Arch. Rational Mech. Anal. 196 (2010) 753–809
Digital Object Identifier (DOI) 10.1007/s00205-009-0254-5

From the Boltzmann Equation to an Incompressible Navier–Stokes–Fourier System

C. David Levermore & Nader Masmoudi

Communicated by Y. Brenier

Abstract

We establish a Navier–Stokes–Fourier limit for solutions of the Boltzmann equation considered over any periodic spatial domain of dimension two or more. We do this for a broad class of collision kernels that relaxes the Grad small deflection cutoff condition for hard potentials and includes for the first time the case of soft potentials. Appropriately scaled families of DiPerna–Lions renormalized solutions are shown to have fluctuations that are compact. Every limit point is governed by a weak solution of a Navier–Stokes–Fourier system for all time.

1. Introduction

We establish a Navier–Stokes–Fourier fluid dynamical limit for the classical Boltzmann equation considered over any periodic spatial domain of dimension two or more. Here the Navier–Stokes–Fourier system governs (ρ, u, θ), the fluctuations of mass density, bulk velocity, and temperature about their spatially homogeneous equilibrium values in a Boussinesq regime. Specifically, after a suitable choice of units, these fluctuations satisfy the incompressibility and Boussinesq relations

$$\nabla_x \cdot u = 0, \quad \rho + \theta = 0; \tag{1.1}$$

while their evolution is determined by the motion and heat equations

$$\begin{aligned} &\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u, \quad u(x, 0) = u^{in}(x), \\ &\tfrac{D+2}{2} \left(\partial_t \theta + u \cdot \nabla_x \theta \right) = \kappa \Delta_x \theta, \quad \theta(x, 0) = \theta^{in}(x), \end{aligned} \tag{1.2}$$

where $\nu > 0$ and $\kappa > 0$ are the coefficients of kinematic viscosity and thermal conductivity.

This work advances the program laid out in [1–3]. One of the central goals of that program is to connect the DiPerna–Lions theory of global renormalized solutions of the Boltzmann equation to the Leray theory of global weak solutions of the incompressible Navier–Stokes–Fourier system (1.1–1.2). The main result of [3] for the Navier–Stokes limit is to recover the motion equation for a discrete-time version of the Boltzmann equation assuming the DiPerna–Lions solutions satisfy the local conservation of momentum and with the aid of a mild compactness assumption. This result falls short of the goal in a number of respects. First, the heat equation was not treated because the heat flux terms could not be controlled. Second, local momentum conservation was assumed because DiPerna–Lions solutions are not known to satisfy the local conservation law of momentum (or energy) that one would formally expect. Third, the discrete-time case was treated in order to avoid having to control the time regularity of the acoustic modes. Fourth, unnatural technical assumptions were made on the Boltzmann collision kernel. Finally, a weak compactness assumption was required to pass to the limit in certain nonlinear terms. The present work removes all of these shortcomings. It builds upon the recent advances found in [12,15–17,29].

In [29] Lions and Masmoudi recover the Navier–Stokes motion equation with the aid of only the local conservation of momentum assumption and the nonlinear weak compactness assumption that were made in [3]. However, they do not recover the heat equation and they retain the same unnatural technical assumptions made in [3] on the collision kernel. There were two key new ingredients in their work. First, they were able to control the time regularity of the acoustic modes by adapting an idea from [27]. Second, they were able to prove that the contribution of the acoustic modes to the limiting motion equation is just an extra gradient term that can be incorporated into the pressure term. There are two reasons they do not recover the heat equation. First, it is unknown whether or not DiPerna–Lions solutions satisfy a local energy conservation law. Second, even if local energy conservation were assumed, the techniques they used to control the momentum flux would fail to control the heat flux.

In [12] Golse and Levermore recover the Stokes–Fourier system (the linearization of (1.1–1.2) about zero). There were two key new ingredients in their work. First, they control the local momentum and energy conservation defects of the DiPerna–Lions solutions with dissipation rate estimates that allowed them to recover these local conservation laws in the limit. Second, they also control the heat flux with dissipation rate estimates. Because they treat the linear Fourier-Stokes case in [12], they did not need either to control the acoustic modes or a compactness assumption, both of which are used to pass to the limit in the nonlinear terms in [29].

At the same time as this work was being carried out, Golse and Saint-Raymond [15,17] were able to recover the Navier–Stokes–Fourier system without making any nonlinear weak compactness hypothesis. In addition to building on the ideas in [29 and 12], their proof uses the entropy dissipation rate to decompose the collision operator in a new way and uses a new L^1 averaging theory [16] (which has its origins in [33,34]) to prove the compactness assumption. Their result in [15,17] is restricted to a narrow class of bounded Boltzmann kernels that only includes the special case of Maxwell molecules with a Grad small deflection cutoff from among all kernels that are classically derived from an interparticle potential. They have recently extended their result to the case of hard potentials [18].

In the present work we also recover the Navier–Stokes–Fourier system, but for a much wider class of collision kernels than was treated in [15,17,18]. In particular, we are able to treat all classical collision kernels with a weak cutoff that arise from inverse power-law potentials. It is the first treatment of soft potentials in this program, and the first treatment small deflection cutoffs that are weaker than those of Grad. It treats all classical collision kernels to which the DiPerna–Lions theory applies. Our result goes beyond the results mentioned above by combining the ingredients from [12,15–17,29] with some new nonlinear estimates. More specifically, here we adapt the control of the acoustic modes found in [29], the dissipation rate controls of both the heat flux and the conservation defects found in [12], the averaging theory found in [16], and the entropy cutoff technique found in [15,17], which traces back to [33,35].

The next section contains preliminary material regarding the Boltzmann equation, including the formal scaling that leads from the Boltzmann equation to the Navier–Stokes–Fourier system. Section 3 states all our technical assumptions regarding the collision kernel. These assumptions are satisfied by all classical collision kernels with a soft cutoff. Section 4 reviews the DiPerna–Lions theory of global solutions for the Boltzmann equation [9] and the Leray theory of global solutions for the Navier–Stokes–Fourier system. Section 5 presents precise statements of our results. Section 6 gives the proofs of our results modulo five results: one that provides our nonlinear compactness, one that removes the local conservation defects, two that control the fluxes, and one that controls the limits in the quadratic terms. Sections 7 through 11 establish these results, thereby completing our proof.

2. Boltzmann equation preliminaries

Our starting point is the Boltzmann equation. In this section we collect the basic facts we will need about it. These will include its nondimensionalization and its formal conservation and dissipation laws.

2.1. The Boltzmann equation

Here we will introduce the Boltzmann equation only so far as to set our notation, which is essentially that of [3]. More complete introductions to the Boltzmann equation can be found in [6,7,10,11]. The state of a fluid composed of identical point particles confined to a spatial domain $\Omega \subset \mathbb{R}^D$ is described at the kinetic level by a mass density F over the single-particle phase space $\mathbb{R}^D \times \Omega$. More specifically, $F(v, x, t)\, \mathrm{d}v\, \mathrm{d}x$ gives the mass of the particles that occupy any infinitesimal volume $\mathrm{d}v\, \mathrm{d}x$ centered at the point $(v, x) \in \mathbb{R}^D \times \Omega$ at the instant of time $t \geqq 0$. To remove complications due to boundaries, we take Ω to be the periodic domain $\mathbb{T}^D = \mathbb{R}^D/\mathbb{L}^D$, where $\mathbb{L}^D \subset \mathbb{R}^D$ is any D-dimensional lattice. We refer to [22,32] for treatments of bounded domains with a Maxwell reflection boundary condition.

If the particles interact only through a repulsive conservative interparticle force with finite range, then at low enough densities this range will be much smaller than the interparticle spacing. In that regime all but binary collisions can be neglected when $D \geqq 2$, and the evolution of $F = F(v, x, t)$ is governed by the classical Boltzmann equation [7]:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad F(v, x, 0) = F^{in}(v, x) \geqq 0. \tag{2.1}$$

The Boltzmann collision operator $\mathcal{B}$ acts only on the v argument of F. It is formally given by

$$\mathcal{B}(F, F) = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (F_1' F' - F_1 F)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, \mathrm{d}v_1, \tag{2.2}$$

where v_1 ranges over $\mathbb{R}^D$ endowed with its Lebesgue measure $\mathrm{d}v_1$ while ω ranges over the unit sphere $\mathbb{S}^{D-1} = \{\omega \in \mathbb{R}^D : |\omega| = 1\}$ endowed with its rotationally invariant measure $d\omega$. The F_1', F', F_1, and F appearing in the integrand designate $F(\cdot, x, t)$ evaluated at the velocities v_1', v', v_1, and v respectively, where the primed velocities are defined by

$$v_1' = v_1 - \omega\, \omega \cdot (v_1 - v), \quad v' = v + \omega\, \omega \cdot (v_1 - v), \tag{2.3}$$

for any given $(\omega, v_1, v) \in \mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$. Quadratic operators like $\mathcal{B}$ are extended by polarization to be bilinear and symmetric.

The unprimed and primed velocities are possible velocities for a pair of particles either before and after, or after and before, they interact through an elastic binary collision. Conservation of momentum and energy for particle pairs during collisions is expressed as

$$v + v_1 = v' + v_1', \quad |v|^2 + |v_1|^2 = |v'|^2 + |v_1'|^2. \tag{2.4}$$

Equation (2.3) represents the general nontrivial solution of these $D + 1$ equations for the $4D$ unknowns v_1', v', v_1, and v in terms of the $3D - 1$ parameters (ω, v_1, v).

The collision kernel b is positive almost everywhere. The Galilean invariance of the collisional physics implies that b has the classical form

$$b(\omega, v_1 - v) = |v_1 - v|\, \Sigma(|\omega \cdot n|, |v_1 - v|), \tag{2.5}$$

where $n = (v_1 - v)/|v_1 - v|$ and Σ is the specific differential cross-section. Technical conditions on b will be imposed in Section 3.

2.2. Nondimensionalized form

We will work with the nondimensionalized form of the Boltzmann equation that was used in [3]. That form is motivated by the fact the Navier–Stokes–Fourier system can be formally derived from the Boltzmann equation through a scaling in which the density F is close to a spatially homogeneous Maxwellian $M = M(v)$ that has the same total mass, momentum, and energy as the initial data F^{in}. By an appropriate choice of a Galilean frame and of mass and velocity units, it can be assumed that this so-called absolute Maxwellian M has the form

$$M(v) \equiv \frac{1}{(2\pi)^{D/2}} \exp(-\tfrac{1}{2}|v|^2). \tag{2.6}$$

This corresponds to the spatially homogeneous fluid state with its density and temperature equal to 1 and bulk velocity equal to 0. This state is consistent with the form of the Navier–Stokes–Fourier system given by (1.1–1.2).

It is natural to introduce the relative kinetic density, $G = G(v, x, t)$, defined by $F = MG$. Recasting the initial-value problem (2.1) for G yields

$$\epsilon\, \partial_t G + v \cdot \nabla_x G = \frac{1}{\epsilon} \mathcal{Q}(G, G), \quad G(v, x, 0) = G^{in}(v, x). \tag{2.7}$$

The positive, nondimensional parameter ϵ is the Knudsen number, which is the ratio of the mean-free-path to the macroscopic length scale determined by setting the volume of $\mathbb{T}^D$ to unity [3]. The collision operator is now given by

$$\mathcal{Q}(G, G) = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (G_1' G' - G_1 G)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1. \tag{2.8}$$

Under the assumptions we will make in Section 3, the nondimensional collision kernel b can be normalized so that

$$\iiint_{\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D} b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v = 1. \tag{2.9}$$

Fluid dynamical regimes are those where the mean free path is small compared to the macroscopic length scales—that is where the Knudsen number ϵ is small. The long-time scaling in (2.7) is consistent with a formal derivation of either the Stokes–Fourier or Navier–Stokes–Fourier systems [2].

