}}$) : For $1\leq i\leq j-1$, $\mathbf{u}^{\mathrm{b}}_{k,i}\cdot\nabla_x\mathrm{d} = \sum_{h=1}^{i}\widetilde{Z}^{\mathrm{b}}_h(\zeta,P_0\phi^{\mathrm{int}}_{k,i-h})$. Taking $\zeta=0$, we deduce that on the boundary, we have $-\mathbf{u}^{\mathrm{b}}_{k,i}\cdot\mathrm{n} = \sum_{h=1}^{i} Z^{\mathrm{b}}_h(P_0\phi^{\mathrm{int}}_{k,i-h})$;

$(\mathbf{P^5_{j-1}})$: For $1 \leq h \leq j-1$, $i\lambda_{k,h} = Q_h(\Psi_{k,0}, \Psi_{k,0})$, where the quadratic form Q_1 and Q_2 are defined in (3.17) and (3.34) respectively, and Q_h for $3 \leq h \leq j-1$ is defined as

$$Q_h(\Psi_{k,0}, \Psi_{l,0}) = \int_{\partial\Omega} Z^{\mathrm{b}}_h(\phi^{\mathrm{int}}_{l,0})\Psi_{k,0}\mathrm{d}\sigma_x + \langle \mathcal{D}(Z^{\mathrm{int}}_{h-2}(\phi^{\mathrm{int}}_{l,0}))|\phi^{\mathrm{int}}_{k,0}\rangle \,. \tag{3.40}$$

$(\mathbf{P^6_{j-1}})$: For $1 \leq i \leq j-1$, $\phi^{\mathrm{int}}_{k,i} = P_0\phi^{\mathrm{int}}_{k,i} + \sum_{h=1}^{i} Z^{\mathrm{int}}_h(P_0\phi^{\mathrm{int}}_{k,i-h})$;

$(\mathbf{P^7_{j-1}})$: The last assumption to check deals with the number of orthogonality conditions needed and specifies what is already determined and what is still not determined in the construction. We distinguish between j cases:

Case 1: $i\lambda_{k,h}$ is a simple eigenvalue of L_h for $0 \leq h \leq j-2$. No orthogonality condition is needed, and every term in the expansion is fully determined;

Case m $(2 \leq m \leq j)$**:** $i\lambda_{k,h}$ is a multiple eigenvalue of L_h for $0 \leq h \leq m-2$, and a simple eigenvalue of L_h for $h = m-1$. (Note: the **case j** means that all the eigenvalues $i\lambda_{k,h}$ for $0 \leq h \leq j-2$ are multiple eigenvalues.)

- We need the orthogonality conditions: For each $0 \leq h \leq m-2$,
$$Q_{h+1}(\Psi_k\,, \Psi_l) = 0\,, \quad \text{for} \quad \Psi_k, \Psi_l \in \mathrm{H}_0 \cap \cdots \cap \mathrm{H}_h\,, \tag{3.41}$$
where for $h \geq 1$, the space $\mathrm{H}_h = \mathrm{H}_h(\lambda_h) = \{\Psi \in \mathrm{H}_1(\lambda_1) \cap \cdots \cap \mathrm{H}_{h-1}(\lambda_{h-1}) : L_h\Psi = i\lambda_h\Psi\}$.
- For $1 \leq h \leq j-m$, $\phi^{\mathrm{int}}_{k,h}$ are completely determined. (for the **case j**, no term is completely determined.)
- For $j-m+1 \leq h \leq j-1$, $(P^{\perp}_0 + \cdots + P^{\perp}_{j-1-h})\phi^{\mathrm{int}}_{k,h}$ are determined.
- For $j-m+1 \leq h \leq j-1$, $P_{j-1-h}\phi^{\mathrm{int}}_{k,h}$ are not determined,

where P_{h-1} is the orthogonal projection on $\mathrm{H}_1(\lambda_1) \cap \cdots \cap \mathrm{H}_{h-1}(\lambda_{h-1})$, and $P_{h-1} = P_h + P^{\perp}_h$, where $P^{\perp}_h$ is the orthogonal projection on $\mathrm{H}_1(\lambda_1) \cap \cdots \cap \mathrm{H}_{h-1}(\lambda_{h-1}) \cap \mathrm{H}^{\perp}_h(\lambda_h)$.

