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ScienceDirect
Physica D 237 (2008) 1663–1675
PHYSICA D
www.elsevier.com/locate/physd

Global well-posedness for 2D polymeric fluid models and growth estimate

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Available online 21 March 2008

Abstract

Motivated by [P. Constantin, N. Masmoudi, Global well-posedness for a Smoluchowski equation coupled with Navier–Stokes equations in 2D, Comm. Math. Phys. 278 (2008) 179–191; F. Lin, Ping Zhang, Zhifei Zhang, On the global existence of smooth solution to the 2-D FENE dumbbell model, Comm. Math. Phys. 277 (2008) 531–553], we prove the global existence of smooth solutions to a coupled microscopic–macroscopic model for polymeric fluid in 2D under the co-rotational assumption. Furthermore, we provide an estimate on the time growth of these solutions. Our method is general and can be applied to the co-rotational FENE model and to the rod-like model without the co-rotational assumption.

Keywords: Nonlinear Fokker–Planck equations; Navier–Stokes equations; Micro–macro interactions; Global well-posedness

1. Introduction

In this paper, we study the global smooth solutions for a coupled microscopic–macroscopic model for polymeric fluid in 2D. The micro-mechanical models for polymeric liquids often consist of beads joined by springs or rods [2,8]. In the simplest case, a molecule configuration can be described by its end-to-end vector Q. Taking into account the elastic effect together with the thermo-fluctuation, the distribution function $\psi(t, x, Q)$ of molecule orientations Q satisfies a Fokker–Planck equation. The convection velocity u satisfies the Navier–Stokes equations with an elastic stress which reflects the microscopic contribution of the polymer molecules to the overall macroscopic flow fields. Mathematically, this system reads (one may check [12] for a formal energetic variational derivation)

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \triangle u + \nabla\cdot\tau, \quad x\in\mathbf{R}^2,\\ \nabla\cdot u = 0, \quad x\in\mathbf{R}^2,\\ \psi_t + u\cdot\nabla\psi = \triangle_Q\,\psi - \nabla_Q\cdot(S(u)Q\psi - \nabla_Q U\psi), \quad (x, Q)\in\mathbf{R}^2\times\mathbf{R}^2, \end{cases} \tag{1.1}$$

where $S(u)$ and the polymer stress τ are given by

$$S(u) = \frac{\nabla u - \nabla u^t}{2}, \quad \tau = \int_{\mathbf{R}^2} \nabla_Q U\otimes Q\psi\,\mathrm{d}Q, \tag{1.2}$$

and $U(Q) = U(|Q|^2)$ is the potential.

Existence results for micro-macro models of polymeric fluids are usually limited to small time existence and uniqueness of strong solutions [17] or global existence of weak solution [16] (see also [15] for the Oldroyd B model). In the setting when the last

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0167-2789/$ - see front matter
doi:10.1016/j.physd.2008.03.020

equation of (1.1) is formulated as a stochastic PDE, we refer to [10] (see also [9] for a polynomial force). With regards to PDE coupled systems, we refer the reader to the earlier work [18].

In the work [12] (see also [17]), the authors proved the global existence of smooth solutions to (1.1) near equilibrium, which is a sort of an extension of a related result for the Oldroyd model [11], and which corresponds to the Hooke dumbbell model. In two space dimensions, Constantin, Fefferman, Titi and Zarnescu [6] proved the global existence of smooth solutions to a coupled nonlinear Fokker–Planck and Navier–Stokes system when the convection velocity u in the Fokker–Planck equation is replaced by a sort of time averaged one. Later this assumption was removed by Constantin and Masmoudi [7]. At the same time, Lin, Zhang and Zhang [13] independently proved the global regularity for the 2D co-rotational FENE model. The main ideas in [7,13] are the so-called losing derivative estimate from [1,5]. In this paper, we treat the case where $Q \in \mathbf{R}^2$. Our observation here is that an adaptation of this method enables us to get not only the global existence of smooth solutions but also the time growth estimate for thus obtained solution as in (1.12). We also point out that there is a new difficulty for the model (1.1) which lies in the fact that in order to get regularities for the solutions of (1.1), we need to do momentum estimate for the distribution function $\psi(t, x, Q)$. One of the main limitation of our result is the fact that we have to take the antisymmetric part of ∇u which explains the terminology co-rotational model. A more physical model would be the case where $S(u) = \nabla u$. However, we do not know how to get global solutions in that case. Besides, our method of proving the time growth estimate also holds for the 2D co-rotational FENE model and the 2D rod-like model. For the 2D rod-like model [7], there is no need for the co-rotational assumption since, we have an L^∞ bound on the extra stress tensor τ.

Let $M \stackrel{\text{def}}{=} \mathrm{e}^{-U}$. After renormalization, we will assume that $\int_{\mathbf{R}^2} M \,\mathrm{d}Q = 1$. In what follows, we will seek the solution of (1.1) with the form

$$\psi = M + \sqrt{M} f. \tag{1.3}$$

We assume that $\int_{\mathbf{R}^2} \psi(0, x, Q)\mathrm{d}Q = 1$; then formally from (1.1), there holds $\int_{\mathbf{R}^2} \psi(t, x, Q)\mathrm{d}Q = 1$, which together with (1.3) yield

$$\int_{\mathbf{R}^2} \sqrt{M} f \,\mathrm{d}Q = 0. \tag{1.4}$$

Plugging (1.3) into (1.1) and using the fact that $S(u)Q \cdot \nabla_Q U = 0$, we get the new system for (u, f):

$$\begin{cases} u_t + u \cdot \nabla u + \nabla p = \triangle u + \nabla \cdot \tau, \quad x \in \mathbf{R}^2, \\ \nabla \cdot u = 0, \quad x \in \mathbf{R}^2, \\ f_t + u \cdot \nabla f + \nabla_Q \cdot (S(u)Qf) - \left(\triangle_Q + \dfrac{\triangle_Q U}{2} - \dfrac{|\nabla_Q U|^2}{4}\right) f = 0, \quad (x, Q) \in \mathbf{R}^2 \times \mathbf{R}^2, \end{cases} \tag{1.5}$$

with the polymer stress τ given by

$$\tau = \int_{\mathbf{R}^2} \nabla_Q U \otimes Q \sqrt{M} f \,\mathrm{d}Q, \tag{1.6}$$

together with the initial conditions:

$$u|_{t=0} = u_0(x), \qquad f|_{t=0} = f_0(x, Q), \quad \text{for } (x, Q) \in \mathbf{R}^2 \times \mathbf{R}^2. \tag{1.7}$$

Before presenting the main result of this paper, we introduce some basic notations that will be used in the rest of the paper. We shall denote $\mathbf{R}^2_x$ by $\mathbf{R}^2$, and $\mathbf{R}^2_Q$ by $\mathbf{D}$. Also as in [12], to avoid additional technicalities, we assume that

$$\begin{gathered} |Q| \le C(1 + |\nabla_Q U|), \qquad \triangle_Q U \le C + \delta |\nabla_Q U|^2 \quad \text{with } \delta < 1, \\ \int_{\mathbf{D}} |\nabla_Q U|^2 \mathrm{e}^{-U} \mathrm{d}Q \le C, \qquad \int_{\mathbf{D}} |Q|^4 \mathrm{e}^{-U} \mathrm{d}Q \le C, \end{gathered} \tag{1.8}$$

and

$$\begin{gathered} |\nabla_Q^k (Q \nabla_Q U)| \le C(|Q||\nabla_Q U| + 1), \qquad \int_{\mathbf{D}} \left|\nabla_Q^k \left(Q \nabla_Q U \mathrm{e}^{-\frac{U}{2}}\right)\right|^2 \mathrm{d}Q \le C, \\ \left|\nabla_Q^k \left(\triangle_Q U - \frac{|\nabla_Q U|^2}{2}\right)\right| \le C(1 + |\nabla_Q U|^2), \end{gathered} \tag{1.9}$$

for all integers $0 \le k \le 2$. More precisely, we only need $k = 0$ in (1.9) for Theorem 1.1, and $0 \le k \le 2$ in (1.9) is required for Theorem 1.2. In particular, these conditions hold for the Hooke model, namely the case $U = |Q|^2/2$.