This nondimensionalization has the normalizations

$$\int_{\mathbb{R}^D} M \mathrm{d}v = 1, \quad \int_{\mathbb{T}^D} \mathrm{d}x = 1, \tag{2.10}$$

associated with the domains $\mathbb{R}^D$ and $\mathbb{T}^D$ respectively, (2.9) associated with the collision kernel b, and

$$\iint_{\mathbb{R}^D \times \mathbb{T}^D} G^{in}\, M \mathrm{d}v\, \mathrm{d}x = 1, \quad \iint_{\mathbb{R}^D \times \mathbb{T}^D} v\, G^{in}\, M \mathrm{d}v\, \mathrm{d}x = 0,$$
$$\iint_{\mathbb{R}^D \times \mathbb{T}^D} \tfrac{1}{2}|v|^2 G^{in}\, M \mathrm{d}v\, \mathrm{d}x = \tfrac{D}{2}. \tag{2.11}$$

associated with the initial data G^{in}.

Because $M \mathrm{d}v$ is a positive unit measure on $\mathbb{R}^D$, we denote by $\langle \xi \rangle$ the average over this measure of any integrable function $\xi = \xi(v)$:

$$\langle \xi \rangle = \int_{\mathbb{R}^D} \xi(v)\, M \mathrm{d}v. \tag{2.12}$$

Because $\mathrm{d}\mu = b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M\mathrm{d}v$ is a positive unit measure on $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, we denote by $\langle\!\langle \Xi \rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$:

$$\langle\!\langle \Xi \rangle\!\rangle = \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \Xi(\omega, v_1, v)\,\mathrm{d}\mu. \tag{2.13}$$

The collisional measure $\mathrm{d}\mu$ is invariant under the transformations

$$(\omega, v_1, v) \mapsto (\omega, v, v_1), \quad (\omega, v_1, v) \mapsto (\omega, v_1', v'). \tag{2.14}$$

These, and compositions of these, are called collisional symmetries.

2.3. Formal conservation and dissipation laws

We now list for later reference the basic conservation and entropy dissipation laws that are formally satisfied by solutions the Boltzmann equation. Derivations of these laws in this nondimensional setting are outlined in [3], and can essentially be found in [6] (Sec. II.6-7), [10] (Sec. 1.4), or [11].

First, if G solves the Boltzmann equation (2.7) then G satisfies local conservation laws of mass, momentum, and energy:

$$\begin{aligned} \epsilon\,\partial_t \langle G \rangle + \nabla_x \cdot \langle v\,G \rangle &= 0,\\ \epsilon\,\partial_t \langle v\,G \rangle + \nabla_x \cdot \langle v \otimes v\,G \rangle &= 0,\\ \epsilon\,\partial_t \langle \tfrac{1}{2}|v|^2 G \rangle + \nabla_x \cdot \langle v \tfrac{1}{2}|v|^2 G \rangle &= 0. \end{aligned} \tag{2.15}$$

Integrating these over space and time while recalling the normalizations (2.11) of G^{in} yields the global conservation laws of mass, momentum, and energy:

$$\begin{aligned} \int_{\mathbb{T}^D} \langle G(t) \rangle\,\mathrm{d}x &= \int_{\mathbb{T}^D} \langle G^{in} \rangle\,\mathrm{d}x = 1,\\ \int_{\mathbb{T}^D} \langle v\,G(t) \rangle\,\mathrm{d}x &= \int_{\mathbb{T}^D} \langle v\,G^{in} \rangle\,\mathrm{d}x = 0,\\ \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G(t) \rangle\,\mathrm{d}x &= \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G^{in} \rangle\,\mathrm{d}x = \tfrac{D}{2}. \end{aligned} \tag{2.16}$$

Second, if G solves the Boltzmann equation (2.7) then G satisfies the local entropy dissipation law

$$\begin{aligned} &\epsilon\,\partial_t \langle (G\log(G) - G + 1) \rangle + \nabla_x \cdot \langle v\,(G\log(G) - G + 1) \rangle\\ &\quad = -\frac{1}{\epsilon} \left\langle\!\!\left\langle \frac{1}{4} \log\left(\frac{G_1' G'}{G_1 G}\right) (G_1' G' - G_1 G) \right\rangle\!\!\right\rangle \leqq 0. \end{aligned} \tag{2.17}$$

Integrating this over space and time gives the global entropy equality

$$H(G(t)) + \frac{1}{\epsilon^2} \int_0^t R(G(s))\,\mathrm{d}s = H(G^{in}), \tag{2.18}$$

where $H(G)$ is the relative entropy functional

$$H(G) = \int_{\mathbb{T}^D} \langle (G \log(G) - G + 1) \rangle \, \mathrm{d}x, \tag{2.19}$$

and $R(G)$ is the entropy dissipation rate functional

$$R(G) = \int_{\mathbb{T}^D} \left\langle\!\!\left\langle \frac{1}{4} \log\left(\frac{G_1' G'}{G_1 G} \right) (G_1' G' - G_1 G) \right\rangle\!\!\right\rangle \mathrm{d}x. \tag{2.20}$$

3. Technical assumptions on the collision kernel

In this section we give all of our additional assumptions regarding the collision kernel b. These assumptions are satisfied by many classical collision kernels. For example, they are satisfied by the collision kernel for hard spheres of mass m and radius r_o, which has the form

$$b(\omega, v_1 - v) = |\omega \cdot (v_1 - v)| \frac{(2r_o)^{D-1}}{2m}. \tag{3.1}$$

They are also satisfied by all the classical collision kernels with a small deflection cutoff that derive from a repulsive intermolecular potential of the form c/r^k with $k > 2\frac{D-1}{D+1}$. Specifically, these kernels have the form

$$b(\omega, v_1 - v) = \hat{b}(\omega \cdot n) \, |v_1 - v|^{\beta} \quad \text{with } \beta = 1 - 2\tfrac{D-1}{k}, \tag{3.2}$$

where $\hat{b}(\omega \cdot n)$ is positive almost everywhere, has even symmetry in ω, and satisfies the small deflection cutoff condition

$$\int_{\mathbb{S}^{D-1}} \hat{b}(\omega \cdot n) \, \mathrm{d}\omega < \infty. \tag{3.3}$$

The condition $k > 2\frac{D-1}{D+1}$ is equivalent to $\beta > -D$, which insures that $b(\omega, v_1 - v)$ is locally integrable with respect to $v_1 - v$. The cases $\beta > 0$, $\beta = 0$, and $\beta < 0$ correspond respectively to the so-called hard, Maxwell, and soft potential cases.

The works of Golse and Saint-Raymond cover the case of hard potentials with a Grad small deflection cutoff [17,18]. The Grad small deflection cutoff is much more restrictive than the cutoff (3.3) which merely guarantees the local integrability of b. Our work therefore relaxes their small deflection cutoff condition for hard potentials and treats for the first time the case of soft potentials. It thereby covers all classical collision kernels to which the DiPerna–Lions theory applies.

We have already stated that the collision kernel b is positive almost everywhere and has the form (2.5)—assumptions clearly met by the hard sphere and inverse power kernels given by (3.1) and (3.2). Our additional five assumptions on b are technical in nature—that is, they are required by our mathematical argument. We therefore examine which of the commonly studied physical collision kernels satisfy these assumptions. We also give some consequences of these assumptions that will play an important role in what follows.

3.1. DiPerna–Lions assumption

Our *first assumption* is that the collision kernel b satisfies the requirements of the DiPerna–Lions theory. That theory requires that b be locally integrable with respect to $\mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v$, and that it moreover satisfies

$$\lim_{|v|\to\infty} \frac{1}{1+|v|^2} \int_K \bar{b}(v_1 - v)\, \mathrm{d}v_1 = 0, \quad \text{for every compact } K \subset \mathbb{R}^D, \tag{3.4}$$

where $\bar{b}$ is defined by

$$\bar{b}(v_1 - v) \equiv \int_{\mathbb{S}^{D-1}} b(\omega, v_1 - v)\, \mathrm{d}\omega. \tag{3.5}$$

Galilean symmetry (2.5) implies that $\bar{b}$ is a function of $|v_1 - v|$ only. This condition is met by the hard sphere kernel (3.1) because in that case $\bar{b}(v_1 - v)$ is proportional to $|v_1 - v|$ and therefore grows like $|v|$ as $|v| \to \infty$. It is also met by the inverse power kernels (3.2) with $\beta > -D$ because in that case $\bar{b}(v_1 - v)$ grows like $|v|^\beta$ as $|v| \to \infty$ and $\beta \leqq 1$.

The DiPerna–Lions assumption implies that the measure $b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v$ is finite. The nondimensional kernel b can therefore be chosen to satisfy the normalization (2.9).

3.2. Attenuation assumption

A major role will be played by the attenuation coefficient a, which is defined by

$$a(v) \equiv \int_{\mathbb{R}^D} \bar{b}(v_1 - v)\, M_1 \mathrm{d}v_1 = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1. \tag{3.6}$$

A few facts about a are readily evident from what we have already assumed. First, a must be positive and locally integrable. Because (3.4) holds, one can show that

$$\lim_{|v|\to\infty} \frac{a(v)}{1+|v|^2} = 0. \tag{3.7}$$

Next, the normalization (2.9) implies that a satisfies

$$\int_{\mathbb{R}^D} a\, M \mathrm{d}v = 1. \tag{3.8}$$

Finally, Galilean symmetry (2.5) implies that a is a function of $|v|$ only.

Our *second assumption* regarding the collision kernel b is that a is bounded below as

$$C_a \big(1 + |v|\big)^\alpha \leqq a(v), \tag{3.9}$$

for some constants $C_a > 0$ and $\alpha \in \mathbb{R}$. This condition is met by the hard sphere kernel (3.1), for which (3.9) is satisfied with $\alpha = 1$, and by all the inverse power kernels (3.2) with $\beta > -D$, for which (3.9) is satisfied with $\alpha = \beta$.

An immediate consequence of the attenuation assumption (3.9) is that $\frac{1}{a}\xi \in L^p(aM\mathrm{d}v)$ for every $p \in [1, \infty)$ whenever $|\xi(v)|$ is bounded above by a polynomial in $|v|$.

3.3. Loss operator assumption

Another major role in what follows will be played by the linearized collision operator $\mathcal{L}$, which is defined formally by

$$\mathcal{L}\tilde{g} \equiv -2\mathcal{Q}(1, \tilde{g}) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \big(\tilde{g} + \tilde{g}_1 - \tilde{g}' - \tilde{g}_1'\big)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1. \tag{3.10}$$

One has the decomposition

$$\frac{1}{a}\mathcal{L} = \mathcal{I} + \mathcal{K}^- - 2\mathcal{K}^+, \tag{3.11}$$

where the loss operator $\mathcal{K}^-$ and gain operator $\mathcal{K}^+$ are formally defined by

$$\mathcal{K}^-\tilde{g} \equiv \frac{1}{a}\int_{\mathbb{R}^D} \tilde{g}_1\, \bar{b}(v_1 - v)\, M_1 \mathrm{d}v_1, \tag{3.12}$$

$$\mathcal{K}^+\tilde{g} \equiv \frac{1}{2a}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \big(\tilde{g}' + \tilde{g}_1'\big)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1. \tag{3.13}$$

By using the Hölder inequality it can be easily shown that for every $p \in [1, \infty]$

$$\mathcal{K}^\pm : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \text{are bounded with } \|\mathcal{K}^\pm\| \leqq 1. \tag{3.14}$$

If then follows from (3.11) that $\frac{1}{a}\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v)$ is bounded with $\|\frac{1}{a}\mathcal{L}\| \leqq 4$ for every $p \in [1, \infty]$.