Remark: Regarding the condition (3.41), actually we have a stronger orthogonality property which is actually equivalent to (3.41), namely : for each $0 \leq h \leq m-2$,

$$Q_{h+1}(\Psi_k\,, \Psi_l) = 0\,, \quad \text{for} \quad l \neq k, \quad \Psi_k, \Psi_l \in \mathrm{H}_0\,. \tag{3.42}$$

Indeed, we just need to use that the L_h leave stable the spaces H_h. Of course, we have to define L_h over the whole space H_0 even if the eigenvalue is simple, but in this case we just take it to be the identity.

In the next subsection, we are going to prove the 7 hypotheses $(\mathbf{P^1_j}) - (\mathbf{P^7_j})$ assuming $\mathbf{P_{i-1}}$ for $i \leq j$.

3.6. General case: order $\sqrt{\varepsilon}^{j-1}$ of the boundary layer.

The order $\sqrt{\varepsilon}^{j-1}$ of the boundary layer gives

$$\begin{aligned} -\mathcal{A}^{\mathrm{d}}\phi^{\mathrm{b}}_{k,j} =& (\mathcal{A}^{\pi} + \mathcal{D}^{\mathrm{b}} - i\lambda_{k,0})\phi^{\mathrm{b}}_{k,j-1} + (\mathcal{F}^{\mathrm{b}} - i\lambda_{k,1})\phi^{\mathrm{b}}_{k,j-2} + (\mathcal{G}^{\mathrm{b}} - i\lambda_{k,2})\phi^{\mathrm{b}}_{k,j-3} \\ & - i\sum_{h=3}^{j-1} \lambda_{k,h}\phi^{\mathrm{b}}_{k,j-1-h}\,. \end{aligned} \tag{3.43}$$

<u>**Step 1**</u> is to solve $\rho^{\mathrm{b}}_{k,j} + \theta^{\mathrm{b}}_{k,j}$ which satisfies the ODE

$$\begin{aligned} -\partial_\zeta(\rho^{\mathrm{b}}_{k,j} + \theta^{\mathrm{b}}_{k,j})|\nabla_x \mathrm{d}|^2 =& \big((\xi + 2\nu)|\nabla_x \mathrm{d}|^2\partial^2_{\zeta\zeta} - i\lambda_{k,0}\big)\,(\mathbf{u}^{\mathrm{b}}_{k,j-1}\cdot\nabla_x \mathrm{d}) + \mathcal{F}^{\mathbf{u}}_{d\pi}(\mathbf{u}^{\mathrm{b}}_{k,j-2}\cdot\nabla_x\pi) \\ & + (\mathcal{F}^{\mathbf{u}}_{dd} - i\lambda_{k,1})(\mathbf{u}^{\mathrm{b}}_{k,j-2}\cdot\nabla_x \mathrm{d}) + G^{\mathbf{u}}_{d\pi}(\mathbf{u}^{\mathrm{b}}_{k,j-3}\cdot\nabla_x\pi) \\ & + (\mathcal{G}^{\mathbf{u}}_{dd} - i\lambda_{k,2})(\mathbf{u}^{\mathrm{b}}_{k,j-3}\cdot\nabla_x \mathrm{d}) - i\sum_{h=3}^{j-2}\lambda_{k,h}(\mathbf{u}^{\mathrm{b}}_{k,j-1-h}\cdot\nabla_x \mathrm{d})\,. \end{aligned} \tag{3.44}$$

Integrating from ζ to ∞, the solution of (3.44) can be represented as

$$\rho^{\mathrm{b}}_{k,j} + \theta^{\mathrm{b}}_{k,j} = \sum_{h=2}^{j} Y^{\mathrm{b}}_h(\zeta, P_0\phi^{\mathrm{int}}_{k,j-h})\,,$$

where Y^{b}_h are linear functions. This corresponds to $(\mathbf{P^1_j})$ in the induction hypothesis.