For the convenience of the readers, let us recall some basic facts about the Littlewood–Paley theory — check [3] and [4] for more details. Let $B \stackrel{\text{def}}{=} \{\xi \in \mathbf{R}^2,\ |\xi| \leq \frac{4}{3}\}$ and $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbf{R}^2,\ \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. Let $\chi \in C_c^\infty(B)$ and $\varphi \in C_c^\infty(\mathcal{C})$ which satisfy

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \xi \in \mathbf{R}^2.$$

We denote $h \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varphi$ and $\tilde{h} \stackrel{\text{def}}{=} \mathcal{F}^{-1}\chi$. Then the dyadic operators Δ_j and S_j can be defined as follows

$$\begin{aligned} &\Delta_j f = \varphi(2^{-j}D)f = 2^{2j}\int_{\mathbf{R}^2} h(2^j y) f(x-y)\,\mathrm{d}y, \quad \text{for } j \geq 0,\\ &S_j f = \chi(2^{-j}D)f = \sum_{-1\leq k\leq j-1} \Delta_k f = 2^{2j}\int_{\mathbf{R}^2} \tilde{h}(2^j y) f(x-y)\,\mathrm{d}y, \quad \text{and} \quad \Delta_{-1} f = S_0 f. \end{aligned} \tag{1.10}$$

With the introduction of Δ_j and S_j, we can define the following space:

Definition 1.1. Let $s \in \mathbf{R}$, we define

$$\tilde{C}^s(\mathbf{R}^2; L^2(\mathbf{D})) = \{f \in \mathcal{S}'(\mathbf{R}^2 \times \mathbf{D}); \quad \|f\|_s < \infty\},$$

with

$$\|f\|_s = \sup_{j\geq -1} 2^{js} \left\| \left(\int_{\mathbf{D}} |\Delta_j f|^2\,\mathrm{d}Q \right)^{\frac{1}{2}} \right\|_{L_x^\infty}$$

Now we state our main result as follows.

Theorem 1.1. *Let $1 < s < 2$. Let $u_0 \in H^1(\mathbf{R}^2) \cap C^s(\mathbf{R}^2)$, $f_0 \in H^1(\mathbf{R}^2; L^2(\mathbf{D})) \cap \tilde{C}^{s-1}(\mathbf{R}^2; L^2(\mathbf{D}))$, and $|Q| f_0 \in L^\infty(\mathbf{R}^2; L^2(\mathbf{D}))$. Then (1.5)–(1.7) has a unique global solution (u, f) such that for any $T > 0$, there hold*

$$\begin{aligned} &u \in C([0, +\infty); H^1(\mathbf{R}^2) \cap C^s(\mathbf{R}^2)) \cap L^2((0, T); H^2(\mathbf{R}^2)),\\ &f \in C([0, +\infty); H^1(\mathbf{R}^2; L^2(\mathbf{D})) \cap \tilde{C}^{s-1}(\mathbf{R}^2; L^2(\mathbf{D}))),\\ &|Q| f \in L^\infty((0, +\infty) \times \mathbf{R}^2; L^2(\mathbf{D})), \quad \text{and} \quad \left(\nabla_Q + \frac{1}{2}\nabla_Q U\right) f \in L^2((0, T); H^1(\mathbf{R}^2; L^2(\mathbf{D}))). \end{aligned} \tag{1.11}$$

Furthermore, there holds

$$\|u(t)\|_{C^s} + \|f(t)\|_{s-1} \leq C_0(1 + \|u_0\|_{C^s} + \|f_0\|_{s-1})^{\exp(C_0 t)}, \quad \forall t < \infty \tag{1.12}$$

where C_0 only depends on $\|u_0\|_{L^2}^2 + \|f_0\|_{L^2}^2 + \|(1 + |Q|) f_0\|_{L^\infty(\mathbf{R}^2; L^2(\mathbf{D}))}^2$.

Remark 1.1. Compared with [7,13], the new ingredient in the above theorem is the time growth estimate (1.12), which is not possible via a Lyapunov functional argument. Actually the method of the proof of (1.12) can be extended to the Rod-like model (without the anti-symmetric assumption on $S(u)$ in (1.1)) and co-rotational FENE model. But as there are already global well-posedness results in [7,13], we chose to present the estimate (1.12) for the model (1.1).

With smoother initial data together with higher momentum integrability on f_0, we can prove further regularity of the solutions (u, f) obtained in Theorem 1.1. For a clear presentation, we will only give a proof of the following theorem

Theorem 1.2. *Under the assumptions of Theorem 1.1, we assume further that $u_0 \in H^2(\mathbf{R}^2)$, $f_0 \in H^2(\mathbf{R}^2; L^2(\mathbf{D}))$ and $|Q| f_0 \in H^1(\mathbf{R}^2; L^2(\mathbf{D}))$, $\sqrt{1 + |Q|^2}\nabla_Q f_0 \in L^\infty(\mathbf{R}^2; L^2(\mathbf{D}))$. Then for any $T > 0$, we have*

$$\begin{aligned} &u \in C_{\text{loc}}([0, +\infty); H^2(\mathbf{R}^2)) \cap L^2((0, T); H^3(\mathbf{R}^2)),\\ &f \in C_{\text{loc}}([0, +\infty); H^2(\mathbf{R}^2; L^2(\mathbf{D}))), \quad \sqrt{1 + |Q|^2}\nabla_Q f \in L^\infty((0, T) \times \mathbf{R}^2; L^2(\mathbf{D})),\\ &\text{and} \quad \left(\nabla_Q + \frac{1}{2}\nabla_Q U\right) f \in L^2((0, T); H^2(\mathbf{R}^2; L^2(\mathbf{D}))). \end{aligned} \tag{1.13}$$

As can be seen from the proof of Theorem 1.2, we can also get higher regularity, namely for any integer s, $s \geq 3$, we have

Theorem 1.3. *Under the assumptions of Theorem 1.1, we assume further that $u_0 \in H^s(\mathbf{R}^2)$, $f_0 \in H^s(\mathbf{R}^2; L^2(\mathbf{D}))$ and $|Q| f_0 \in H^1(\mathbf{R}^2; L^2(\mathbf{D}))$, $\sqrt{1+|Q|^2}\nabla_Q f_0 \in L^\infty(\mathbf{R}^2; L^2(\mathbf{D}))$. Then for any $T > 0$, we have*

$$\begin{aligned} & u \in C_{\mathrm{loc}}([0,+\infty); H^s(\mathbf{R}^2)) \cap L^2((0,T); H^{s+1}(\mathbf{R}^2)), \\ & f \in C_{\mathrm{loc}}([0,+\infty); H^s(\mathbf{R}^2; L^2(\mathbf{D}))), \quad \sqrt{1+|Q|^2}\nabla_Q f \in L^\infty((0,T)\times \mathbf{R}^2; L^2(\mathbf{D})), \\ & \textit{and} \quad \left(\nabla_Q + \frac{1}{2}\nabla_Q U\right) f \in L^2((0,T); H^s(\mathbf{R}^2; L^2(\mathbf{D}))). \end{aligned} \tag{1.14}$$

The proof of Theorem 1.3 will be left to the reader (see also Remark 2.1)

Let us end this section by some notations used in the sequel:

Notations: We will denote $[a; b]$ the commutator between a and b, ∂^α the derivatives with respect to x variables, and

$$\|\phi\|_{L^p} = \left(\int_{\mathbf{R}^2\times\mathbf{D}} |\phi(x,Q)|^p \,\mathrm{d}x\,\mathrm{d}Q\right)^{\frac{1}{p}}, \quad |g|_s = \left(\int_{\mathbf{R}^2\times\mathbf{D}} \sum_{|\alpha|\le s} |\partial^\alpha g|^2 \,\mathrm{d}x\,\mathrm{d}Q\right)^{\frac{1}{2}},$$

and $\|u\|_{H^s}$ the standard Sobolev norm of u when u depends only on the x variable. We shall use the convention (f,g) to denote both the inner product on $\mathbf{R}^2$, $\int_{\mathbf{R}^2} fg\,\mathrm{d}x$, and on $\mathbf{R}^2\times\mathbf{D}$, $\int_{\mathbf{R}^2\times\mathbf{D}} fg\,\mathrm{d}x\,\mathrm{d}q$. And we will denote $C_{\dots}$ a uniform positive constant which depends on the listed variables, but may change from line to line.