Our *third assumption* regarding b is that there exists $s \in (1, \infty]$ and $C_b \in (0, \infty)$ such that

$$\left(\int_{\mathbb{R}^D} \left|\frac{\bar{b}(v_1 - v)}{a(v_1)\, a(v)}\right|^s a(v_1)\, M_1 \mathrm{d}v_1\right)^{\frac{1}{s}} \leqq C_b. \tag{3.15}$$

Because this bound is uniform in v, we may take C_b to be the supremum over v of the left-hand side of (3.15). This condition is met by the hard sphere kernel (3.1) and by the inverse power kernels (3.2) with $\beta > -D$. For hard and Maxwell potentials $(2(D-1) \leqq k)$ condition (3.15) is satisfied with $s = \infty$, taking the form

$$\frac{\bar{b}(v_1 - v)}{a(v_1)\, a(v)} \leqq C_b. \tag{3.16}$$

This is very similar to the bound assumed in [12]. For soft potentials $(2\frac{D-1}{D+1} < k < 2(D-1))$ condition (3.15) is satisfied for every s in the interval $1 < s < D/(\frac{2(D-1)}{k} - 1)$. The case $s \in (1, \infty)$ in (3.15) therefore allows these soft potentials to be considered.

Our third assumption (3.15) has several immediate implications for the loss operator $\mathcal{K}^-$, which we now express formally as

$$\mathcal{K}^-\tilde{g} = \int_{\mathbb{R}^D} K^-(v_1, v)\, \tilde{g}_1\, a_1 M_1 \mathrm{d}v_1, \quad \text{where } K^-(v_1, v) = \frac{\bar{b}(v_1 - v)}{a(v_1)\, a(v)}. \tag{3.17}$$

It is clear that K^- is symmetric ($K^-(v_1, v) = K^-(v, v_1)$) and positive almost everywhere. An interpolation argument shows that whenever there exist $p, q, r, t \in [1, \infty]$ with $r \leqq t$ such that

$$\frac{1}{p} + \frac{1}{r} + \frac{1}{t} = 1 + \frac{1}{q}, \quad \frac{1}{p} + \frac{1}{p^*} = 1, \quad p^*, q \in [r, t], \tag{3.18}$$

$$C_{rt} \equiv \left(\int_{\mathbb{R}^D} \left(\int_{\mathbb{R}^D} |K^-(v_1, v)|^r \, a_1 M_1 \mathrm{d}v_1 \right)^{\frac{t}{r}} aM\mathrm{d}v \right)^{\frac{1}{t}} < \infty, \tag{3.19}$$

then

$$\mathcal{K}^- : L^p(aM\mathrm{d}v) \to L^q(aM\mathrm{d}v) \quad \text{is bounded with } \|\mathcal{K}^-\| \leqq C_{rt}. \tag{3.20}$$

Moreover, whenever r and t in (3.19) are both finite (that is—when $[r, t] \subset [1, \infty)$) then

$$\mathcal{K}^- : L^p(aM\mathrm{d}v) \to L^q(aM\mathrm{d}v) \quad \text{is compact.} \tag{3.21}$$

Assertion (3.21) follows from assertion (3.20) because when $[r, t] \subset [1, \infty)$ then the expression for C_{rt} given in (3.19) defines the norm for a Banach space of kernels in which finite-rank kernels are dense.

Because $aM\mathrm{d}v$ is a unit measure, the assumed bound (3.15) clearly implies that (3.19) holds with $C_{rt} \leqq C_b$ for every $r \in [1, s]$ and $t \in [1, \infty]$. Three immediate consequences of this are

$$\mathcal{K}^- : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \text{is compact for every } p \in (1, \infty), \tag{3.22}$$

$$\mathcal{K}^- : L^p(aM\mathrm{d}v) \to L^{p^*}(aM\mathrm{d}v) \text{ is bounded with } \|\mathcal{K}^-\| \leqq C_b \text{ for every } p^* \in [1, 2s], \tag{3.23}$$

$$\mathcal{K}^- : L^p(aM\mathrm{d}v) \to L^{p^*}(aM\mathrm{d}v) \quad \text{is compact for every } p^* \in [1, 2s). \tag{3.24}$$

The first of these follows from (3.21) by setting $q = p$ in (3.18) and choosing $r \in (1, s]$ such that $p, p^* \in [r, r^*]$ and then setting $t = r^*$. The second follows from (3.20) by setting $q = p^*$ in (3.18) and observing that as r ranges over $[1, s]$ while t ranges over $[1, \infty]$ then p^* given by $\frac{1}{p^*} = \frac{1}{2}(\frac{1}{r} + \frac{1}{t})$ will range over $[1, 2s]$. The last follows from (3.21) by excluding the cases $r = \infty$ and $t = \infty$ from the preceding consideration.

3.4. Gain operator assumption

Our *fourth assumption* regarding b is that

$$\mathcal{K}^+ : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v) \quad \text{is compact.} \tag{3.25}$$

This condition is met by the hard sphere kernel (3.1) and by the cutoff inverse power kernels (3.2) that derive from a repulsive intermolecular potential of the form c/r^k. For $D = 3$ this fact was demonstrated by HILBERT [20] for hard spheres, by GRAD [19] for hard potentials ($k \geqq 4$) with a Grad small deflection cutoff, and by GOLSE and Poupaud [14] for soft potentials with $k > 2$ with a Grad small deflection cutoff. For general D this fact has recently been demonstrated by Sun [25,36] for kernels (3.2) that satisfy the small deflection cutoff (3.3). Even when $D = 3$ this extends the result of Golse and Poupaud for soft potentials with Grad cutoffs to $k > 1$.

An immediate consequence of our fourth assumption (3.25) on the gain operator $\mathcal{K}^+$ is that

$$\mathcal{K}^+ : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \text{is compact for every } p \in (1,\infty). \tag{3.26}$$

This assertion follows from (3.14) and (3.25) by interpolation.

When our gain operator assumption (3.25) is combined with our loss operator assumption (3.15), we conclude that

$$\frac{1}{a}\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \text{is Fredholm for every } p \in (1,\infty). \tag{3.27}$$

This assertion follows from the decomposition $\frac{1}{a}\mathcal{L} = \mathcal{I} + \mathcal{K}^- - 2\mathcal{K}^+$ given by (3.11) because the operators $\mathcal{K}^-$ and $\mathcal{K}^+$ are compact by (3.22) and (3.26).

3.5. Saturated kernel assumption

Our *fifth assumption* regarding b is that for every $\delta > 0$ there exists C_δ such that $\bar{b}$ satisfies

$$\frac{\bar{b}(v_1 - v)}{1 + \delta \dfrac{\bar{b}(v_1 - v)}{1 + |v_1 - v|^2}} \leqq C_\delta \big(1 + a(v_1)\big)\big(1 + a(v)\big) \quad \text{for every } v_1, v \in \mathbb{R}^D. \tag{3.28}$$

This condition is met by the hard sphere kernel (3.1) and by the inverse power kernels (3.2). For hard and Maxwell potentials it follows from (3.16) that condition (3.28) is satisfied with $C_\delta = C_b$. For soft potentials one has $\bar{b}(v_1 - v) = C_\beta |v_1 - v|^\beta$ for some $\beta < 0$. In that case the left-hand side of (3.28) is bounded above, whereby (3.28) holds with C_δ equal to this bound.

3.6. Null spaces

Here we characterize the null space of the Fredholm operator $\frac{1}{a}\mathcal{L}$ considered over $L^p(aM\mathrm{d}v)$ for every $p \in (1,\infty)$. One can use the collisional symmetries (2.14) to show that $\frac{1}{a}\mathcal{L}$ is formally symmetric and nonnegative definite with respect to the $L^2(aM\mathrm{d}v)$ inner product. In particular, for every $\tilde{g} \in L^2(aM\mathrm{d}v)$ one shows that

$$\langle \tilde{g}\, \mathcal{L}\tilde{g} \rangle = \tfrac{1}{4} \big\langle\!\big\langle \big(\tilde{g} + \tilde{g}_1 - \tilde{g}' - \tilde{g}_1'\big)^2 \big\rangle\!\big\rangle \geqq 0. \tag{3.29}$$

One can show (see [6], Chapter IV.1) that the null space of $\frac{1}{a}\mathcal{L} : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v)$ is $\mathrm{Null}(\mathcal{L}) \equiv \mathrm{span}\{1, v_1, \ldots, v_D, |v|^2\}$. Moreover, our first, third, and fourth assumptions combine to show that for every $p \in (1,\infty)$

$$\text{the null space of } \frac{1}{a}\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \text{is } \mathrm{Null}(\mathcal{L}). \tag{3.30}$$

Indeed, because assumption (3.4) implies that $\mathrm{Null}(\mathcal{L}) \subset L^p(aM\mathrm{d}v)$ for every $p \in [1,\infty)$, it is clear that $\mathrm{Null}(\mathcal{L})$ is contained in the null space of $\frac{1}{a}\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v)$ for every $p \in [1,\infty)$. Because $L^p(aM\mathrm{d}v) \subset L^2(aM\mathrm{d}v)$ for every $p \in [2,\infty)$, and because the null space of $\frac{1}{a}\mathcal{L} : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v)$ is given by $\mathrm{Null}(\mathcal{L})$, it therefore follows that the null space of $\frac{1}{a}\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v)$ is also given by $\mathrm{Null}(\mathcal{L})$ for every $p \in [2,\infty)$. Now observe that the adjoint of $\frac{1}{a}\mathcal{L}$ over $L^p(aM\mathrm{d}v)$ is $\frac{1}{a}\mathcal{L}$ over $L^{p^*}(aM\mathrm{d}v)$. Because by (3.27) these operators are Fredholm, their null spaces for must have the same dimension. In particular, when $p \in (1,2]$ the dimension of the null space must be equal to the dimension of $\mathrm{Null}(\mathcal{L})$. It therefore follows that the null space of $\frac{1}{a}\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(aM\mathrm{d}v)$ is also given by $\mathrm{Null}(\mathcal{L})$ for every $p \in (1,2]$.

3.7. Coercivity

We will make use of some coercivity estimates the operator $\mathcal{L}$ satisfies. If we let $\lambda > 0$ be the smallest nonzero eigenvalue of $\frac{1}{a}\mathcal{L}$ considered over $L^2(aM\mathrm{d}v)$ then one has the coercivity estimate

$$\lambda \big\langle a\,(\mathcal{P}_a^\perp \tilde{g})^2 \big\rangle \leqq \langle \tilde{g}\mathcal{L}\tilde{g}\rangle \quad \text{for every } \tilde{g} \in L^2(aM\mathrm{d}v). \tag{3.31}$$

Here $\mathcal{P}_a^\perp = \mathcal{I} - \mathcal{P}_a$ and $\mathcal{P}_a$ is the orthogonal projection from $L^2(aM\mathrm{d}v)$ onto $\mathrm{Null}(\mathcal{L})$, which is given by

$$\mathcal{P}_a\tilde{g} = \langle a\,\tilde{g}\rangle + \frac{1}{\frac{1}{D}\langle a\,|v|^2\rangle}\, v\cdot\langle a\,v\,\tilde{g}\rangle + \frac{|v|^2 - \langle a\,|v|^2\rangle}{\langle a\,|v|^4\rangle - \langle a\,|v|^2\rangle^2}\Big\langle\big(|v|^2 - \langle a\,|v|^2\rangle\big)\,\tilde{g}\Big\rangle. \tag{3.32}$$

This follows from the Fredholm property (3.27), the fact that $\frac{1}{a}\mathcal{L}$ is symmetric in $L^2(aM\mathrm{d}v)$, and the characterization of $\mathrm{Null}(\mathcal{L})$ given by (3.30).