<u>**Step 2**</u> is to solve $\mathbf{u}^{\mathrm{b}}_{k,j-1}\cdot\nabla_x\pi$ which satisfies the ODE

$$\begin{aligned}\mathcal{L}f^{\mathrm{b}} =& (i\lambda_{k,1} - \mathcal{F}^{\mathbf{u}}_{\pi\pi})(\mathbf{u}^{\mathrm{b}}_{k,j-2}\cdot\nabla_x\pi)\\ &+ (i\lambda_{k,2} - \mathcal{G}^{\mathbf{u}}_{\pi\pi})(\mathbf{u}^{\mathrm{b}}_{k,j-3}\cdot\nabla_x\pi) - \mathcal{F}^{\mathbf{u}}_{\pi d}(\mathbf{u}^{\mathrm{b}}_{k,j-2}\cdot\nabla_x \mathrm{d})\\ &+ i\lambda_{k,3}(\mathbf{u}^{\mathrm{b}}_{k,j-4}\cdot\nabla_x\pi) - \mathcal{G}^{\mathbf{u}}_{\pi d}(\mathbf{u}^{\mathrm{b}}_{k,j-3}\cdot\nabla_x \mathrm{d})\\ &+ i\sum_{h=4}^{j-1}\lambda_{k,h}(\mathbf{u}^{\mathrm{b}}_{k,j-1-h}\cdot\nabla_x\pi) - \partial_{\pi^\alpha}(\rho^{\mathrm{b}}_{k,j-1} + \theta^{\mathrm{b}}_{k,j-1})|\nabla_x\pi^\alpha|^2\,,\end{aligned} \tag{3.45}$$

with boundary conditions

$$f^{\mathrm{b}}(\zeta = 0) = -\mathbf{u}^{\mathrm{int}}_{k,j-1}\cdot\nabla_x\pi\,, \quad f^{\mathrm{b}}(\zeta\to\infty) = 0\,.$$

Note that $\mathbf{u}^{\mathrm{int}}_{k,j-1}\cdot\nabla_x\pi$ is taken value on the boundary $\partial\Omega$. Suppose that we already have $\phi^{\mathrm{int}}_{k,j-1} = P_0\phi^{\mathrm{int}}_{k,j-1} + \sum_{h=1}^{j-1} Z^{\mathrm{int}}_h(P_0\phi^{\mathrm{int}}_{k,j-1-h})$. The solution of (3.45) can be represented as $\mathbf{u}^{\mathrm{b}}_{k,j-1}\cdot\nabla_x\pi = f^{\mathrm{b}}_1 + f^{\mathrm{b}}_2 + \cdots + f^{\mathrm{b}}_j$, where f^b_1 satisfies the ODE

$$\begin{aligned}\mathcal{L}f^{\mathrm{b}}_1 &= 0\,,\\ f^{\mathrm{b}}_1(\zeta=0) &= -(P_0\phi^{\mathrm{int}}_{k,j-1})_{\mathbf{u}}\cdot\nabla_x\pi\,,\end{aligned} \tag{3.46}$$

whose solution is $f^{\mathrm{b}}_1 = \widetilde{Z}^{\mathrm{b,u}}_0(P_0\phi^{\mathrm{int}}_{k,j-1})$, and f^b_2 satisfies the ODE

$$\begin{aligned}\mathcal{L}f^{\mathrm{b}}_2 &= (i\lambda_{k,1} - \mathcal{F}^{\mathbf{u}}_{\pi\pi})(\widetilde{Z}^{\mathrm{b,u}}_0(P_0\phi^{\mathrm{int}}_{k,j-2}))\,,\\ f^{\mathrm{b}}_2(\zeta=0) &= -\big(Z^{\mathrm{int}}_1(P_0\phi^{\mathrm{int}}_{k,j-2})\big)_{\mathbf{u}}\cdot\nabla_x\pi\,.\end{aligned} \tag{3.47}$$

Noticing that this equation has exactly the same structure with the equation (3.27), we infer that the solution is $f^b_2 = \widetilde{Z}^{\mathrm{b,u}}_1(\zeta, P_0\phi^{\mathrm{int}}_{k,j-2})$.