2. Proof of Theorems 1.1 and 1.2

Let us first recall the following lemma from [12], which will be constantly used in the sequel.

Lemma 2.1. *Let f satisfy $\int_{\mathbf{D}} f\sqrt{M}\mathrm{d}Q = 0$, and U satisfy (1.8). Then*

$$\begin{aligned} & \|f\|_{L^2} + \|\nabla_Q U f\|_{L^2} + \|\nabla_Q f\|_{L^2} \le C \left\| \left(\nabla_Q f + \frac{1}{2}\nabla_Q U f\right)\right\|_{L^2}, \\ & \|\nabla_Q U Q f\|_{L^2} \le C \left\|(1+|Q|^2)^{\frac{1}{2}} \left(\nabla_Q f + \frac{1}{2}\nabla_Q U f\right)\right\|_{L^2}. \end{aligned}$$

Now let us present the proof of Theorem 1.1.

Proof of Theorem 1.1. As the existence of solutions to (1.5)–(1.7) can be essentially proved from the *a priori* estimates, for simplicity, we only present the *a priori* estimate for smooth enough solutions to (1.5)–(1.7).

Step 1. The estimate of $\|u(t)\|_{H^1}$ and $|f|_1$.

Standard energy estimate applied to the first equation of (1.5) gives

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^1}^2 + \|\nabla u\|_{H^1}^2 &= \sum_{|\alpha|\le 1} \left[-([\partial^\alpha; u]\cdot\nabla u, \partial^\alpha u) + (\partial^\alpha \mathtt{div}\tau, \partial^\alpha u)\right] \\ &\le C(\|\nabla u\|_{L^\infty}\|u\|_{H^1}^2 + |f|_1\|\nabla u\|_{H^1}), \end{aligned} \tag{2.1}$$

where we used (1.6) and (1.9) such that

$$\|\tau\|_{H^1} \le \left(\int_{\mathbf{D}} \left|\nabla_Q U \otimes Q \mathrm{e}^{-\frac{U}{2}}\right|^2 \mathrm{d}Q\right)^{\frac{1}{2}} |f|_1 \le C|f|_1.$$

Similarly, we apply ∂^α with $|\alpha| \le 1$ to the third equation of (1.5), and take $L^2(\mathbf{R}^2\times\mathbf{D})$ inner product of the resulting equation with $\partial^\alpha f$ to get

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|f|_1^2 + \left[|\nabla_Q f|_1^2 - \sum_{|\alpha|\le 1}\frac{1}{2}\left(\left(\triangle_Q U - \frac{1}{2}|\nabla_Q U|^2\right)\partial^\alpha f, \partial^\alpha f\right)\right] \\ & \quad = -\sum_{|\alpha|\le 1}\left[(\partial^\alpha(u\cdot\nabla f), \partial^\alpha f) + (\partial^\alpha \nabla_Q\cdot(S(u)Qf), \partial^\alpha f)\right]. \end{aligned} \tag{2.2}$$

By integration by parts, we get

$$-\frac{1}{2}(\triangle_Q U \partial^\alpha f, \partial^\alpha f) = (\nabla_Q U \partial^\alpha f, \nabla_Q \partial^\alpha f).$$

Thus the dissipation due to the Fokker–Planck kinetic dynamics is given by

$$|\nabla_Q f|_1^2 - \sum_{|\alpha|\le 1} \frac{1}{2}\left(\left(\triangle_Q U - \frac{1}{2}|\nabla_Q U|^2\right)\partial^\alpha f, \partial^\alpha f\right) = \left|\nabla_Q f + \frac{1}{2}\nabla_Q U f\right|_1^2.$$

Since $\operatorname{div} u = 0$, we have

$$\sum_{|\alpha|\le 1}(\partial^\alpha(u\cdot\nabla f), \partial^\alpha f) = \sum_{|\alpha|\le 1}(\partial^\alpha u\cdot\nabla f, \partial^\alpha f) \le \|\nabla u\|_{L^\infty}|f|_1^2.$$

Then, we notice that $\operatorname{div} u = 0$ implies that $\operatorname{div}_Q(S(u)Q) = 0$. Hence, we get

$$\begin{aligned}\sum_{|\alpha|\le 1}(\partial^\alpha\nabla_Q\cdot(S(u)Qf), \partial^\alpha f) &= -\sum_{|\alpha|\le 1}((\partial^\alpha S(u))Qf, \partial^\alpha\nabla_Q f)\\ &\le \left\|\left(\int_{\mathbf{D}}|Qf|^2\mathrm{d}Q\right)^{\frac{1}{2}}\right\|_{L^\infty}\|\nabla u\|_{H^1}|\nabla_Q f|_1.\end{aligned}$$

Therefore, for any $\delta > 0$, we get by summing (2.1) and $\delta\times$ (2.2)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^1}^2 + \delta|f|_1^2) + \|\nabla u\|_{H^1}^2 + 2\delta\left|\nabla_Q f + \frac{1}{2}\nabla_Q U f\right|_1^2\\ &\quad\le C(1+\|\nabla u\|_{L^\infty})(\|u\|_{H^1}^2 + |f|_1^2) + \delta\left\|\left(\int_{\mathbf{D}}|Qf|^2\mathrm{d}Q\right)^{\frac{1}{2}}\right\|_{L^\infty}\|\nabla u\|_{H^1}|\nabla_Q f|_1.\end{aligned} \tag{2.3}$$

Step 2. The estimate of $\|\int_{\mathbf{D}}|Qf|^2\mathrm{d}Q\|_{L^\infty}$.

Multiplying the f equation in (1.5) by f, we get by integrating the resulting equation over $\mathbf{D}$ that

$$\frac{1}{2}\partial_t\int_{\mathbf{D}}|f(t)|^2\,\mathrm{d}Q + \frac{1}{2}u\cdot\nabla\int_{\mathbf{D}}|f(t)|^2\,\mathrm{d}Q + \int_{\mathbf{D}}\left|\nabla_Q f + \frac{1}{2}\nabla_Q U f\right|^2\mathrm{d}Q = -\int_{\mathbf{D}}\nabla_Q\cdot(S(u)Qf)f\,\mathrm{d}Q = 0, \tag{2.4}$$

which together with the standard L^p estimate of the transport equation gives

$$\begin{aligned}&\left\|\left(\int_{\mathbf{D}}|f(t,x,Q)|^2\,\mathrm{d}Q\right)^{\frac{1}{2}}\right\|_{L^p} \le \left\|\left(\int_{\mathbf{D}}|f_0(x,Q)|^2\,\mathrm{d}Q\right)^{\frac{1}{2}}\right\|_{L^p}, \quad \text{for } p = 2,\infty, \quad \text{and}\\ &\int_0^\infty\int_{\mathbf{R}^2}\int_{\mathbf{D}}\left|\nabla_Q f + \frac{1}{2}\nabla_Q U f\right|^2\,\mathrm{d}Q\,\mathrm{d}x\,\mathrm{d}t \le 2\int_{\mathbf{R}^2}\int_{\mathbf{D}}|\ f_0\ |^2\ \mathrm{d}Q\,\mathrm{d}x.\end{aligned} \tag{2.5}$$

Similar to (2.4), we have

$$\begin{aligned}&\frac{1}{2}\partial_t\int_{\mathbf{D}}|Q|^2|f(t)|^2\,\mathrm{d}Q + \frac{1}{2}u\cdot\nabla\int_{\mathbf{D}}|Q|^2|f(t)|^2\,\mathrm{d}Q\\ &\quad-\int_{\mathbf{D}}\left(\triangle_Q + \frac{1}{2}\triangle_Q U - \frac{1}{4}|\nabla_Q U|^2\right)f|Q|^2f\mathrm{d}Q = -\int_{\mathbf{D}}\nabla_Q\cdot(S(u)Qf)|Q|^2f\,\mathrm{d}Q.\end{aligned} \tag{2.6}$$