One can show that for some $\ell > 0$ the operator $\mathcal{L}$ satisfies the coercivity estimate

$$\ell \big\langle a\,(\mathcal{P}^\perp \tilde{g})^2 \big\rangle \leqq \langle \tilde{g}\mathcal{L}\tilde{g}\rangle \quad \text{for every } \tilde{g} \in L^2(aM\mathrm{d}v). \tag{3.33}$$

Here $\mathcal{P}^\perp = \mathcal{I} - \mathcal{P}$ and $\mathcal{P}$ is the orthogonal projection from $L^2(M\mathrm{d}v)$ onto $\mathrm{Null}(\mathcal{L})$, which is given by

$$\mathcal{P}\tilde{g} = \langle\tilde{g}\rangle + v\cdot\langle v\,\tilde{g}\rangle + \big(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\big)\big\langle(\tfrac{1}{D}|v|^2 - 1)\,\tilde{g}\big\rangle. \tag{3.34}$$

Indeed, assumption (3.9) ensures that $\mathcal{P}$ and $\mathcal{P}^\perp$ are bounded as linear operators from $L^2(aM\mathrm{d}v)$ into itself. Because $\mathcal{P}^\perp = \mathcal{P}^\perp\mathcal{P}_a^\perp$, we then have that every $\tilde{g} \in L^2(aM\mathrm{d}v)$ satisfies

$$\big\|\mathcal{P}^\perp\tilde{g}\big\|_{L^2(aM\mathrm{d}v)} = \big\|\mathcal{P}^\perp\mathcal{P}_a^\perp\tilde{g}\big\|_{L^2(aM\mathrm{d}v)} \leqq \big\|\mathcal{P}^\perp\big\|_{L^2(aM\mathrm{d}v)}\big\|\mathcal{P}_a^\perp\tilde{g}\big\|_{L^2(aM\mathrm{d}v)},$$

where $\|\mathcal{P}^\perp\|_{L^2(aM\mathrm{d}v)}$ denotes the operator norm of $\mathcal{P}^\perp$. It therefore follows from (3.31) that we may take $\ell = \lambda/\|\mathcal{P}^\perp\|^2_{L^2(aM\mathrm{d}v)}$ in (3.33).

3.8. Psuedoinverse

We use a particular pseudoinverse of $\mathcal{L}$ defined as follows. The Fredholm property (3.27) implies that for every $p \in (1, \infty)$

$$\mathcal{L} : L^p(aM\mathrm{d}v) \to L^p(a^{1-p}M\mathrm{d}v) \quad \text{is bounded,}$$

and that for every $\xi \in L^p(a^{1-p}M\mathrm{d}v)$ there exists a unique $\hat{\xi} \in L^p(aM\mathrm{d}v)$ such that

$$\mathcal{L}\hat{\xi} = \mathcal{P}^{\perp}\xi, \quad \mathcal{P}\hat{\xi} = 0. \tag{3.35}$$

For every $\xi \in L^p(a^{1-p}M\mathrm{d}v)$ we define $\mathcal{L}^{-1}\xi = \hat{\xi}$ where $\hat{\xi}$ is determined above. This defines an operator $\mathcal{L}^{-1}$ such that

$$\begin{gathered} \mathcal{L}^{-1} : L^p(a^{1-p}M\mathrm{d}v) \to L^p(aM\mathrm{d}v) \quad \text{is bounded,} \\ \mathcal{L}^{-1}\mathcal{L} = \mathcal{P}^{\perp} \quad \text{over } L^p(aM\mathrm{d}v), \quad \mathcal{L}\mathcal{L}^{-1} = \mathcal{P}^{\perp} \quad \text{over } L^p(a^{1-p}M\mathrm{d}v), \end{gathered} \tag{3.36}$$

and $\mathrm{Null}(\mathcal{L}^{-1}) = \mathrm{Null}(\mathcal{L})$. The operator $\mathcal{L}^{-1}$ is the unique pseudoinverse of $\mathcal{L}$ with these properties.

4. Global solutions

In order to mathematically justify the Navier–Stokes–Fourier limit of the Boltzmann equation, we must make precise: (1) the notion of solution for the Boltzmann equation, and (2) the notion of solution for the Navier–Stokes–Fourier system. Ideally, these solutions should be global while the bounds should be physically natural. We therefore work in the setting of DiPerna–Lions renormalized solutions for the Boltzmann equation, and in the setting of Leray solutions for the Navier–Stokes–Fourier system. These theories have the virtues of considering physically natural classes of initial data, and consequently, of yielding global solutions.

4.1. DiPerna–Lions solutions

DiPerna and Lions [9] proved the global existence of a type of weak solution to the Boltzmann equation over the whole space $\mathbb{R}^D$ for any initial data satisfying natural physical bounds. As they pointed out, with only slight modifications their theory can be extended to the periodic box $\mathbb{T}^D$. Their original theory has been strengthened, most notably in [26,30]. Here we give a version of their theory relevant to this paper.

The DiPerna–Lions theory does not yield solutions that are known to solve the Boltzmann in the usual weak sense. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial-value problems that are obtained by multiplying the Boltzmann equation in (2.7) by $\Gamma'(G)$, where Γ' is the derivative of an admissible function Γ:

$$\big(\epsilon\,\partial_t + v\cdot\nabla_x\big)\Gamma(G) = \frac{1}{\epsilon}\,\Gamma'(G)\mathcal{Q}(G,G), \quad G(v,x,0) = G^{in}(v,x) \geqq 0. \tag{4.1}$$

A function $\Gamma : [0,\infty) \to \mathbb{R}$ is called admissible if it is continuously differentiable and for some constant $C_\Gamma < \infty$ its derivative satisfies

$$\big|\Gamma'(Z)\big| \leqq \frac{C_\Gamma}{\sqrt{1+Z}} \quad \text{for every } Z \geqq 0. \tag{4.2}$$

The solutions lie in $C([0,\infty); w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x))$, where the prefix "$w$-" on a space indicates that the space is endowed with its weak topology. We say that $G \geqq 0$ is a weak solution of (4.1) provided that it is initially equal to G^{in}, and that it satisfies (4.1) in the sense that for every $Y \in L^\infty(\mathrm{d}v; C^1(\mathbb{T}^D))$ and every $[t_1,t_2] \subset [0,\infty)$ it satisfies

$$\epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_2))\,Y\rangle\,\mathrm{d}x - \epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_1))\,Y\rangle\,\mathrm{d}x - \int_{t_1}^{t_2}\int_{\mathbb{T}^D} \langle \Gamma(G)\,v\cdot\nabla_x Y\rangle\,\mathrm{d}x\,\mathrm{d}t$$
$$= \frac{1}{\epsilon}\int_{t_1}^{t_2}\int_{\mathbb{T}^D} \big\langle \Gamma'(G)\mathcal{Q}(G,G)\,Y\big\rangle\,\mathrm{d}x\,\mathrm{d}t. \tag{4.3}$$

If G is a weak solution of (4.1) for one such Γ with $\Gamma' > 0$, and if G satisfies certain bounds, then it is a weak solution of (4.1) for every admissible Γ. Such solutions are called *renormalized solutions* of the Boltzmann equation (2.7).

Specifically, cast in our setting, the theory of renormalized solutions yields the following.

Theorem 4.1 (DiPerna–Lions renormalized solutions). *Let b satisfy*

$$\lim_{|v|\to\infty} \frac{1}{1+|v|^2} \int_{\mathbb{S}^{D-1}\times K} b(\omega, v_1 - v)\,\mathrm{d}\omega\,\mathrm{d}v_1 = 0, \quad \textit{for every compact } K \subset \mathbb{R}^D. \tag{4.4}$$

Given any initial data G^{in} in the entropy class

$$E(M\mathrm{d}v\,\mathrm{d}x) = \big\{G^{in} \geqq 0 \;:\; H(G^{in}) < \infty\big\}, \tag{4.5}$$

there exists at least one $G \geqq 0$ in $C([0,\infty); w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x))$ that for every admissible function Γ is a weak solution of (4.1). This solution satisfies a weak form of the local conservation law of mass

$$\epsilon\,\partial_t\langle G\rangle + \nabla_x\cdot\langle v\,G\rangle = 0. \tag{4.6}$$

Moreover, there exists a matrix-valued distribution W such that $W\,\mathrm{d}x$ is nonnegative definite measure and G and W satisfy a weak form of the local conservation law of momentum

$$\epsilon\,\partial_t\langle v\,G\rangle + \nabla_x\cdot\langle v\otimes v\,G\rangle + \nabla_x\cdot W = 0, \tag{4.7}$$

and for every $t > 0$, the global energy equality

$$\int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G(t)\rangle \,\mathrm{d}x + \int_{\mathbb{T}^D} \tfrac{1}{2}\,\mathrm{tr}(W(t))\,\mathrm{d}x = \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G^{in}\rangle \,\mathrm{d}x, \tag{4.8}$$

and the global entropy inequality

$$H(G(t)) + \int_{\mathbb{T}^D} \tfrac{1}{2}\,\mathrm{tr}(W(t))\,\mathrm{d}x + \frac{1}{\epsilon^2}\int_0^t R(G(s))\,\mathrm{d}s \leqq H(G^{in}). \tag{4.9}$$

DiPerna–Lions renormalized solutions are not known to satisfy many properties that one would formally expect to be satisfied by solutions of the Boltzmann equation. In particular, the theory does not assert either the local conservation of momentum in (2.15), the global conservation of energy in (2.16), the global entropy equality (2.18), or even a local entropy inequality; nor does it assert the uniqueness of the solution. Nevertheless, as shown in [17], it provides enough control to establish a Navier–Stokes–Fourier limit theorem for bounded collision kernels and, as shown here, to do so for a much larger class of collision kernels.

4.2. Leray solutions

The DiPerna–Lions theory has many similarities with the Leray theory of global weak solutions of the initial-value problem for Navier–Stokes type systems [8,23]. For the Navier–Stokes–Fourier system (1.1–1.2) with mean zero initial data, the Leray theory is set in the following Hilbert spaces of vector- and scalar-valued functions:

$$\begin{aligned}
\mathbb{H}_v &= \left\{ w \in L^2(\mathrm{d}x;\mathbb{R}^D) \;:\; \nabla_x \cdot w = 0,\ \int_{\mathbb{T}^D} w\,\mathrm{d}x = 0 \right\},\\
\mathbb{H}_s &= \left\{ \chi \in L^2(\mathrm{d}x;\mathbb{R}) \;:\; \int_{\mathbb{T}^D} \chi\,\mathrm{d}x = 0 \right\},\\
\mathbb{V}_v &= \left\{ w \in \mathbb{H}_v \;:\; \int_{\mathbb{T}^D} |\nabla_x w|^2\,\mathrm{d}x < \infty \right\},\\
\mathbb{V}_s &= \left\{ \chi \in \mathbb{H}_s \;:\; \int_{\mathbb{T}^D} |\nabla_x \chi|^2\,\mathrm{d}x < \infty \right\}.
\end{aligned}$$

Let $\mathbb{H} = \mathbb{H}_v \oplus \mathbb{H}_s$ and $\mathbb{V} = \mathbb{V}_v \oplus \mathbb{V}_s$. Then in our setting the Leray theory yields the following.