Remark: This is a key point in solving the ansatz: because $P_0\phi^{\mathrm{int}}_{k,j-2}$ is a linear combination of $\phi^{\mathrm{int}}_{l,0}$, so in (3.47) the equation and the boundary conditions are the *same* linear combination with those of (3.27). Thus, the solution of (3.47) is also the *same* linear combination with that of (3.27).

Furthermore, f^{b}_h, for $h = 3, 4, \cdots, j$ satisfies the ODE

$$\begin{aligned}\mathcal{L}f^{\mathrm{b}}_h &= g^{\mathrm{b}}_h(P_0\phi^{\mathrm{int}}_{k,j-h})\,,\\ f^{\mathrm{b}}_h(\zeta=0) &= -\big(Z^{\mathrm{int}}_1(P_0\phi^{\mathrm{int}}_{k,j-h})\big)_{\mathbf{u}}\cdot\nabla_x\pi\,.\end{aligned} \tag{3.48}$$

where $g^{\mathrm{b}}_h(P_0\phi^{\mathrm{int}}_{k,j-h})$ denotes the terms in the right-hand side of (3.45) in which $P_0\phi^{\mathrm{int}}_{k,j-h}$ appears. The solution of (3.48) is $f^{\mathrm{b}}_h = \widetilde{Z}^{\mathrm{b,u}}_{h-1}(\zeta, P_0\phi^{\mathrm{int}}_{k,j-h})$. Thus

$$\mathbf{u}^{\mathrm{b}}_{k,j-1}\cdot\nabla_x\pi = \sum_{h=0}^{j-1}\widetilde{Z}^{\mathrm{b,u}}_h(\zeta, P_0\phi^{\mathrm{int}}_{k,j-1-h})\,. \tag{3.49}$$

This corresponds to the $(\mathbf{P^2_j})$ in the induction hypothesis.

Step 3 is to solve $\theta^{\mathrm{b}}_{k,j-1}$ which can be similarly obtained

$$\theta^{\mathrm{b}}_{k,j-1} = \sum_{h=0}^{j-1} \widetilde{Z}^{\mathrm{b},\theta}_h(\zeta, P_0\phi^{\mathrm{int}}_{k,j-1-h})\,. \tag{3.50}$$

This corresponds to $(\mathbf{P^3_j})$ in the induction hypothesis.

Step 4 is to solve $\mathbf{u}^{\mathrm{b}}_{k,j}\cdot\nabla_x \mathrm{d}$ which is $\sum_{h=1}^{j} \widetilde{Z}^{\mathrm{b}}_h(\zeta, P_0\phi^{\mathrm{int}}_{k,j-h})$. Consequently, taking $\zeta = 0$, we get the boundary value

$$-\mathbf{u}^{\mathrm{b}}_{k,j}\cdot\mathrm{n} = \sum_{h=1}^{j} Z^{\mathrm{b}}_h(P_0\phi^{\mathrm{int}}_{k,j-h})\,. \tag{3.51}$$

This corresponds to $(\mathbf{P^4_j})$ in the induction hypothesis.

3.7. General case: order $\sqrt{\varepsilon}^j$ of the interior.

The order $\sqrt{\varepsilon}^j$ of the interior part in (3.1) reads

$$\begin{aligned}(\mathcal{A} - i\lambda_{k,0})\phi^{\mathrm{int}}_{k,j} &= i\lambda_{k,1}\phi^{\mathrm{int}}_{k,j-1} + (i\lambda_{k,2} - \mathcal{D})\phi^{\mathrm{int}}_{k,j-2} + \sum_{h=3}^{j} i\lambda_{k,h}\phi^{\mathrm{int}}_{k,j-h}\,,\\ \mathbf{u}^{\mathrm{int}}_{k,j}\cdot\mathrm{n} = -\mathbf{u}^{\mathrm{b}}_{k,j}\cdot\mathrm{n} &= \sum_{h=1}^{j} Z^{\mathrm{b}}_h(P_0\phi^{\mathrm{int}}_{k,j-h})\,.\end{aligned} \tag{3.52}$$