Moreover, we get by integration by parts

$$\begin{aligned}-\int_{\mathbf{D}}\left(\triangle_Q + \frac{1}{2}\triangle_Q U - \frac{1}{4}|\nabla_Q U|^2\right)f|Q|^2f\mathrm{d}Q &= \int_{\mathbf{D}}|Q|^2\left|\nabla_Q f + \frac{1}{2}\nabla_Q U f\right|^2\mathrm{d}Q\\ &\quad-2\int_{\mathbf{D}}|f|^2\mathrm{d}Q + \int_{\mathbf{D}}\nabla_Q U\cdot Q|f|^2\mathrm{d}Q.\end{aligned}$$

Using the fact that $\operatorname{div}_Q(S(u)Q) = 0$ and $S(u)$ is an antisymmetric matrix, we get

$$\begin{aligned}\int_{\mathbf{D}}\nabla_Q\cdot(S(u)Qf)|Q|^2f\,\mathrm{d}Q &= \frac{1}{2}\int_{\mathbf{D}}S(u)Q\cdot\nabla_Q|f|^2|Q|^2\,\mathrm{d}Q\\ &= -\int_{\mathbf{D}}S(u)Q\cdot Q|f|^2\,\mathrm{d}Q = 0.\end{aligned}$$

So we get by summing up (2.4) and $\eta\times$ (2.6) that

$$\frac{1}{2}\partial_t\int_{\mathbf{D}}(1+\eta|Q|^2)|f(t)|^2\,\mathrm{d}Q+\frac{1}{2}u\cdot\nabla\int_{\mathbf{D}}(1+\eta|Q|^2)|f(t)|^2\,\mathrm{d}Q+\int_{\mathbf{D}}(1+\eta|Q|^2)\left|\nabla_Q f+\frac{1}{2}\nabla_Q U f\right|^2\mathrm{d}Q$$
$$=2\eta\int_{\mathbf{D}}|f|^2\mathrm{d}Q-\eta\int_{\mathbf{D}}\nabla_Q U\cdot Q|f|^2\mathrm{d}Q. \tag{2.7}$$

On the other hand, thanks to (1.8) and Lemma 2.1, we have

$$\int_{\mathbf{D}}(2-\nabla_Q U\cdot Q)|f|^2\mathrm{d}Q\le C\int_{\mathbf{D}}(1+|\nabla_Q U|^2)|f|^2\,\mathrm{d}Q\le C\int_{\mathbf{D}}\left|\nabla_Q f+\frac{1}{2}\nabla_Q U f\right|^2\mathrm{d}Q$$

Therefore, by taking η sufficiently small, we deduce from (2.7) that

$$\int_{\mathbf{D}}(1+\eta|Q|^2)|f(t,\Phi_t(x),Q)|^2\,\mathrm{d}Q+\int_0^t\int_{\mathbf{D}}(1+\eta|Q|^2)\left|\left(\nabla_Q+\frac{1}{2}\nabla_Q U\right)f(t,\Phi_{t'}(x),Q)\right|^2\mathrm{d}Q\mathrm{d}t'$$
$$\le\int_{\mathbf{D}}(1+\eta|Q|^2)|f_0(x,Q)|^2\mathrm{d}Q, \tag{2.8}$$

where $\Phi_t(x)$ is determined by

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x)=u(t,\Phi_t(x)),\\ \Phi_t(x)|_{t=0}=x.\end{cases}$$

Thanks to Lemma 2.1 and (2.8), we get by taking δ small enough in (2.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^1}^2+\delta|f|_1^2)+\frac{1}{2}\|\nabla u\|_{H^1}^2+\delta\left|\nabla_Q f+\frac{1}{2}\nabla_Q U f\right|_1^2\le C(1+\|\nabla u\|_{L^\infty})(\|u\|_{H^1}^2+|f|_1^2). \tag{2.9}$$

Step 3. Here, we estimate (u,f) in $C^s\times\tilde{C}^{s-1}$ by using some estimates in the spirit of the losing derivative estimate in [5]. Given any $0\le T_0<T$, let us first recall the following estimate of u from [13]: for any $\lambda>0$, there holds

$$M^s_{\lambda,[T_0,T]}(u)\le C\|u(T_0)\|_{C^s}+C\left(\frac{1}{\lambda}+\|u\|_{L^4_{[T_0,T]}(L^4)}\right)M^s_{\lambda,[T_0,T]}(u)+CM^{s-1}_{\lambda,[T_0,T]}(\tau), \tag{2.10}$$

where $M^s_{\lambda,[T_0,T]}(u)\overset{\mathrm{def}}{=}\sup_{j\ge-1,t\in[T_0,T]}2^{js-\Phi_\lambda(t)}\|\Delta_j u(t)\|_{L^\infty}$ with

$$\Phi_\lambda(t,t')\overset{\mathrm{def}}{=}\lambda\int_{t'}^t(1+\|\nabla u(t'')\|_{L^\infty})\mathrm{d}t''+\lambda\int_{t'}^t\left\|\int_{\mathbf{D}}(1+|Q|^2)|f(t'')|^2\mathrm{d}Q\right\|_{L^\infty}\mathrm{d}t'',$$
$$\Phi_\lambda(t)\overset{\mathrm{def}}{=}\Phi_\lambda(t,T_0).$$

On the other hand, thanks to (1.9), we have

$$M^{s-1}_{\lambda,[T_0,T]}(\tau)\le C\sup_{j\ge-1,t\in[T_0,T]}2^{j(s-1)-\Phi_\lambda(t)}\left\|\left(\int_{\mathbf{D}}|\Delta_j f(t,x,Q)|^2\mathrm{d}Q\right)^{\frac{1}{2}}\right\|_{L^\infty}. \tag{2.11}$$

This reduces the estimate of τ to that of f. In order to do so, let us set $f_j\overset{\mathrm{def}}{=}\Delta_j f$. Then we get by applying Δ_j to the f equation in (1.5) that

$$\partial_t f_j+u\cdot\nabla f_j-\left(\triangle_Q+\frac{\triangle_Q U}{2}-\frac{|\nabla_Q U|^2}{4}\right)f_j=-\left[\Delta_j(u\cdot\nabla f)-u\cdot\nabla f_j\right]-\Delta_j\left[\nabla_Q\cdot(S(u)Qf)\right].$$

Multiplying the above equation by f_j, and integrating the resulting equation over $\mathbf{D}$, we obtain

$$\frac{1}{2}\partial_t\int_{\mathbf{D}}|f_j|^2\mathrm{d}Q+\frac{1}{2}u\cdot\nabla\int_{\mathbf{D}}|f_j|^2\mathrm{d}Q+\int_{\mathbf{D}}\left|\nabla_Q f_j+\frac{1}{2}\nabla_Q U f_j\right|^2\mathrm{d}Q$$
$$=-\int_{\mathbf{D}}\left[\Delta_j(u\cdot\nabla f)-u\cdot\nabla f_j\right]f_j\mathrm{d}Q-\int_{\mathbf{D}}\Delta_j\left[\nabla_Q\cdot(S(u)Qf)\right]f_j\mathrm{d}Q$$
$$\overset{\mathrm{def}}{=}I+II.$$

Set $F_j(t,x) \stackrel{\text{def}}{=} \int_{\mathbf{D}} |f_j(t,x,Q)|^2 \mathrm{d}Q$. Then the proof of (4.13) in [13] ensures

$$|I| \le C F_j(t)^{\frac{1}{2}} \left[2^j \sum_{j' \gtrsim j} \|\Delta_{j'} u\|_{L^\infty} \left\| \left(\int_{\mathbf{D}} |f|^2 \mathrm{d}Q \right)^{\frac{1}{2}} \right\|_{L^\infty} + \|\nabla u\|_{L^\infty} \sum_{|j'-j| \le 5} F_{j'}(t)^{\frac{1}{2}} \right],$$

and the proof of (4.14) in [13] gives

$$|II| \le C \left\| \left(\int_{\mathbf{D}} |Q|^2 |f|^2 \mathrm{d}Q \right)^{\frac{1}{2}} \right\|_{L^\infty} \left(\int_{\mathbf{D}} |\nabla_Q f_j|^2 \mathrm{d}Q \right)^{\frac{1}{2}} \times \left[\sum_{j' \gtrsim j} 2^{j'} \|\Delta_{j'} u\|_{L^\infty} + 2^{-j} \sum_{j' \lesssim j} 2^{2j'} \|\Delta_{j'} u\|_{L^\infty} \right].$$