Theorem 4.2 (Leray Solutions). *Given any initial data $(u^{in}, \theta^{in}) \in \mathbb{H}$, there exists at least one $(u, \theta) \in C([0,\infty); w\text{-}\mathbb{H}) \cap L^2(\mathrm{d}t; \mathbb{V})$ that is a weak solution of the Navier–Stokes–Fourier system (1.1–1.2) in the sense that for every $(w, \chi) \in \mathbb{H} \cap C^1(\mathbb{T}^D)$ and every $[t_1, t_2] \subset [0,\infty)$ it satisfies*

$$\int_{\mathbb{T}^D} w\cdot u(t_2)\,\mathrm{d}x - \int_{\mathbb{T}^D} w\cdot u(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_{\mathbb{T}^D} \nabla_x w : (u\otimes u)\,\mathrm{d}x\,\mathrm{d}t = -\nu\int_{t_1}^{t_2}\int_{\mathbb{T}^D} \nabla_x w : \nabla_x u\,\mathrm{d}x\,\mathrm{d}t, \tag{4.10}$$

$$\int_{\mathbb{T}^D} \chi\,\theta(t_2)\,\mathrm{d}x - \int_{\mathbb{T}^D} \chi\,\theta(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_{\mathbb{T}^D} \nabla_x \chi\cdot(u\,\theta)\,\mathrm{d}x\,\mathrm{d}t = -\tfrac{2}{D+2}\,\kappa\int_{t_1}^{t_2}\int_{\mathbb{T}^D} \nabla_x\chi\cdot\nabla_x\theta\,\mathrm{d}x\,\mathrm{d}t. \tag{4.11}$$

Moreover, for every $t > 0$, (u,θ) satisfies the dissipation inequalities

$$\int_{\mathbb{T}^D} \tfrac{1}{2}|u(t)|^2\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^D} \nu|\nabla_x u|^2\,\mathrm{d}x\,\mathrm{d}s \leqq \int_{\mathbb{T}^D} \tfrac{1}{2}|u^{in}|^2\,\mathrm{d}x, \tag{4.12}$$

$$\int_{\mathbb{T}^D} \tfrac{D+2}{4}|\theta(t)|^2\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^D} \kappa|\nabla_x \theta|^2\,\mathrm{d}x\,\mathrm{d}s \leqq \int_{\mathbb{T}^D} \tfrac{D+2}{4}|\theta^{in}|^2\,\mathrm{d}x. \tag{4.13}$$

By arguing formally from the Navier–Stokes–Fourier system (1.1–1.2), one would expect these inequalities to be equalities. However, that is not asserted by the Leray theory. Also, as was the case for the DiPerna–Lions theory, the Leray theory does not assert uniqueness of the solution.

Because the role of the dissipation inequalities (4.12) and (4.13) is to provide a priori estimates, the existence theory also works if they are replaced by the single dissipation inequality

$$\int_{\mathbb{T}^D} \tfrac{1}{2}|u(t)|^2 + \tfrac{D+2}{4}|\theta(t)|^2\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^D} \nu|\nabla_x u|^2 + \kappa|\nabla_x\theta|^2\,\mathrm{d}x\,\mathrm{d}s \leqq \int_{\mathbb{T}^D} \tfrac{1}{2}|u^{in}|^2 + \tfrac{D+2}{4}|\theta^{in}|^2\,\mathrm{d}x. \tag{4.14}$$

It is this version of the Leray theory that we will obtain in the limit.

5. Main results

5.1. Formal derivation

The Navier–Stokes–Fourier system (1.1–1.2) can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the kinetic density F about the absolute Maxwellian M are scaled to be on the order of ϵ. More precisely, we consider families of initial data G_ϵ^{in} for and families of solutions G_ϵ to the scaled Boltzmann initial-value problem (2.7) that are parameterized by the Knudsen number ϵ and have the form

$$G_\epsilon^{in} = 1 + \epsilon\, g_\epsilon^{in}, \quad G_\epsilon = 1 + \epsilon\, g_\epsilon, \tag{5.1}$$

One sees from the Boltzmann equation (2.7) satisfied by G_ϵ that the fluctuations g_ϵ satisfy

$$\epsilon\, \partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L} g_\epsilon = \mathcal{Q}(g_\epsilon, g_\epsilon), \tag{5.2}$$

where $\mathcal{L}$ is the linearized collision operator defined by (3.10).

A formal derivation in the style of [2] can be carried out by assuming that $g_\epsilon \to g$ with $g \in L^\infty(\mathrm{d}t; L^2(M\mathrm{d}v\,\mathrm{d}x))$, and that all formally small terms vanish. One finds that g has the infinitesimal Maxwellian form

$$g = v\cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta, \tag{5.3}$$

where (u, θ) solves the Navier–Stokes–Fourier system (1.1–1.2) with the coefficients of kinematic viscosity and thermal conductivity given in terms of the linearized collision operator $\mathcal{L}$, and the matrix-valued function A and the vector-valued function B defined by

$$A(v) = v\otimes v - \tfrac{1}{D}|v|^2 I, \quad B(v) = \tfrac{1}{2}|v|^2 v - \tfrac{D+2}{2} v. \tag{5.4}$$

Because each entry of A and B is in $L^2(a^{-1}M\mathrm{d}v)$, we can define the matrix-valued function $\widehat{A} \in L^2(aM\mathrm{d}v; \mathbb{R}^{D\times D})$ and the vector-valued function $\widehat{B} \in L^2(aM\mathrm{d}v; \mathbb{R}^D)$ by

$$\widehat{A} = \mathcal{L}^{-1}A, \quad \text{and} \quad \widehat{B} = \mathcal{L}^{-1}B, \tag{5.5}$$

where $\mathcal{L}^{-1}$ is the unique pseudoinverse of $\mathcal{L}$ that satisfies (3.36). Because $\mathcal{P}A = 0$ and $\mathcal{P}B = 0$, it follows from (3.35) that $\widehat{A}$ and $\widehat{B}$ are respectively the unique solutions of

$$\mathcal{L}\widehat{A} = A, \quad \mathcal{P}\widehat{A} = 0, \quad \text{and} \quad \mathcal{L}\widehat{B} = B, \quad \mathcal{P}\widehat{B} = 0. \tag{5.6}$$

Because each entry of A and B is in $L^p(a^{1-p}M\mathrm{d}v)$ for every $p \in (1, \infty)$, each entry of $\widehat{A}$ and $\widehat{B}$ is therefore in $L^p(aM\mathrm{d}v)$ for every $p \in (1, \infty)$. The coefficients of kinematic viscosity ν and thermal conductivity κ are given by

$$\nu = \tfrac{1}{(D-1)(D+2)}\left\langle \widehat{A} : \mathcal{L}\widehat{A}\right\rangle, \quad \kappa = \tfrac{1}{D}\left\langle \widehat{B}\cdot\mathcal{L}\widehat{B}\right\rangle. \tag{5.7}$$

In this section we state our main results, which proves this formal relationship.

5.2. Statement of the main theorem

Our main theorem is the following.

Theorem 5.1. *Let the collision kernel b satisfy the assumptions of Sect. 3.*

Let G_ϵ^{in} be a family in the entropy class $E(M\mathrm{d}v\,\mathrm{d}x)$ given by (4.5) that satisfies the normalizations (2.11) and the bound

$$H(G_\epsilon^{in}) \leqq C^{in}\epsilon^2, \tag{5.8}$$

for some positive constant C^{in}. Let g_ϵ^{in} be the associated family of fluctuations given by (5.1). Assume that for some $(u^{in}, \theta^{in}) \in \mathbb{H}$ the family g_ϵ^{in} satisfies

$$\lim_{\epsilon\to 0}\left(\Pi\langle v\, g_\epsilon^{in}\rangle, \langle(\tfrac{1}{D+2}|v|^2 - 1)\, g_\epsilon^{in}\rangle\right) = (u^{in}, \theta^{in}) \quad \textit{in the sense of distributions.} \tag{5.9}$$

Let G_ϵ be any family of DiPerna–Lions renormalized solutions of the Boltzmann equation (2.7) *that have G_ϵ^{in} as initial values. Let g_ϵ be the family of fluctuations given by* (5.1).

Then the family g_ϵ is relatively compact in w-$L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\sigma M \mathrm{d}v\,\mathrm{d}x))$, where $\sigma = 1 + |v|^2$. Every limit point g of g_ϵ in w-$L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\sigma M \mathrm{d}v\,\mathrm{d}x))$ has the infinitesimal Maxwellian form (5.3) *where $(u, \theta) \in C([0,\infty);\, w\text{-}\mathbb{H}) \cap L^2(\mathrm{d}t;\, \mathbb{V})$ is a Leray solution with initial data (u^{in}, θ^{in}) of the Navier–Stokes–Fourier system* (1.1–1.2) *with ν and κ given by* (5.7). *More specifically, (u, θ) satisfies the weak form of the Navier–Stokes–Fourier system given by* (4.10–4.11) *and the dissipation inequality*

$$\int_{\mathbb{T}^D} \tfrac{1}{2}|u(t)|^2 + \tfrac{D+2}{4}|\theta(t)|^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{T}^D} \nu |\nabla_x u|^2 + \kappa |\nabla_x \theta|^2 \,\mathrm{d}x\,\mathrm{d}s \leqq \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2} H(G_\epsilon^{in}) \leqq C^{in}. \tag{5.10}$$

Moreover, every subsequence g_{ϵ_k} of g_ϵ that converges to g as $\epsilon_k \to 0$ also satisfies

$$\Pi \langle v\, g_{\epsilon_k} \rangle \to u \quad \text{in } C([0,\infty);\, \mathcal{D}'(\mathbb{T}^D;\mathbb{R}^D)), \tag{5.11}$$

$$\langle (\tfrac{1}{D+2}|v|^2 - 1)\, g_{\epsilon_k} \rangle \to \theta \quad \text{in } C([0,\infty);\, w\text{-}L^1(\mathrm{d}x;\mathbb{R})). \tag{5.12}$$

where Π is the orthogonal projection from $L^2(\mathrm{d}x;\mathbb{R}^D)$ onto divergence-free vector fields.

In the next section we will reduce the proof of our theorem to a sequence of propositions, the proofs of which will be given in subsequent sections.

Remark. The dissipation inequality (5.10) is just (4.14) with the right-hand side replaced by the lim inf. We can recover (4.14) in the limit by replacing (5.8) and (5.9) with the hypothesis

$$g_\epsilon^{in} \to v\cdot u^{in} + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta^{in} \quad \text{entropically at order } \epsilon \text{ as } \epsilon \to 0. \tag{5.13}$$

The notion of *entropic convergence*, was introduced in [3] and is defined as follows.

Definition 5.1. Let G_ϵ be a family in the entropy class $E(M\mathrm{d}v\,\mathrm{d}x)$ given by (4.5) and let g_ϵ be the associated family of fluctuations given by (5.1). The family g_ϵ is said to *converge entropically at order* ϵ to some $g \in L^2(M\mathrm{d}v\,\mathrm{d}x)$ if and only if

$$g_\epsilon \to g \text{ in } w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x), \quad \text{and} \quad \lim_{\epsilon\to 0}\frac{1}{\epsilon^2} H(G_\epsilon) = \int_{\mathbb{T}^D} \tfrac{1}{2}\langle g^2\rangle \,\mathrm{d}x. \tag{5.14}$$

Proposition 4.11 of [3] showed that entropic convergence is stronger than norm convergence in $L^1(\sigma M\mathrm{d}v\,\mathrm{d}x)$). It is therefore a natural tool for obtaining strong convergence results for fluctuations about an absolute Maxwellian [4,5,11,12,17, 24,27,28]. With the addition of hypothesis (5.13), it is clear from (5.10) and (5.14) that (4.14) is recovered. Moreover, one can prove in the style of Theorem 6.2 of

[12] that if (4.14) is an equality for every $t \in [0, T]$ then as $\epsilon \to 0$ one obtains the strong convergences

$$g_\epsilon(t) \to v \cdot u(t) + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta(t) \quad \text{entropically at order } \epsilon \text{ for every } t \in [0, T],$$
$$\frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{\epsilon^2 (1 + \frac{1}{3}\epsilon\, g_\epsilon)^{\frac{1}{2}}} \to \Phi : \nabla_x u + \Psi \cdot \nabla_x \theta \quad \text{in } L^1([0, T]; L^1((\sigma + \sigma_1)\mathrm{d}\mu\, \mathrm{d}x)),$$

where $\Phi = A + A_1 - A' + A'_1$ and $\Psi = B + B_1 - B' + B'_1$. In particular, one obtains these strong convergences for so long as (u, θ) is a classical solution of the Navier–Stokes–Fourier system.

Remark. The Stokes–Fourier and acoustic limit results of [12] can be extended to the class of collision kernels considered here. Moreover, their scaling hypotheses can be weakened to $G_\epsilon = 1 + \delta_\epsilon g_\epsilon$ where $\delta_\epsilon = o(\epsilon)$ as $\epsilon \to 0$, and $\delta_\epsilon = O\left(\epsilon^{\frac{1}{2}}\right)$ as $\epsilon \to 0$ respectively, which is formally optimal in the first case but not in the second [21].