The first solvability condition for the equation (3.52) is obtained by taking the inner product with $\phi^{\mathrm{int}}_{k,0}$, which is

$$\begin{aligned} i\lambda_{k,j} =& \int_{\partial\Omega} Z^{\mathrm{b}}_1(P_0\phi^{\mathrm{int}}_{k,j-1})\Psi_{k,0}\mathrm{d}\sigma_x + \int_{\partial\Omega} Z^{\mathrm{b}}_2(P_0\phi^{\mathrm{int}}_{k,j-2})\Psi_{k,0}\mathrm{d}\sigma_x\\ &+ \sum_{h=3}^{j}\left\{\int_{\partial\Omega} Z^{\mathrm{b}}_h(P_0\phi^{\mathrm{int}}_{k,j-h})\Psi_{k,0}\mathrm{d}\sigma_x + \langle \mathcal{D}(Z^{\mathrm{int}}_{h-2}(P_0\phi^{\mathrm{int}}_{k,j-h}))|\phi^{\mathrm{int}}_{k,0}\rangle\right\}\,. \end{aligned} \tag{3.53}$$

Because of the orthogonality conditions (3.41) for $1 \le h \le j-1$, and the expressions of Q_1, Q_2 and Q_h defined in (3.17), (3.34) and (3.40), only the last two terms in (3.53) are non-zero, i.e.

$$\begin{aligned} i\lambda_{k,j} =& \int_{\partial\Omega} Z^{\mathrm{b}}_j(\phi^{\mathrm{int}}_{k,0})\Psi_{k,0}\mathrm{d}\sigma_x + \langle \mathcal{D}(Z^{\mathrm{int}}_{j-2}(\phi^{\mathrm{int}}_{k,0}))|\phi^{\mathrm{int}}_{k,0}\rangle\\ =& Q_j(\Psi_{k,0}, \Psi_{k,0})\,, \end{aligned} \tag{3.54}$$

which is *completely* determined. This corresponds to the $(\mathbf{P^5_j})$ in the induction hypothesis.

To solve the equation (3.52), we need to consider $j+1$ cases:

Case 1: $i\lambda_{k,h}$ is a simple eigenvalue of L_h for $0 \le h \le j-1$. No orthogonality condition is needed, and every term is fully determined;

Case m $(2 \le m \le j+1)$**:** $i\lambda_{k,h}$ is a multiple eigenvalue of L_h for $0 \le h \le m-2$, and a simple eigenvalue of L_h for $m-1 \le h \le j-1$.

We only consider the **case j+1** here, i.e. all the eigenvalues $i\lambda_{k,h}$ are multiple. The other cases are simpler. Taking the inner product with $\phi^{\mathrm{int}}_{l,0}$, for $l \neq k$, $\lambda_{l,0} = \lambda_{k,0}$, which is

$$\sum_{h=1}^{j-1} i\lambda_{k,h} a_{kl,j-h} = \sum_{h=1}^{j-1} Q_h(P_0\phi^{\mathrm{int}}_{k,j-h}, \Psi_{l,0}) + Q_j(\Psi_{k,0}, \Psi_{l,0})\,. \tag{3.55}$$

If $\Psi_{k,0}, \Psi_{l,0} \in \mathrm{H}_1(\lambda_1) \cap \mathrm{H}_2(\lambda_2) \cap \cdots \cap \mathrm{H}_{j-1}(\lambda_{j-1})$, then because of the orthogonality condition (3.41) for $1 \le h \le j-2$,

$$\begin{aligned} Q_h(P_0\phi^{\mathrm{int}}_{k,j-h}, \Psi_{l,0}) &= Q_h(P_{h-1}\phi^{\mathrm{int}}_{k,j-h}, \Psi_{l,0}) + Q_h(\sum_{\delta=1}^{h-1} P_\delta^\perp \phi^{\mathrm{int}}_{k,j-h}, \Psi_{l,0}) \\ &= i\lambda_{l,h} a_{kl,j-h}\,. \end{aligned}$$