Therefore, we obtain

$$\begin{aligned}
&\partial_t \int_{\mathbf{D}} |f_j|^2 \mathrm{d}Q + u \cdot \nabla \int_{\mathbf{D}} |f_j|^2 \mathrm{d}Q + 2 \int_{\mathbf{D}} \left| \nabla_Q f_j + \frac{1}{2} \nabla_Q U f_j \right|^2 \mathrm{d}Q \\
&\quad \le C F_j(t)^{\frac{1}{2}} \left[2^j \sum_{j' \gtrsim j} \|\Delta_{j'} u\|_{L^\infty} \left\| \left(\int_{\mathbf{D}} |f|^2 \mathrm{d}Q \right)^{\frac{1}{2}} \right\|_{L^\infty} + \|\nabla u\|_{L^\infty} \sum_{|j'-j| \le 5} F_{j'}(t)^{\frac{1}{2}} \right] \\
&\qquad + C_\epsilon \left\| \int_{\mathbf{D}} |Q|^2 |f|^2 \mathrm{d}Q \right\|_{L^\infty} \left[\sum_{j' \gtrsim j} 2^{j'} \|\Delta_{j'} u\|_{L^\infty} + 2^{-j} \sum_{j' \lesssim j} 2^{2j'} \|\Delta_{j'} u\|_{L^\infty} \right]^2 + \epsilon \int_{\mathbf{D}} |\nabla_Q f_j|^2 \mathrm{d}Q,
\end{aligned}$$

which together with Lemma 2.1 gives

$$\begin{aligned}
\partial_t \int_{\mathbf{D}} |f_j|^2 \mathrm{d}Q + u \cdot \nabla \int_{\mathbf{D}} |f_j|^2 \mathrm{d}Q &\le C F_j(t)^{\frac{1}{2}} \left[2^j \sum_{j' \gtrsim j} \|\Delta_{j'} u\|_{L^\infty} \left\| \left(\int_{\mathbf{D}} |f|^2 \mathrm{d}Q \right)^{\frac{1}{2}} \right\|_{L^\infty} + \|\nabla u\|_{L^\infty} \sum_{|j'-j| \le 5} F_{j'}(t)^{\frac{1}{2}} \right] \\
&\quad + C \left\| \int_{\mathbf{D}} |Q|^2 |f|^2 \mathrm{d}Q \right\|_{L^\infty} \left[\sum_{j' \gtrsim j} 2^{j'} \|\Delta_{j'} u\|_{L^\infty} + 2^{-j} \sum_{j' \lesssim j} 2^{2j'} \|\Delta_{j'} u\|_{L^\infty} \right]^2. \qquad (2.12)
\end{aligned}$$

Set

$$M^{(s-1)}_{\lambda,[T_0,T]}(f) = \left(\sup_{j \ge -1, t \in [T_0,T]} 2^{2j(s-1) - 2\Phi_\lambda(t)} \left\| \int_{\mathbf{D}} |f_j|^2 \mathrm{d}Q \right\|_{L^\infty} \right)^{\frac{1}{2}}.$$

Using that $1 < s < 2$ and using the standard L^p estimate for the transport equations, we get after multiplying (2.12) with $2^{2j(s-1)-2\Phi_\lambda(t)}$ that

$$\begin{aligned}
[M^{(s-1)}_{\lambda,[T_0,T]}(f)]^2 &\le \sup_{j \ge -1} 2^{2j(s-1)} \|F_j(T_0)\|_{L^\infty} + C [M^s_{\lambda,[T_0,T]}(u)]^2 \sup_{t \in [T_0,T]} \int_{T_0}^t 2^{-2\Phi_\lambda(t,t')} \left\| \int_{\mathbf{D}} (1+|Q|^2) |f(t')|^2 \mathrm{d}Q \right\|_{L^\infty} \mathrm{d}t' \\
&\quad + C [M^{(s-1)}_{\lambda,[T_0,T]}(f)]^2 \sup_{t \in [T_0,T]} \int_{T_0}^t 2^{-2\Phi_\lambda(t,t')} (1 + \|\nabla u(t')\|_{L^\infty}) \mathrm{d}t' \\
&\le \|f(T_0)\|_{s-1} + \frac{C}{\lambda} ([M^{(s-1)}_{\lambda,[T_0,T]}(f)]^2 + [M^s_{\lambda,[T_0,T]}(u)]^2),
\end{aligned}$$

from which, we deduce for λ sufficiently large that

$$[M^{(s-1)}_{\lambda,[T_0,T]}(f)]^2 \le C \|f(T_0)\|^2_{s-1} + \frac{C}{\lambda} [M^s_{\lambda,[T_0,T]}(u)]^2. \qquad (2.13)$$

Plugging (2.13) in (2.11) and using (2.10), we get

$$\begin{aligned}
&M^{s-1}_{\lambda,[T_0,T]}(\tau) \le C \|f(T_0)\|_{s-1} + \frac{C}{\lambda^{\frac{1}{2}}} M^s_{\lambda,[T_0,T]}(u), \quad \text{and} \\
&M^s_{\lambda,[T_0,T]}(u) \le C(\|u(T_0)\|_{C^s} + \|f(T_0)\|_{s-1}) + C \|u\|_{L^4_{[T_0,T]}(L^4)} M^s_{\lambda,[T_0,T]}(u). \qquad (2.14)
\end{aligned}$$

Step 4. The closed estimates of $\|u(t)\|_{H^1}$, $|f(t)|_1$, $\|u(t)\|_{C^s}$ and $\|f(t)\|_{s-1}$.

First, thanks to (2.9), we get by applying Gronwall inequality that

$$\|u(t)\|_{H^1}^2 + \delta|f(t)|_1^2 + \frac{1}{2}\int_0^t \|\nabla u(t')\|_{H^1}^2 \mathrm{d}t' + \delta\int_0^t \left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right) f(t')\right|_1^2 \mathrm{d}t'$$
$$\leq (\|u_0\|_{H^1}^2 + \delta|f_0|_1^2)\exp\left(C\int_0^t (1+\|\nabla u(t')\|_{L^\infty})\mathrm{d}t'\right). \tag{2.15}$$

On the other hand, taking $T_0 = 0$ in (2.14), we find for $\lambda \gg 1$

$$M_\lambda^s(u)(T) \leq CB_0 + C\|u\|_{L^4_{[0,T]}(L^4)} M_\lambda^s(u)(T), \tag{2.16}$$

where $M_\lambda^s(u)(T) = M_{\lambda,[0,T]}^s(u)$, and $B_0 \overset{\text{def}}{=} \|u_0\|_{C^s} + \|f_0\|_{s-1}$. Next using (1.6), (1.9) and (2.5) and applying Lemma 2.1, we get that

$$\int_0^\infty \|\tau\|_{L^2}^2 \,\mathrm{d}t \leq \|f_0\|_{L^2}^2.$$

This bound, together with the proof of Theorem 3.1 of [14] imply that $u \in C([0,T]; L^2(\mathbf{R}^2)) \cap L^2([0,T]; H^1(\mathbf{R}^2))$ for any $T < \infty$. Moreover, there holds

$$\|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{L^2}^2 \mathrm{d}t' \leq \|u_0\|_{L^2}^2 + \int_0^t \|\tau(t')\|_{L^2}^2 \mathrm{d}t' \leq C(\|u_0\|_{L^2}^2 + \|f_0\|_{L^2}^2), \tag{2.17}$$

from which and the standard interpolation inequality we get

$$\|u\|_{L^4_{[0,T]}(L^4)} \leq C\|u\|_{L^\infty_{[0,T]}(L^2)}^{\frac{1}{2}} \|\nabla u\|_{L^2_{[0,T]}(L^2)}^{\frac{1}{2}}.$$