6. Proof of main theorem

In order to clarify the structure of the proof, we defer the proofs of many technical details to later sections.

6.1. Fluctuations

Because the family G_ϵ satisfies the entropy inequality

$$H\left(G_\epsilon(t)\right) + \frac{1}{\epsilon^2}\int_0^t R\left(G_\epsilon(s)\right) \mathrm{d}s \leqq H\left(G_\epsilon^{in}\right) \leqq C^{in}\epsilon^2, \tag{6.1}$$

Proposition 3.1 (1) of [3] implies that the family

$$\sigma g_\epsilon \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(M\mathrm{d}v\, \mathrm{d}x)), \tag{6.2}$$

where $\sigma = 1 + |v|^2$. We will show that every limit point of the family g_ϵ is governed by a Leray solution of the Navier–Stokes–Fourier system.

We will also consider the associated family of scaled collision integrands defined by

$$q_\epsilon = \frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{\epsilon^2}. \tag{6.3}$$

The entropy inequality (6.1) and Proposition 3.4 (1) of [3] imply that the family

$$\sigma \frac{q_\epsilon}{\sqrt{n_\epsilon}} \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}\mu\, \mathrm{d}x)), \tag{6.4}$$

where $n_\epsilon = 1 + \frac{1}{3}\epsilon\, g_\epsilon$, and $\mathrm{d}\mu = b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v$ is a positive unit measure.

Consider any convergent subsequence of the family g_ϵ, still abusively denoted g_ϵ, such that the sequence $q_\epsilon/\sqrt{n_\epsilon}$ also converges. Let g be the w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\sigma M \mathrm{d}v\, \mathrm{d}x))$ limit point of the sequence g_ϵ, and q be the w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\sigma \mathrm{d}\mu\, \mathrm{d}x))$ limit point of the sequence $q_\epsilon/\sqrt{n_\epsilon}$. Then the entropy inequality (6.1) and Proposition 3.8 of [3] imply that g is an infinitesimal Maxwellian given by

$$g = \rho + v\cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\theta, \tag{6.5}$$

for some $(\rho, u, \theta) \in L^\infty(\mathrm{d}t; L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))$ that for every $t \geqq 0$ satisfies

$$\begin{aligned}\int_{\mathbb{T}^D} &\tfrac{1}{2}|\rho(t)|^2 + \tfrac{1}{2}|u(t)|^2 + \tfrac{D}{4}|\theta(t)|^2\,\mathrm{d}x \\ &\leqq \int_{\mathbb{T}^D} \tfrac{1}{2}\langle |g(t)|^2\rangle\,\mathrm{d}x \leqq \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2} H\big(G_\epsilon(t)\big),\end{aligned} \tag{6.6}$$

while Proposition 3.4 (2) of [3] implies that $q \in L^2(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t)$. We will show that (ρ, u, θ) is a Leray solution of the Navier–Stokes–Fourier system (1.1–1.2) with initial data (u^{in}, θ^{in}).

6.2. *Nonlinear compactness by averaging*

Key to our proof is the fact the sequence

$$a\,\frac{g_\epsilon^2}{n_\epsilon} \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(M \mathrm{d}v\,\mathrm{d}x)), \tag{6.7}$$

where $n_\epsilon = 1 + \frac{1}{3}\epsilon\, g_\epsilon$. We establish this fact in Sect. 7 by employing the entropy inequality (6.1) and the L^1 velocity averaging theory of Golse and Saint-Raymond [16]. They used this averaging theory to prove analogous compactness results while establishing Navier–Stokes–Fourier limits for collision kernels with a Grad cutoff that derive from hard potentials [17,18].

6.3. *Approximate conservation laws*

In order to prove our main theorem we have to pass to the limit in approximate local and global conservation laws built from the renormalized Boltzmann equation (4.1). We choose to use the normalization of that equation given by

$$\Gamma(Z) = \frac{Z-1}{1+(Z-1)^2}. \tag{6.8}$$

After dividing by ϵ, Eq. (4.1) becomes

$$\epsilon\,\partial_t \tilde{g}_\epsilon + v\cdot\nabla_x \tilde{g}_\epsilon = \frac{1}{\epsilon^2}\,\Gamma'(G_\epsilon)\,\mathcal{Q}(G_\epsilon, G_\epsilon), \tag{6.9}$$

where $\tilde{g}_\epsilon = \Gamma(G_\epsilon)/\epsilon$. By introducing $N_\epsilon = 1 + \epsilon^2 g_\epsilon^2$, we can write

$$\tilde{g}_\epsilon = \frac{g_\epsilon}{N_\epsilon}, \quad \Gamma'(G_\epsilon) = \frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}. \tag{6.10}$$

When the moment of the renormalized Boltzmann equation (6.9) is formally taken with respect to any $\zeta \in \operatorname{span}\{1, v_1, \ldots, v_D, |v|^2\}$, one obtains

$$\partial_t \langle \zeta\, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle v\, \zeta\, \tilde{g}_\epsilon \rangle = \frac{1}{\epsilon} \left\langle\!\left\langle \zeta\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\right\rangle. \tag{6.11}$$

This fails to be a local conservation law because the so-called *conservation defect* on the right-hand side is generally nonzero.

It can be shown from (4.3) that every DiPerna–Lions solution satisfies (6.11) in the sense that for every $\chi \in C^1(\mathbb{T}^D)$ and every $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned}\int_{\mathbb{T}^D} \chi\, \langle \zeta\, \tilde{g}_\epsilon(t_2) \rangle \, \mathrm{d}x - \int_{\mathbb{T}^D} \chi\, \langle \zeta\, \tilde{g}_\epsilon(t_1) \rangle \, \mathrm{d}x &= \int_{t_1}^{t_2} \int_{\mathbb{T}^D} \frac{1}{\epsilon} \nabla_x \chi \cdot \langle v\, \zeta\, \tilde{g}_\epsilon \rangle \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_{t_1}^{t_2} \int_{\mathbb{T}^D} \chi\, \frac{1}{\epsilon} \left\langle\!\left\langle \zeta\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\right\rangle \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{6.12}$$

This is the sense in which we understand (6.11) is satisfied. Approximate global conservation laws are obtained by setting $\chi = 1$ above.

The fact that the conservation defect term on the right-hand side of (6.12) vanishes as $\epsilon \to 0$ follows from the fact χ is bounded, the fact ζ is a collision invariant, and the compactness result (6.7). Specifically, we show that

$$\frac{1}{\epsilon} \left\langle\!\left\langle \zeta\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\right\rangle \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)) \text{ as } \epsilon \to 0. \tag{6.13}$$

This fact is established by Theorem 8.1, which is stated and proved in Sect. 8 using the compactness result (6.7).

6.4. Establishing the global conservation laws

By (6.13) the right-hand side of (6.12) vanishes with ϵ uniformly over all $[t_1, t_2]$ contained in any bounded interval of time. By setting $\chi = 1$ it follows that the sequence

$$\int_{\mathbb{T}^D} \langle \zeta\, \tilde{g}_\epsilon(t) \rangle \, \mathrm{d}x \quad \text{is equicontinuous in } C([0, \infty); \mathbb{R}). \tag{6.14}$$

The fact $\tilde{g}_\epsilon = \Gamma(G_\epsilon)/\epsilon$ with Γ given by (6.8) and the entropy bound (6.1) imply that

$$\int_{\mathbb{T}^D} \left\langle \tilde{g}_\epsilon(t)^2 \right\rangle \mathrm{d}x \leqq 3C^{in} \quad \text{for every } t \geqq 0. \tag{6.15}$$

It then follows from the Cauchy–Schwarz inequality that for every $t \geqq 0$ the sequence in (6.14) is also equibounded in $C([0, \infty); \mathbb{R})$. The Arzela–Ascoli theorem then implies that the sequence

$$\int_{\mathbb{T}^D} \langle \zeta\, \tilde{g}_\epsilon(t) \rangle \, \mathrm{d}x \quad \text{is relatively compact in } C([0, \infty); \mathbb{R}).$$

Now setting $t_1 = 0$ and $\chi = 1$ in (6.12), letting $\epsilon \to 0$, and using the normalization (2.16) shows that for every $t \geqq 0$ we have the limiting global conservation law

$$\int_{\mathbb{T}^D} \langle \zeta\, g(t) \rangle \,\mathrm{d}x = \int_{\mathbb{T}^D} \langle \zeta\, g(0) \rangle \,\mathrm{d}x = 0. \tag{6.16}$$

The infinitesimal Maxwellian form (6.5) then implies that, as stated by the Main Theorem 5.1,

$$\int_{\mathbb{T}^D} \rho \,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^D} u \,\mathrm{d}x = 0, \quad \int_{\mathbb{T}^D} \theta \,\mathrm{d}x = 0. \tag{6.17}$$

6.5. Establishing the incompressibility and Boussinesq relations

We know that g is a infinitesimal Maxwellian of the form (6.5) parameterized by its associated (fluctuation of) velocity field u, mass density ρ, and temperature θ. By proceeding as in the proof of Proposition 4.2 of [3], multiply (6.12) by ϵ, pass to the limit, and use the infinitesimal Maxwellian of the form to see that these functions satisfy

$$\nabla_x \cdot u = 0, \quad \nabla_x(\rho + \theta) = 0. \tag{6.18}$$

The first of these is the incompressibility relation while the second is a weak form of the Boussinesq relation. By (6.17) we see that

$$\int_{\mathbb{T}^D} \rho + \theta \,\mathrm{d}x = 0.$$

In other words, for every $t \geqq 0$ the function $(\rho + \theta)(\,\cdot\,, t)$ is an element of $L^2(\mathrm{d}x)$ that is orthogonal to the constants. Because (6.18) states that $\nabla_x(\rho + \theta) = 0$, a classical argument based on Fourier series shows that

$$\rho + \theta = 0 \quad \text{for almost every } (x, t) \in \mathbb{T}^D \times [0, \infty). \tag{6.19}$$

By (6.5), this implies that both the incompressibility and Boussinesq relations (1.1) hold, and that g is of the form (5.3) as stated by the Main Theorem 5.1.

6.6. Establishing the dissipation inequality

By passing to the limit in the weak form of (6.9) while using the facts that $\Gamma'(G_\epsilon)\, q_\epsilon \to q$ in w-$L^1_{loc}(\mathrm{d}t;\, w$-$L^1(\sigma \mathrm{d}\mu\,\mathrm{d}x))$ and that $g_\epsilon \to g$ in w-$L^1_{loc}(\mathrm{d}t;\, w$-$L^1(\sigma M \mathrm{d}v\,\mathrm{d}x))$ as $\epsilon \to 0$, we find that

$$v \cdot \nabla_x g = \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} q\, b(\omega, v_1 - v) \,\mathrm{d}\omega\, M_1 \mathrm{d}v_1. \tag{6.20}$$

It then follows from (6.5) and (6.18) that for every $\hat{\xi} \in L^2(aM\mathrm{d}v)$ we have

$$\langle\!\langle \hat{\xi}\, q \rangle\!\rangle = \langle \hat{\xi}\, A \rangle : \nabla_x u + \langle \hat{\xi}\, B \rangle \cdot \nabla_x \theta. \tag{6.21}$$

Then by arguing as in the proof of Proposition 4.6 of [3] we obtain

$$\int_0^t \int_{\mathbb{T}^D} \nu|\nabla_x u|^2 + \kappa|\nabla_x \theta|^2 \,\mathrm{d}x\,\mathrm{d}s \leqq \int_0^t \int_{\mathbb{T}^D} \tfrac{1}{4}\langle\!\langle q^2 \rangle\!\rangle \,\mathrm{d}x\,\mathrm{d}s$$
$$\leqq \liminf_{\epsilon\to 0} \frac{1}{\epsilon^4} \int_0^t R\big(G_\epsilon(s)\big)\,\mathrm{d}s, \tag{6.22}$$

where ν and κ are given by (5.7). The dissipation inequality (5.10) asserted by the Main Theorem 5.1 follows by combining (6.1), (6.6), and (6.22).