For $h = j-1$, $Q_{j-1}(P_0\phi^{\mathrm{int}}_{k,1}, \Psi_{l,0}) = i\lambda_{l,j-2}a_{kl,1} + Q_{j-1}(P_{j-2}^\perp\phi^{\mathrm{int}}_{k,1}, \Psi_{l,0})$. Thus, the identity (3.55) implies that we need the orthogonality condition that for $k \neq l$,

$$Q_j(\Psi_{k,0}\,, \Psi_{l,0}) = \int_\Omega L_j(\Psi_{k,0})\Psi_{l,0}\,\mathrm{d}x = 0\,, \quad \text{for} \quad \Psi_{k,0}, \Psi_{l,0} \in \mathrm{H}_1 \cap \cdots \cap \mathrm{H}_{j-1}\,, \tag{3.56}$$

where the symmetric operator L_j is defined by $L_j\Psi_{l,0} = i\lambda_{l,j}\Psi_{l,0}$, for $\Psi_{l,0} \in \mathrm{H}_1 \cap \cdots \cap \mathrm{H}_{j-1}$.

If $\Psi_{k,0}, \Psi_{l,0} \in \mathrm{H}_1(\lambda_1) \cap \mathrm{H}_2(\lambda_2) \cap \cdots \cap \mathrm{H}_{j-2}(\lambda_{j-2}) \cap \mathrm{H}^\perp_{j-1}(\lambda_{j-1})$, i.e $\lambda_{k,h} = \lambda_{l,h}$ for $0 \le h \le j-2$, but $\lambda_{k,j-1} \neq \lambda_{l,j-1}$, from the identity (3.55), for these k, l, $a_{kl,1}$ can be determined by

$$a_{kl,1} = \tfrac{1}{i\lambda_{k,j-1} - i\lambda_{l,j-1}} Q_j(\Psi_{k,0}, \Psi_{l,0})\,.$$

This means that $(P_0^\perp + P_1^\perp + \cdots + P_{j-1}^\perp)\phi^{\mathrm{int}}_{k,1}$ is completely determined, but $P_{j-1}\phi^{\mathrm{int}}_{k,1}$ is still left as undetermined.

If $\Psi_{k,0}, \Psi_{l,0} \in \mathrm{H}_1 \cap \cdots \cap \mathrm{H}_{j-3} \cap \mathrm{H}^\perp_{j-2}$,

$$Q_{j-1}(P_{j-1}^\perp\phi^{\mathrm{int}}_{k,1}\,, \Psi_{l,0}) + Q_j(\Psi_{k,0}\,, \Psi_{l,0}) = (i\lambda^k_{j-2,0} - i\lambda^l_{j-2,0})a_{kl,2} + i\lambda_{k,j-1}a_{kl,1}\,, \tag{3.57}$$

from which $a_{kl,2}$ thus $P^\perp_{j-2}\phi^{\mathrm{int}}_{k,2}$ is completely determined.

Under these solvability conditions, the equation (3.52) can be solved in the following way: $\phi^{\mathrm{int}}_{k,j} = \phi^1 + \phi^2 + \cdots + \phi^j$, where ϕ^h satisfies the equation

$$\begin{aligned} (\mathcal{A} - i\lambda_{k,0})\phi^h &= q^{\mathrm{int}}_h\,, \\ \mathbf{u}^h\cdot\mathrm{n} &= Z^{\mathrm{b}}_0(P_0\phi^{\mathrm{int}}_{k,j-h})\,. \end{aligned} \tag{3.58}$$

where q^{int}_h is the summation of all the terms which include $P_0\phi^{\mathrm{int}}_{k,j-h}$ in the right-hand side of (3.52). For example, ϕ^1 satisfies the equation

$$\begin{aligned} (\mathcal{A} - i\lambda_{k,0})\phi^1 &= i\lambda_{k,1}P_0\phi^{\mathrm{int}}_{k,j-1}\,, \\ \mathbf{u}^1\cdot\mathrm{n} &= Z^{\mathrm{b}}_0(P_0\phi^{\mathrm{int}}_{k,j-1})\,. \end{aligned} \tag{3.59}$$