Hence, we deduce that there exists a small number δ which only depends on $(\|u_0\|_{L^2}^2 + \|f_0\|_{L^2}^2)$ such that if $T < \delta$, then $\|u\|_{L^4_{[0,T]}(L^4)} \leq \frac{1}{2}$. Hence, using (2.16) we get that

$$M_\lambda^s(u)(T) \leq 2CB_0. \tag{2.18}$$

With (2.18), we get by repeating step 4 of [13] and using (2.13) that

$$\sup_{t\in[0,T]} \|u(t)\|_{C^s} \leq C(T) \quad \text{and} \quad \sup_{t\in[0,T]} \|f(t)\|_{s-1} \leq C(T)$$

for some constant $C(T)$ which depends on B_0 and $\|(1+|Q|)f_0\|_{L^\infty(\mathbf{R}^2;L^2(\mathbf{D}))}$. Moreover, (2.15) ensures that (1.5)–(1.7) has a unique solution (u, f) on $[0, T]$ with

$$\|u(t)\|_{H^1\cap C^s}^2 + |f(t)|_1^2 + \|f(t)\|_{s-1}^2 + \int_0^t \|\nabla u(t')\|_{H^1}^2 \mathrm{d}t' + \int_0^t \left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right) f(t')\right|_1^2 \mathrm{d}t'$$
$$\leq C\left(T, \|u_0\|_{H^1\cap C^s}, |f_0|_1, \|f_0\|_{s-1}, \|\sqrt{1+|Q|^2} f_0\|_{L_x^\infty(L_Q^2)}\right), \quad \text{and} \tag{2.19}$$
$$\|\sqrt{1+|Q|^2} f(t)\|_{L_x^\infty(L_Q^2)} \leq C\|\sqrt{1+|Q|^2} f_0\|_{L_x^\infty(L_Q^2)}.$$

Now let us denote T^* the maximal time of existence of (u, f) such that (2.19) holds. Then if $T^* < \infty$, we have

$$\lim_{t\to T^*} \|u(t)\|_{C^s} = \infty. \tag{2.20}$$

Otherwise, $\sup_{t\in[0,T^*]} \|u(t)\|_{C^s} < \infty$. Then thanks to (2.15), we have

$$\sup_{t\in[0,T^*]} [\|u(t)\|_{H^1}^2 + |f(t)|_1^2] + \int_0^{T^*} \left[\|\nabla u(t)\|_{H^1}^2 + \left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right) f(t)\right|_1^2\right] \mathrm{d}t < \infty,$$

and (2.13) gives

$$M_{\lambda,[0,T^*]}^{(s-1)}(f) < \infty.$$

Notice that from (2.8), we deduce that

$$\Phi_\lambda(T^*) \leq C\lambda T^*\left[1 + \sup_{t\in[0,T^*]} \|u(t)\|_{C^s} + \left\|\int_{\mathbf{D}} (1+|Q|^2)|f_0(x,Q)|^2 \,\mathrm{d}Q\right\|_{L^\infty}\right],$$

therefore, we obtain

$$\sup_{t\in[0,T^*]} \|f(t)\|_{s-1} < \infty,$$

which contradicts the assumption that T^* is the maximal time of existence.

In what follows, we shall prove that

$$\sup_{t\in[0,T^*)} \|u(t)\|_{C^s} \le C\left(T^*, \|u_0\|_{H^1\cap C^s}, \|f_0\|_{s-1}, \|\sqrt{1+|Q|^2} f_0\|_{L_x^\infty(L_Q^2)}\right), \tag{2.21}$$

which contradicts (2.20), and therefore $T^* = \infty$. The main idea of the proof of (2.21) follows the argument of step 4 of [13] (see also [7]). The new observation here is that not only can we prove (2.21), but we also get a time growth estimate for $\|u(t)\|_{C^s}$ and $\|f(t)\|_{s-1}$ for t large enough. For completeness, we are going to present the details here.

Let $T < T^*$ with $T^* - T \ll 1$. Then thanks to (2.17) and the proof of Theorem 3.3 of [5] (namely, Lemma 3.5 of [5]), we conclude that for any $\varepsilon > 0$, there exists $\delta_0 > 0$ which only depends on $C(\|u_0\|_{L^2}^2 + \|f_0\|_{L^2}^2)$ such that for $T_0 < T$ and $0 < T - T_0 \le \delta_0$, there holds

$$\|\nabla u\|_{\widetilde{L}^1_{[T_0,T]}(C^0)} \stackrel{\text{def}}{=} \sup_{j\ge -1} 2^j \|\Delta_j u\|_{L^1([T_0,T];L^\infty(\mathbf{R}^2))} < \varepsilon. \tag{2.22}$$

On the other hand, we can always take $\delta_0 \le \delta$ and hence, thanks to (2.13) and (2.14), a proof similar to the proof of (2.18) ensures that

$$M^s_{\lambda,[T_0,T]}(u) + M^{(s-1)}_{\lambda,[T_0,T]}(f) \le C B(T_0),$$

where $B(t) \stackrel{\text{def}}{=} \|u(t)\|_{C^s} + \|f(t)\|_{s-1}$.

Therefore, we get by using (2.8) that

$$B(t) \le C_0 B(T_0) \exp\left(C_0 \int_{T_0}^t (1 + \|\nabla u(t')\|_{L^\infty})\, \mathrm{d}t'\right), \quad \text{for } t \in [T_0, T],$$

where C_0 depends on $\|u_0\|_{L^2}^2 + \|f_0\|_{L^2}^2$, $\|(1+|Q|)f_0\|^2_{L^\infty(\mathbf{R}^2;L^2(\mathbf{D}))}$ and may change from one line to an other. It follows that

$$\log(e + B(t)) \le C_0 + \log(e + B(T_0)) + C_0 \int_{T_0}^t (1 + \|\nabla u(t')\|_{L^\infty}) \mathrm{d}t', \quad \text{for } t \in [T_0, T], \tag{2.23}$$

Next, we will assume that $\delta_0/2 \le T - T_0 \le \delta_0$ which allows us to replace the first C_0 of the right hand side by $C_0(T - T_0)$. Using Littlewood–Paley theory, for any integer N, which will be determined later on, we can decompose u into three parts: low frequency part, middle frequency part, and high frequency part:

$$u = \Delta_{-1} u + \sum_{j=0}^N \Delta_j u + \sum_{j>N} \Delta_j u,$$

which implies

$$\|\nabla u\|_{L^\infty} \le C\|u\|_{L^2} + \sum_{j=0}^N \|\Delta_j \nabla u\|_{L^\infty} + C 2^{-N(s-1)} \|u\|_{C^s}, \quad \text{for } s > 1,$$

from which, we deduce that

$$\begin{aligned}\int_{T_0}^t \|\nabla u(t')\|_{L^\infty} \mathrm{d}t' \le{}& C\|u\|_{L^1_{[T_0,t]}(L^2)} + (N+1)\|\nabla u\|_{\widetilde{L}^1_{[T_0,t]}(C^0)} \\ &+ C(T - T_0) 2^{-N(s-1)} \|u\|_{L^\infty_{[T_0,t]}(C^s)}, \quad \text{for } T_0 \le t \le T.\end{aligned} \tag{2.24}$$

Now we choose N such that $1 \le 2^{-N(s-1)} \|u\|_{L^\infty_{[T_0,t]}(C^s)} \le 2$, i.e.

$$N \sim \frac{\log(e + \|u\|_{L^\infty_{[T_0,t]}(C^s)})}{(s-1)\log 2}.$$

Then, we get by substituting (2.24) into (2.23) that

$$\log(e + \max_{t\in[T_0,T]} B(t)) \le \log(e + B(T_0)) + C_0\|u\|_{L^1_{[T_0,T]}(L^2)} + C_0(T - T_0) + C_0 \|\nabla u\|_{\widetilde{L}^1_{[T_0,T]}(C^0)} \log(e + \max_{t\in[T_0,T]} B(t)).$$

Thus, if we choose ε small enough, we get by using (2.17) and (2.22) that

$$\begin{aligned}\log(e+\max_{t\in[T_0,T]}B(t)) &\le 2\log(e+B(T_0))+C_0(1+\|u_0\|_{L^2}+\|f_0\|_{L^2})(T-T_0)\\ &\le 2\log(e+B(T_0))+C_0(T-T_0)\end{aligned}\tag{2.25}$$

for $\delta_0/2<T-T_0<\delta_0$. This implies that

$$\log(e+B(t))\le 2^{2t/\delta_0}(\log(e+B(0))+C_0),$$

for any $t<\infty$. This proves (1.12) and the proof of Theorem 1.1 is ended. ∎

Now let us turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. As the regularity of the solution (u,f) constructed in Theorem 1.1 can essentially be obtained by estimates, here we only present the *a priori* estimates for smooth enough solutions of (1.5)–(1.7).