6.7. Approximate dynamical equations

The difficulty in passing to the limit in (6.11) is that the fluxes are order $1/\epsilon$. This difficulty is overcome by using the same strategy as in the formal derivations [1,2]. First, we pass to the limit when $\zeta = v_i$ for $i = 1, \ldots, D$ or when $\zeta = (\frac{1}{2}|v|^2 - \frac{D+2}{2})$. In other words, we pass to the limit is the approximate motion and heat equations

$$\partial_t \langle v\, \tilde{g}_\epsilon \rangle + \tfrac{1}{\epsilon}\, \nabla_x \cdot \langle A\, \tilde{g}_\epsilon \rangle + \tfrac{1}{\epsilon}\, \nabla_x \langle \tfrac{1}{D}|v|^2 \tilde{g}_\epsilon \rangle = \tfrac{1}{\epsilon} \langle\!\langle v\, \Gamma'(G_\epsilon)\, q_\epsilon \rangle\!\rangle, \tag{6.23}$$
$$\partial_t \langle (\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\, \tilde{g}_\epsilon \rangle + \tfrac{1}{\epsilon}\, \nabla_x \cdot \langle B\, \tilde{g}_\epsilon \rangle = \tfrac{1}{\epsilon} \langle\!\langle (\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\, \Gamma'(G_\epsilon)\, q_\epsilon \rangle\!\rangle. \tag{6.24}$$

Second, the approximate momentum equation (6.23) will be integrated against divergence-free test functions. The last term in its flux will thereby be eliminated, and we only have to pass to the limit in the flux terms above that involve A and B—namely, in the terms

$$\frac{1}{\epsilon}\langle A\, \tilde{g}_\epsilon \rangle, \quad \frac{1}{\epsilon}\langle B\, \tilde{g}_\epsilon \rangle. \tag{6.25}$$

Recall that $A = \mathcal{L}\widehat{A}$ and $B = \mathcal{L}\widehat{B}$ where $\widehat{A}$ and $\widehat{B}$ are defined by (5.5) and that each entry of $\widehat{A}$ and $\widehat{B}$ is in $L^p(aM\mathrm{d}v)$ for every $p \in [1, \infty)$.

6.8. Compactness of the flux terms

Let $s \in (1, \infty]$ be from the assumed bound (3.15) on b. Let $p = 2 + \frac{1}{s-1}$, so that $p = 2$ when $s = \infty$. Let $\hat{\xi} \in L^p(aM\mathrm{d}v)$ and set $\xi = \mathcal{L}\hat{\xi}$. We claim that the sequence of moments

$$\frac{1}{\epsilon}\langle \xi\, \tilde{g}_\epsilon \rangle \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)). \tag{6.26}$$

Because each entry of the flux terms (6.25) has this form, it follows that

$$\text{the entries of } \frac{1}{\epsilon}\langle A\, \tilde{g}_\epsilon \rangle \text{ and } \frac{1}{\epsilon}\langle B\, \tilde{g}_\epsilon \rangle \text{ are}$$
$$\text{relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)). \tag{6.27}$$

Claim (6.26) will follow from the observation that

$$\langle \xi\, \tilde{g}_\epsilon \rangle = \big\langle \big(\mathcal{L}\hat{\xi}\big)\, \tilde{g}_\epsilon \big\rangle = \big\langle \hat{\xi}\, \mathcal{L}\tilde{g}_\epsilon \big\rangle = \big\langle\!\big\langle \hat{\xi}\, \big(\tilde{g}_\epsilon + \tilde{g}_{\epsilon 1} - \tilde{g}'_\epsilon - \tilde{g}'_{\epsilon 1}\big) \big\rangle\!\big\rangle, \tag{6.28}$$

once we establish the fact that the sequence

$$\frac{1}{\epsilon}\,\hat{\xi}\,\big(\tilde{g}_\epsilon + \tilde{g}_{\epsilon 1} - \tilde{g}'_\epsilon - \tilde{g}'_{\epsilon 1}\big) \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x)). \tag{6.29}$$

To establish assertion (6.29), define the symmetrically normalized collision integrand $\tilde{q}_\epsilon$ by

$$\tilde{q}_\epsilon = \frac{q_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon} = \frac{1}{\epsilon^2}\,\frac{G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}, \tag{6.30}$$

and define T_ϵ by

$$\frac{1}{\epsilon}\big(\tilde{g}_\epsilon + \tilde{g}_{\epsilon 1} - \tilde{g}'_\epsilon - \tilde{g}'_{\epsilon 1}\big) = \tilde{g}'_{\epsilon 1}\tilde{g}'_\epsilon - \tilde{g}_{\epsilon 1}\tilde{g}_\epsilon - \tilde{q}_\epsilon + T_\epsilon. \tag{6.31}$$

The compactness result (6.7) and Lemma 9.2 imply that the sequences

$$\hat{\xi}\,\tilde{g}'_{\epsilon 1}\tilde{g}'_\epsilon \quad \text{and} \quad \hat{\xi}\,\tilde{g}_{\epsilon 1}\tilde{g}_\epsilon \quad \text{are relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x)).$$

The compactness result (6.7), Lemma 9.2, and Proposition 10.1 imply that

$$\hat{\xi}\,T_\epsilon \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t;\, L^1(\mathrm{d}\mu\,\mathrm{d}x)) \text{ as } \epsilon \to 0. \tag{6.32}$$

Finally, Lemma 10.2 of [12] implies the sequence $\tilde{q}_\epsilon$ is bounded in $L^2(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t)$. It follows that the sequence

$$\hat{\xi}\,\tilde{q}_\epsilon \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x)). \tag{6.33}$$

We have thereby established assertion (6.29).

6.9. Convergence of the density terms

The densities term corresponding to (6.23) and (6.24) are

$$\Pi\langle v\,\tilde{g}_\epsilon\rangle \quad \text{and} \quad \langle(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\,\tilde{g}_\epsilon\rangle. \tag{6.34}$$

Here Π is the Leray projection onto divergence-free vector fields in $L^2(\mathrm{d}x;\mathbb{R}^D)$. The sequences (6.34) are convergent in $w\text{-}L^2_{loc}(\mathrm{d}t;\, w\text{-}L^2(\mathrm{d}x))$.

We use the Arzela–Ascoli theorem to establish that these sequences are convergent in $C([0,\infty);\, w\text{-}L^2(\mathrm{d}x))$. Indeed, it follows from the L^2 bound (6.15) and the Cauchy–Schwarz inequality that for every $t \geqq 0$ the sequences (6.34) are pointwise relatively compact in $w\text{-}L^2(\mathrm{d}x)$. That they are also equicontinuous follows from the weak forms (6.12) of the approximate motion and heat equations upon noting that the flux terms are relatively compact in $w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}x))$ by (6.27) while the conservation defects vanish by (6.13). It then follows from the Arzela–Ascoli theorem that the sequences (6.34) are relatively compact in $C([0,\infty);\, w\text{-}L^2(\mathrm{d}x))$. Because they are convergent in the weaker $w\text{-}L^2_{loc}(\mathrm{d}t;\, w\text{-}L^2(\mathrm{d}x))$ topology, they must be convergent in $C([0,\infty); w\text{-}L^2(\mathrm{d}x))$. We thereby conclude that as $\epsilon \to 0$ one has

$$\begin{aligned} \Pi\langle v\,\tilde{g}_\epsilon\rangle &\to u && \text{in } C([0,\infty); w\text{-}L^2(\mathrm{d}x;\mathbb{R}^D)),\\ \langle(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\,\tilde{g}_\epsilon\rangle &\to \tfrac{D+2}{2}\theta && \text{in } C([0,\infty); w\text{-}L^2(\mathrm{d}x)). \end{aligned} \tag{6.35}$$

The limits asserted in (5.11) and (5.12) of Theorem 5.1 then follow. Moreover, by combining these results with (6.1), (6.6), and (6.22), we conclude that $(u,\theta) \in C([0,\infty); w\text{-}\mathbb{H}) \cap L^2(\mathrm{d}t;\mathbb{V})$, as asserted by Theorem 5.1. By hypothesis (5.9) of Theorem 5.1 we also can argue that

$$u(x,0) = u^{in}(x), \quad \theta(x,0) = \theta^{in}(x), \quad \text{for almost every } x, \tag{6.36}$$

as asserted by Theorem 5.1.

6.10. Convergence of the flux terms

Upon placing (6.31) into the right-hand side of (6.28), the moments (6.26) decompose as

$$\frac{1}{\epsilon}\,\langle \xi\,\tilde{g}_\epsilon\rangle = \left\langle\!\!\left\langle \hat{\xi}\,\left(\tilde{g}'_{\epsilon 1}\tilde{g}'_\epsilon - \tilde{g}_{\epsilon 1}\tilde{g}_\epsilon\right)\right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \hat{\xi}\,\tilde{q}_\epsilon\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \hat{\xi}\,T_\epsilon\right\rangle\!\!\right\rangle. \tag{6.37}$$

The first term in this decomposition is quadratic in $\tilde{g}_\epsilon$, the second is linear in $\tilde{q}_\epsilon$, while the last is a remainder that vanishes as $\epsilon \to 0$ by (6.32).

Because the sequence $\tilde{q}_\epsilon$ is bounded in $L^2(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t)$, we conclude from (6.4) and (6.30) that $\tilde{q}_\epsilon \to q$ in $w\text{-}L^2(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t)$. This fact allows us to pass to the limit in the linear term in the decomposition (6.37). Indeed, by using (6.21) we see that as $\epsilon \to 0$ one has

$$\left\langle\!\!\left\langle \hat{\xi}\,\tilde{q}_\epsilon\right\rangle\!\!\right\rangle \to \left\langle\!\!\left\langle \hat{\xi}\,q\right\rangle\!\!\right\rangle = \langle \hat{\xi}\,A\rangle : \nabla_x u + \langle \hat{\xi}\,B\rangle \cdot \nabla_x \theta \quad \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x)). \tag{6.38}$$

In particular, we see that as $\epsilon \to 0$ one has

$$\begin{aligned} \left\langle\!\!\left\langle \widehat{A}\,\tilde{q}_\epsilon\right\rangle\!\!\right\rangle &\to \nu\left[\nabla_x u + (\nabla_x u)^T\right] && \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x;\mathbb{R}^{D\times D})),\\ \left\langle\!\!\left\langle \widehat{B}\,\tilde{q}_\epsilon\right\rangle\!\!\right\rangle &\to \kappa\,\nabla_x\theta && \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x;\mathbb{R}^{D})), \end{aligned}$$

where ν and κ are given by (5.7).

We now focus on the passage to the limit in the quadratic term in the decomposition (6.37). This term has the equivalent form

$$\left\langle\!\!\left\langle \hat{\xi}\,\left(\tilde{g}'_{\epsilon 1}\tilde{g}'_\epsilon - \tilde{g}_{\epsilon 1}\tilde{g}_\epsilon\right)\right\rangle\!\!\right\rangle = \left\langle \hat{\xi}\,\mathcal{Q}(\tilde{g}_\epsilon,\tilde{g}_\epsilon)\right\rangle. \tag{6.39}$$

This passage to the limit is the most difficult part of establishing the limit of the flux.