Comparing with the equation (3.37), we know the solution of the equation (3.59) in $\mathrm{Null}(\mathcal{A} - i\lambda_{k,0})^\perp$ is $Z^{\mathrm{int}}_1(P_0\phi^{\mathrm{int}}_{k,j-1})$. ϕ^2 satisfies the equation

$$\begin{aligned} (\mathcal{A} - i\lambda_{k,0})\phi^2 &= (i\lambda_{k,2} - \mathcal{D})(P_0\phi^{\mathrm{int}}_{k,j-2}) + i\lambda_{k,1}Z^{\mathrm{int}}_1(P_0\phi^{\mathrm{int}}_{k,j-2})\,, \\ \mathbf{u}^2\cdot\mathrm{n} &= Z^{\mathrm{b}}_1(\phi^{\mathrm{int}}_{k,j-2})\,. \end{aligned} \tag{3.60}$$

Comparing with the equation (3.38), we know the solution of the equation (3.60) in $\mathrm{Null}(\mathcal{A} - i\lambda_{k,0})^\perp$ is $Z^{\mathrm{int}}_2(P_0\phi^{\mathrm{int}}_{k,j-2})$. We can continue the similar process and finally the solution of (3.52) is

$$\phi^{\mathrm{int}}_{k,j} = P_0\phi^{\mathrm{int}}_{k,j} + \sum_{h=1}^{j} Z^{\mathrm{int}}_h(P_0\phi^{\mathrm{int}}_{k,j-h})\,,$$

i.e. $\phi^{\mathrm{int}}_{k,j}$ is determined modulo $P_0\phi^{\mathrm{int}}_{k,j}, P_{j-1}\phi^{\mathrm{int}}_{k,1}, P_{j-2}\phi^{\mathrm{int}}_{k,2}, \cdots, P_1\phi^{\mathrm{int}}_{k,j-1}$ which are undetermined at this stage. This corresponds to the $(\mathbf{P^6_j})$ and $(\mathbf{P^7_j})$ in the induction hypothesis.

We can now inductively continue the process, namely go to the order $\sqrt{\varepsilon}^j$ of the boundary layer, then the order $\sqrt{\varepsilon}^{j+1}$ of the interior, and so on. We should do this at least till the order $N+2$ where N is the precision of the error in (1.16). Note however, that for a given

$\lambda = \lambda_{k,0}$, we may only need to construct a small number of the L_j if after few steps all the eigenvalues become simple, namely if for some j all the eigenvalues of L_j are simple on the space $\mathrm{H}_1(\lambda_1) \cap \cdots \cap \mathrm{H}_{j-1}(\lambda_{j-1})$. It is clear that if the eigenvalues become simple for some $j \leq N+2$, then the orthogonality condition (3.41) allows to determine the eigenfuctions Ψ_k uniquely. If the process does not end, then we just need to satisfy the condition till the order $N+2$ which yield a none unique choice of eigenfunctions. Also, in this case, we set all the undetermined pieces of the eigenfunction, namely those left undetermined in the second point of (7) to be zero.

The last step is the error estimate (1.16). Simple calculations show that the leading order terms in $R^{\pm}_{k,\varepsilon,N}$ are

$$\left\{ (\mathcal{A}^\pi + \mathcal{D}^{\mathrm{b}})\phi^{\mathrm{b}}_{k,N} + \mathcal{F}^{\mathrm{b}}\phi^{\mathrm{b}}_{k,N-1} + \mathcal{G}^{\mathrm{b}}\phi^{\mathrm{b}}_{k,N-2} - \sum_{h=0}^{N} i\lambda_h \phi^{\mathrm{b}}_{k,N-h} \right\} \sqrt{\varepsilon}^N .$$

After a simple change of coordinates, the error estimate for $\|R^{\pm}_{k,\varepsilon,N}\|_{L^p(\Omega)}$, namely (1.16) follows.

4. Conclusions

In this paper we give a more precise construction of the boundary layer that is used to prove the strong decay of the acoustic waves in the paper [4]. Indeed, the construction in [4] is not completely correct in the case we have eigenvalues which are not simple and some extra conditions should be imposed on the basis of eigenvectors chosen. There is an extra interesting algebraic structure based on a sequence of symmetric operators that should be used in this choice. We show in this paper that this construction can be extended to the full acoustic operator which includes the heat conductivity effect. Interestingly, in this non-isentropic case, a geometric condition which is related to the Schiffer's conjecture is not needed because of the additional dissipation effect from the heat conductivity.