Step 1. The estimate of $|\sqrt{1+|Q|^2}f(t)|_1$.

We first get by a similar proof of (2.2) that

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}|Qf|_1^2-\sum_{|\alpha|\le1}\int_{\mathbf D}\left(\left(-\Delta_Q+\frac12\triangle_Q U-\frac14|\nabla_Q U|^2\right)\partial^\alpha f,|Q|^2\partial^\alpha f\right)\mathrm dQ\\ &\quad=-\sum_{|\alpha|\le1}\left[(\partial^\alpha(u\cdot\nabla f),|Q|^2\partial^\alpha f)+(\partial^\alpha\nabla_Q\cdot(S(u)Qf),|Q|^2\partial^\alpha f)\right].\end{aligned}$$

It is easy to observe that

$$\sum_{|\alpha|\le1}(\partial^\alpha(u\cdot\nabla f),|Q|^2\partial^\alpha f)|\le\|\nabla u\|_{L^\infty}|Qf|_1^2,$$

and as $S(u)Q\cdot Q=0$, we get by integration by parts and using (1.8) that

$$\begin{aligned}\sum_{|\alpha|\le1}|(\partial^\alpha\nabla_Q\cdot(S(u)Qf),|Q|^2\partial^\alpha f)|&=\sum_{|\alpha|=1}|((\partial^\alpha S(u))Q\cdot\nabla_Q f,|Q|^2\partial^\alpha f)|\\ &\le\left\|\left(\int_{\mathbf D}|Q\nabla_Q f|^2\mathrm dQ\right)^{\frac12}\right\|_{L^\infty}\|\nabla u\|_{H^1}||Q|^2f|_1\\ &\le C\|Q\nabla_Q f\|_{L^\infty_x(L^2_Q)}\|\nabla u\|_{H^1}|(1+|Q\nabla_Q U|)f|_1.\end{aligned}$$

Therefore, thanks to Lemma 2.1, (2.4) and (2.7), we obtain

$$\begin{aligned}&\frac{\mathrm d}{\mathrm dt}|\sqrt{1+|Q|^2}f|_1^2+\left|\sqrt{1+|Q|^2}\left(\nabla_Q f+\frac12\nabla_Q Uf\right)\right|_1^2\\ &\quad\le C(\|\nabla u\|_{L^\infty}|\sqrt{1+|Q|^2}f|_1^2+\|Q\nabla_Q f\|^2_{L^\infty_x(L^2_Q)}\|\nabla u\|^2_{H^1}).\end{aligned}\tag{2.26}$$

This reduces the estimate of $|\sqrt{1+|Q|^2}f|_1$ to that of $\|\sqrt{1+|Q|^2}\nabla_Q f\|_{L^\infty_x(L^2_Q)}$. In order to do so, we first apply ∇_Q to the third equation of (1.5) and integrate the resulting equation over $\mathbf D$ to get

$$\begin{aligned}&\frac12\partial_t\int_{\mathbf D}|\nabla_Q f|^2\mathrm dQ+\frac12 u\cdot\nabla\int_{\mathbf D}|\nabla_Q f|^2\mathrm dQ+\int_{\mathbf D}\nabla_Q(S(u)Q\cdot\nabla_Q f)\cdot\nabla_Q f\mathrm dQ\\ &\quad-\int_{\mathbf D}\left[\nabla_Q\left(\triangle_Q+\frac12\triangle_Q U-\frac14|\nabla_Q U|^2\right)f\right]\cdot\nabla_Q f\mathrm dQ=0.\end{aligned}$$

Similar to (2.6), we can get by integration by parts

$$\int_{\mathbf D}\nabla_Q(S(u)Q\cdot\nabla_Q f)\cdot\nabla_Q f\mathrm dQ=0,$$

and

$$\begin{aligned}&-\int_{\mathbf D}\left[\nabla_Q\left(\triangle_Q+\frac12\triangle_Q U-\frac14|\nabla_Q U|^2\right)f\right]\cdot\nabla_Q f\mathrm dQ\\ &\quad=\int_{\mathbf D}\left|\left(\nabla_Q+\frac12\nabla_Q U\right)\nabla_Q f\right|^2\mathrm dQ+\frac14\int_{\mathbf D}\left[\triangle_Q\left(\triangle_Q U-\frac12|\nabla_Q U|^2\right)\right]|f|^2\mathrm dQ.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\partial_t \int_{\mathbf{D}} |\nabla_Q f|^2 \mathrm{d}Q + u\cdot\nabla \int_{\mathbf{D}} |\nabla_Q f|^2 \mathrm{d}Q + 2\int_{\mathbf{D}} \left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right)\nabla_Q f\right|^2 \mathrm{d}Q \\
&\quad = -\frac{1}{2}\int_{\mathbf{D}} \left[\triangle_Q\left(\triangle_Q U - \frac{1}{2}|\nabla_Q U|^2\right)\right]|f|^2 \mathrm{d}Q \\
&\quad \le C\int_{\mathbf{D}} (1+|\nabla_Q U|^2)|f|^2 \mathrm{d}Q, \qquad (2.27)
\end{aligned}$$

where in the last step we used the assumption (1.9) on U.

On the other hand, we get by a similar proof of (2.27) that

$$\begin{aligned}
&\partial_t \int_{\mathbf{D}} |Q|^2|\nabla_Q f|^2 \mathrm{d}Q + u\cdot\nabla \int_{\mathbf{D}} |Q|^2|\nabla_Q f|^2 \mathrm{d}Q + 2\int_{\mathbf{D}} |Q|^2\left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right)\nabla_Q f\right|^2 \mathrm{d}Q \\
&\quad \le C\int_{\mathbf{D}} (1+|\nabla_Q U Q|)|\nabla_Q f|^2 \mathrm{d}Q + C\int_{\mathbf{D}} (1+|\nabla_Q U|^2|Q|^2)|f|^2 \mathrm{d}Q. \qquad (2.28)
\end{aligned}$$

So, for any $\eta > 0$, we get by summing up (2.4), $\eta\times$ (2.27) and $\eta^2\times$ (2.28) that

$$\begin{aligned}
&\partial_t \int_{\mathbf{D}} [|f|^2 + \eta(1+\eta|Q|^2)|\nabla_Q f|^2]\mathrm{d}Q + u\cdot\nabla\int_{\mathbf{D}} [|f|^2 + \eta(1+\eta|Q|^2)|\nabla_Q f|^2]\mathrm{d}Q \\
&\quad +2\int_{\mathbf{D}} \left[\left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right)f\right|^2 + \eta(1+\eta|Q|^2)\left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right)\nabla_Q f\right|^2\right]\mathrm{d}Q \\
&\quad \le C\left[\eta\int_{\mathbf{D}} (1+|\nabla_Q U|^2)|f|^2 \mathrm{d}Q + \eta^2\left(\int_{\mathbf{D}} (1+|\nabla_Q U\cdot Q|)|\nabla_Q f|^2\mathrm{d}Q + C\int_{\mathbf{D}}(1+|\nabla_Q U|^2|Q|^2)|f|^2\mathrm{d}Q\right)\right]. \qquad (2.29)
\end{aligned}$$