In the present paper, only the Dirichlet boundary condition is treated. In fact, we can also generalize to the Robin boundary condition, for example, the Navier-slip boundary condition: $[(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T)\cdot\mathbf{n}]^{\tan} + \chi\varepsilon^\beta \mathbf{u}^{\tan} = 0$. Similar phenomena will happen. However, the instantaneous damping effect on the acoustic waves depends the value of β. We will consider this in the forthcoming paper on the incompressible limit of the isentropic compressible fluids with this Navier-slip boundary condition.

These higher order orthogonality conditions in the case of the multiple eigenvalues of the Neumann Laplace operator in Ω are also needed in the hydrodynamic limit of the Boltzmann equation [7]. In [7], kinetic and viscous fluid boundary layers are coupled when the Boltzmann equation is imposed the Maxwell boundary condition which includes the both specular reflection and the boundary diffusion with the accommodation coefficient $\chi\varepsilon^\beta$ as ratio.

References

[1] T. Alazard, A minicourse on the low Mach number limit. *Discrete Contin. Dyn. Syst. Ser. S* **1** (2008), no. 3, 365-404.

[2] J.-Y. Chemin, B. Desjardins; I. Gallagher; E. Grenier, *Mathematical geophysics. An introduction to rotating fluids and the Navier-Stokes equations.* Oxford Lecture Series in Mathematics and its Applications, **32**. The Clarendon Press, Oxford University Press, Oxford, 2006.

[3] B. Desjardins, E. Grenier, Low Mach number limit of compressible viscous flows in the whole space, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **455** (1999), no. 1986, 2271-2279.

[4] B. Desjardins; E. Grenier; P. -L. Lions; N. Masmoudi, *Incompressible limit for solutions to the isentropic Navier-Stokes equations with Dirichlet boundary conditions.* J. Math. Pures Appl. **78**, 1999, 461-471.

[5] E. Feireisl; I. Gallagher; D. Gerard-Varet; A. Novotny, Multi-scale analysis of compressible viscous and rotating fluids. *Comm. Math. Phys.* **314** (2012), no. 3, 641-670.

[6] E. Grenier, N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data. *Comm. Partial Differential Equations* **22** (1997), no. $5-6$, p. $953-975$.

[7] N. Jiang; N. Masmoudi, Boundary layers and hydrodynamic limits of Boltzmann Equation (I): incompressible Navier-Stokes-Fourier limit. (to be submitted to *Comm. Pure Appl. Math*).

[8] C. D. Levermore; W. R. Sun; K. Trivisa, A low Mach number limit of a dispersive Navier-Stokes system. *SIAM J. Math. Anal.* **44** (2012), no. 3, 1760-1807.

[9] P.L. Lions; N. Masmoudi, Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl.* **77** (1998), p. 585-627.

[10] P.L. Lions; N. Masmoudi, Une approche locale de la limite incompressible. *C. R. Acad. Sci. Paris Sr. I Math.* **329** (1999), no. 5, 387-392.

[11] N. Masmoudi, Ekman layers of rotating fluids: the case of general initial data. *Comm. Pure Appl. Math.* **53** (2000), no. 4, 432-483.

[12] N. Masmoudi Examples of singular limits in hydrodynamics. *Handbook of differential equations: evolutionary equations. Vol. III*, 195-275, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007.

[13] L. Simon, *Theorems on Regularity and Singularity of Energy Minimizing Maps*, Based on lecture notes by Norbert Hungerbhler. Lectures in Mathematics ETH Zrich. Birkhuser Verlag, Basel, 1996.

Mathematical Sciences Center, Jin Chunyuan West Building, Beijing, 100084
E-mail address: njiang@math.tsinghua.edu.cn

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, Partially supported by NSF grant DMS-1211806
E-mail address: masmoudi@cims.nyu.edu