Thanks to Lemma 2.1, we get by taking η small enough in (2.29) that

$$\begin{aligned}
&\partial_t \int_{\mathbf{D}} [|f|^2 + \eta(1+\eta|Q|^2)|\nabla_Q f|^2]\mathrm{d}Q + u\cdot\nabla\int_{\mathbf{D}} [|f|^2 + \eta(1+\eta|Q|^2)|\nabla_Q f|^2]\mathrm{d}Q \\
&\quad +\int_{\mathbf{D}} \left[\left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right)f\right|^2 + \eta(1+\eta|Q|^2)\left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right)\nabla_Q f\right|^2\right]\mathrm{d}Q \\
&\quad \le C\int_{\mathbf{D}} (1+|Q|^2)\left|\left(\nabla_Q + \frac{1}{2}\nabla_Q U\right)f\right|^2 \mathrm{d}Q,
\end{aligned}$$

from which and (2.8), we deduce

$$\begin{aligned}
\int_{\mathbf{D}} [|f|^2 + \eta(1+\eta|Q|^2)|\nabla_Q f|^2]\mathrm{d}Q &\le \left\|\int_{\mathbf{D}} [|f_0|^2 + \eta(1+\eta|Q|^2)|\nabla_Q f_0|^2]\mathrm{d}Q\right\|_{L^\infty} \\
&\quad + C\int_0^t\int_{\mathbf{D}} (1+|Q|^2)|\nabla_Q f(t', \Phi_{t'}(x), Q)|^2\,\mathrm{d}Q\mathrm{d}t' \\
&\le C\left\|\int_{\mathbf{D}} [(1+|Q|^2)|f_0|^2 + \eta(1+\eta|Q|^2)|\nabla_Q f_0|^2]\mathrm{d}Q\right\|_{L^\infty}, \qquad (2.30)
\end{aligned}$$

which together Theorem 1.1 and (2.26) implies that

$$|\sqrt{1+|Q|^2} f(t)|_1^2 + \int_0^t \left|\sqrt{1+|Q|^2}\left(\nabla_Q f + \frac{1}{2}\nabla_Q U f\right)(t')\right|_1^2 \mathrm{d}t' \le C(t), \quad \text{for } \forall t < \infty, \qquad (2.31)$$

where $C(t) < \infty$ for any fixed $t < \infty$.

Step 2. The estimate of $|f(t)|_2$.

We first get by a similar proof of (2.1) and (2.2) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{H^2}^2 + \|\nabla u\|_{H^2}^2 \le C(\|\nabla u\|_{L^\infty}\|u\|_{H^2}^2 + |f|_2\|\nabla u\|_{H^2}), \qquad (2.32)$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^2 f\|_{L^2}^2 + \left\|\nabla_Q\partial^2 f + \frac{1}{2}\nabla_Q U\partial^2 f\right\|_{L^2}^2 = -\left[(\partial^2(u\cdot\nabla f), \partial^2 f) + (\partial^2\nabla_Q\cdot(S(u)Qf), \partial^2 f)\right]. \qquad (2.33)$$

As div $u = 0$, we get by applying Lemma 3.1 of [13] that

$$|(\partial^2(u\cdot\nabla f), \partial^2 f)| = |([\partial^2; u]\nabla f, \partial^2 f)| \\ \le C(\|\nabla u\|_{L^\infty}|f|_2 + \|f\|_{L^\infty_x(L^2_Q)}\|\nabla u\|_{H^2})|f|_2.$$

Note that $\nabla_Q\cdot(S(u)Q) = 0$, we have

$$(\partial^2(S(u)Q\nabla_Q f), \partial^2 f) = (\partial^2 S(u)Qf, \nabla_Q\partial^2 f) + (\partial S(u)Q\partial f, \nabla_Q\partial^2 f),$$

Observe that $\dot W^{1,1}(\mathbf{R}^2)\hookrightarrow L^2(\mathbf{R}^2)$, we obtain

$$\|Q\partial f\|_{L^4_x(L^2_Q)} \le C\left(\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}|Q\partial f\, Q\partial^2 f|\,\mathrm{d}Q\,\mathrm{d}x\right)^{\frac12} \le C|Qf|_1^{\frac12}\|Q\partial^2 f\|_{L^2}^{\frac12},$$

from which, we deduce that

$$\|\partial S(u)Q\partial f\|_{L^2} \le \|\partial S(u)\|_{L^4}\|Q\partial f\|_{L^4_x(L^2_Q)} \le C(\|\nabla u\|_{L^\infty}|Qf|_1)^{\frac12}(\|\nabla u\|_{H^2}\|Q\partial^2 f\|_{L^2})^{\frac12},$$

and

$$|(\partial^2(S(u)Q\nabla_Q f), \partial^2 f)| \le C(t)(\|Qf\|_{L^\infty_x(L^2_Q)}\|\nabla u\|_{H^2} + (\|\nabla u\|_{H^2}\|Q\partial^2 f\|_{L^2})^{\frac12})\|\nabla_Q\partial^2 f\|_{L^2},$$

where we used Theorem 1.1 and (2.31). Therefore, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^2 f\|_{L^2}^2 + 2\left\|\left(\nabla_Q + \frac12\nabla_Q U\right)\partial^2 f(t)\right\|_{L^2}^2 \le C_\varepsilon(t)(|f|_2^2 + \|\nabla u\|_{H^2}^2) + \varepsilon(\|Q\partial^2 f\|_{L^2}^2 + \|\nabla_Q\partial^2 f\|_{L^2}^2).$$

Lemma 2.1 applied gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^2 f\|_{L^2}^2 + \frac32\left\|\left(\nabla_Q + \frac12\nabla_Q U\right)\partial^2 f(t)\right\|_{L^2}^2 \le C_\varepsilon(t)(|f|_2^2 + \|\nabla u\|_{H^2}^2) \tag{2.34}$$

by taking ε small enough.

Now let us fix any $T < \infty$ and $\delta > 0$ such that $\delta\times C_\varepsilon(T) \le \frac12$. Then we get by summing up (2.32) and $\delta\times$ (2.34) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{H^2}^2 + \delta\|\partial^2 f\|_{L^2}^2) + \|\nabla u\|_{H^2}^2 + \delta\left\|\left(\nabla_Q + \frac12\nabla_Q U\right)\partial^2 f\right\|_{L^2}^2 \le C_\varepsilon(t)(\|u\|_{H^2}^2 + \delta\|f\|_2^2),$$

for $t \le T$. Gronwall inequality applied gives

$$\sup_{t\in[0,T]}(\|u(t)\|_{H^2}^2 + \delta\|\partial^2 f(t)\|_{L^2}^2) + \int_0^T\left[\|\nabla u(t)\|_{H^2}^2 + \delta\left\|\left(\nabla_Q + \frac12\nabla_Q U\right)\partial^2 f(t)\right\|_{L^2}^2\right]\mathrm{d}t \le C(T).$$

Due to the arbitrariness of T, this completes the proof of Theorem 1.2. ■

Remark 2.1. In this remark, we explain how we can use the same argument to prove Theorem 1.3 when $s \ge 3$. The only difficulty is to control $(\partial^s(S(u)Q\nabla_Q f), \partial^s f)$. We have to use Leibniz formula to compute the first terms. If the s derivatives hit on f, then the term cancel as above. The term $(\partial^{s-1}S(u)Q\partial f, \nabla_Q\partial^s f)$ can be treated exactly as above. Let us only explain how to treat $(\partial S(u)Q\partial^{s-1}f, \nabla_Q\partial^s f)$. We can control this term by putting $\partial S(u)$ in $L^\infty_t L^4_x$, putting $Q\partial^{s-1}f$ in $L^2_t L^4_x L^2_Q$ and putting $\nabla_Q\partial^s f$ in $L^2_t L^2_x L^2_Q$. We leave the details to the reader.

Acknowledgments

Nader Masmoudi is partially supported by and NSF grant DMS 0703145. This work was carried out when Ping Zhang visited Courant Institute of New York University. He would like to thank the hospitality of the Institute. Ping Zhang is partially supported by NSF of China under Grant 10525101 and 10421101, National 973 project and the innovation grant from Chinese Academy of Sciences; and Zhifei Zhang is supported by NSF of China under Grant 10601002.

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