6.10.1. Approximation by infinitesimal Maxwellians

We decompose $\tilde{g}_\epsilon$ into its infinitesimal Maxwellian $\mathcal{P}\tilde{g}_\epsilon$ and its deviation $\mathcal{P}^\perp\tilde{g}_\epsilon$ as

$$\tilde{g}_\epsilon = \mathcal{P}\tilde{g}_\epsilon + \mathcal{P}^\perp\tilde{g}_\epsilon, \tag{6.40}$$

where $\mathcal{P}$ is the orthogonal projection from $L^2(M\mathrm{d}v)$ onto $\mathrm{Null}(\mathcal{L})$ given by (3.34) and $\mathcal{P}^\perp$ is its complement given by $\mathcal{P}^\perp = \mathcal{I} - \mathcal{P}$. We claim that

$$\mathcal{P}^\perp\tilde{g}_\epsilon \to 0 \quad \text{in } L^2_{loc}(\mathrm{d}t; L^2(aM\mathrm{d}v\mathrm{d}x)). \tag{6.41}$$

Indeed, consider the identity

$$\langle \tilde{g}_\epsilon\, \mathcal{L}\tilde{g}_\epsilon\rangle = \left\langle\!\!\left\langle \epsilon\, \tilde{g}_\epsilon \left(\frac{\tilde{g}_\epsilon + \tilde{g}_{\epsilon 1} - \tilde{g}'_\epsilon - \tilde{g}'_{\epsilon 1}}{\epsilon} \right) \right\rangle\!\!\right\rangle.$$

The term inside the parentheses above is relatively compact in w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}\mu\, \mathrm{d}x))$ by (6.29). Because $\epsilon\, \tilde{g}_\epsilon$ is bounded and vanishes almost everywhere as $\epsilon \to 0$, we see that

$$\lim_{\epsilon\to 0} \langle \tilde{g}_\epsilon\, \mathcal{L}\tilde{g}_\epsilon\rangle = 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)).$$

The coercivity bound (3.33) then immediately implies that

$$\lim_{\epsilon\to 0} \left\langle a \left(\mathcal{P}^\perp\tilde{g}_\epsilon\right)^2 \right\rangle = 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)),$$

which establishes claim (6.41).

6.10.2. Quadratic approximation by infinitesimal Maxwellians

When decomposition (6.40) is placed into the quadratic term (6.39), it yields

$$\left\langle \hat{\xi}\, \mathcal{Q}(\tilde{g}_\epsilon, \tilde{g}_\epsilon)\right\rangle = \left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}\tilde{g}_\epsilon)\right\rangle + 2\left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}^\perp\tilde{g}_\epsilon)\right\rangle + \left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}^\perp\tilde{g}_\epsilon, \mathcal{P}^\perp\tilde{g}_\epsilon)\right\rangle. \tag{6.42}$$

Here we show the last two terms above vanish as $\epsilon \to 0$. Recall that the collision kernel b satisfies assumption (3.15) for some $C_b < \infty$ and $s \in (1, \infty]$. Let $p = 2 + \frac{1}{s-1}$, so $p = 2$ when $s = \infty$. Lemma 9.1 of Sect. 9 then combines with (6.39) to yield the basic quadratic estimate

$$\left|\left\langle \hat{\xi}\, \mathcal{Q}(\tilde{g}, \tilde{h})\right\rangle\right| \leqq 2\, C_b^{\frac{1}{p^*}}\, \langle a\, |\hat{\xi}|^p\rangle^{\frac{1}{p}}\, \langle a\, \tilde{g}^2\rangle^{\frac{1}{2}}\, \langle a\, \tilde{h}^2\rangle^{\frac{1}{2}}, \tag{6.43}$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\tilde{g}, \tilde{h} \in L^2(aM\mathrm{d}v)$. It follows from (6.41) and the basic estimate (6.43) that

$$\left.\begin{array}{l} \left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}^\perp\tilde{g}_\epsilon)\right\rangle \to \quad 0 \\ \left\langle \hat{\xi}\, \mathcal{Q}(\mathcal{P}^\perp\tilde{g}_\epsilon, \mathcal{P}^\perp\tilde{g}_\epsilon)\right\rangle \to 0 \end{array}\right\} \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)).$$

So all that remains to be done is to pass to the limit in the first term on the right-hand side of (6.42)—the one that is quadratic in $\mathcal{P}\tilde{g}_\epsilon$.

6.10.3. Passing to the limit

The infinitesimal Maxwellian $\mathcal{P}\tilde{g}_\epsilon$ has the form

$$\mathcal{P}\tilde{g}_\epsilon = \tilde{\rho}_\epsilon + v\cdot\tilde{u}_\epsilon + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\tilde{\theta}_\epsilon , \tag{6.44}$$

where $\tilde{\rho}_\epsilon$, $\tilde{u}_\epsilon$, and $\tilde{\theta}_\epsilon$ are defined by

$$\tilde{\rho}_\epsilon = \langle \tilde{g}_\epsilon\rangle, \quad \tilde{u}_\epsilon = \langle v\, \tilde{g}_\epsilon\rangle, \quad \tilde{\theta}_\epsilon = \langle (\tfrac{1}{D}|v|^2 - 1)\, \tilde{g}_\epsilon\rangle . \tag{6.45}$$

One can show [2] that if $\tilde{g}$ is an infinitesimal Maxwellian then it satisfies the identity

$$\langle \hat{\xi}\ \mathcal{Q}(\tilde{g},\tilde{g})\rangle = \tfrac{1}{2}\langle \hat{\xi}\ \mathcal{L}(\tilde{g}^2)\rangle = \tfrac{1}{2}\langle \mathcal{L}\hat{\xi}\ \mathcal{P}^{\perp}(\tilde{g}^2)\rangle . \tag{6.46}$$

Because $\mathcal{P}\tilde{g}_\epsilon$ is an infinitesimal Maxwellian, we can use the identity (6.46) to express the first term on the right-hand side of (6.42) as

$$\begin{aligned}\big\langle \hat{\xi}\ \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}\tilde{g}_\epsilon)\big\rangle &= \tfrac{1}{2}\big\langle \xi\ \mathcal{P}^{\perp}\big(\mathcal{P}\tilde{g}_\epsilon\big)^2\big\rangle \\ &= \tfrac{1}{2}\langle \xi\ A\rangle : (\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon) + \langle \xi\ B\rangle \cdot \tilde{u}_\epsilon\tilde{\theta}_\epsilon + \tfrac{1}{2}\langle \xi\ C\rangle\, \tilde{\theta}_\epsilon^2 ,\end{aligned} \tag{6.47}$$

where $C(v) = \frac{1}{4}|v|^4 - \frac{D+2}{2}|v|^2 + \frac{D(D+2)}{4}$. It is easily checked that C is in $\mathrm{Null}(\mathcal{L})^{\perp}$. We thereby have reduced the problem to passing to the limit in the terms

$$\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon , \quad \tilde{u}_\epsilon\tilde{\theta}_\epsilon , \quad \tilde{\theta}_\epsilon^2 . \tag{6.48}$$

We are unable to pass to the limit in the above terms in full generality. However, Proposition 11.1 of Sect. 11 yields that

$$\left.\begin{aligned}&\lim_{\epsilon\to 0} \Pi\nabla_x\cdot\big(\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon\big) = \Pi\nabla_x\cdot(u\otimes u)\\ &\lim_{\epsilon\to 0} \nabla_x\cdot\big(\tilde{\theta}_\epsilon\, \tilde{u}_\epsilon\big) = \nabla_x\cdot(\theta\, u)\end{aligned}\right\}\quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)),$$

where Π is the Leray projection onto divergence-free vector fields in $L^2(\mathrm{d}x;\mathbb{R}^D)$. It follows that

$$\left.\begin{aligned}&\lim_{\epsilon\to 0} \Pi\nabla_x\cdot\Big\langle \widehat{A}\ \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}\tilde{g}_\epsilon)\Big\rangle = \Pi\nabla_x\cdot(u\otimes u)\\ &\lim_{\epsilon\to 0} \nabla_x\cdot\Big\langle \widehat{B}\ \mathcal{Q}(\mathcal{P}\tilde{g}_\epsilon, \mathcal{P}\tilde{g}_\epsilon)\Big\rangle = \tfrac{D+2}{2}\nabla_x\cdot(\theta\, u)\end{aligned}\right\}\quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)).$$

We thereby obtain the limiting fluxes for the Navier–Stokes–Fourier motion and heat equations, thereby completing the proof of the Main Theorem 5.1. □

7. Compactness from averaging

Here we establish the compactness assertion (6.7) with the following proposition.

Proposition 7.1. *Under the hypotheses of Theorem 5.1, the sequence*

$$a\,\frac{g_\epsilon^2}{n_\epsilon} \quad \textit{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x)). \tag{7.1}$$

Proof. Let γ_ϵ be defined by $\sqrt{G_\epsilon} = 1 + \epsilon\,\gamma_\epsilon$. Then $g_\epsilon = 2\gamma_\epsilon + \epsilon\,\gamma_\epsilon{}^2$ and

$$\frac{g_\epsilon^2}{n_\epsilon} = \gamma_\epsilon{}^2 \frac{4 + 4\epsilon\,\gamma_\epsilon + \epsilon^2\gamma_\epsilon{}^2}{1 + \frac{2}{3}\epsilon\,\gamma_\epsilon + \frac{1}{3}\epsilon^2\gamma_\epsilon{}^2}.$$

Because $\epsilon\,\gamma_\epsilon \geqq -1$, it follows from the above that

$$\tfrac{3}{2}\gamma_\epsilon{}^2 \leqq \frac{g_\epsilon^2}{n_\epsilon} \leqq \tfrac{9}{2}\gamma_\epsilon{}^2. \tag{7.2}$$

Assertion (7.1) is therefore equivalent to the assertion that the sequence

$$a\gamma_\epsilon{}^2 \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x)). \tag{7.3}$$

We will prove these equivalent assertions in two steps.

We begin by showing that the sequence $a\gamma_\epsilon{}^2$ lies in $L^1_{loc}(\mathrm{d}t;\, L^1(M\mathrm{d}v\,\mathrm{d}x)$ and is equi-integrable in v. Specifically, we show that the sequence

$$a\gamma_\epsilon{}^2 \quad \text{is bounded in } L^1_{loc}(\mathrm{d}t;\, L^1(M\mathrm{d}v\,\mathrm{d}x)), \tag{7.4}$$

and that for every $[0, T] \subset [0, \infty)$

$$\lim_{\eta\to 0} \int_0^T \int_{\mathbb{T}^D} \sup_{\langle \mathbf{1}_S\rangle < \eta} \langle \mathbf{1}_S\, a\gamma_\epsilon{}^2\rangle \,\mathrm{d}x\,\mathrm{d}t = 0 \quad \text{uniformly in } \epsilon, \tag{7.5}$$

where the supremum is taken over all measurable $S \subset \mathbb{R}^D \times \mathbb{T}^D \times [0, T]$. These assertions are established below by Proposition 7.2.

It follows from (7.2) and (7.5) that

$$\lim_{R\to\infty} \mathbf{1}_{\{|v|>R\}}\, a\, \frac{g_\epsilon^2}{n_\epsilon} = 0 \quad \text{in } L^1_{loc}(\mathrm{d}t;\, L^1(M\mathrm{d}v\,\mathrm{d}x)) \quad \text{uniformly in } \epsilon. \tag{7.6}$$

Because a is bounded above and below by positive constants over every set of the form $\{|v| \leqq R\}$ for some $R > 0$, the relative compactness assertion (7.1) will follow once we prove that for every $R > 0$ the sequence

$$\mathbf{1}_{\{|v|\leqq R\}}\, \frac{g_\epsilon^2}{n_\epsilon} \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x)). \tag{7.7}$$

This fact is established below by Proposition 7.3, thereby proving Proposition 7.1. □

The proof of Proposition 7.2 rests solely on the entropy bound (6.1). A key tool is the relative entropy cutoff that was first introduced by Saint-Raymond in her study of the incompressible Euler limit [35]. The proof of Proposition 7.3 will also rest on the L^1 velocity averaging theory of GOLSE and SAINT-RAYMOND [16].

7.1. Equi-integrability proposition

The equi-integrability in v of the family $a\gamma_\epsilon{}^2$ asserted in (7.4) and (7.5) is established by the following proposition.

Proposition 7.2. *Under the hypotheses of Theorem 5.1, the sequence*

$$a\gamma_\epsilon{}^2 \quad \textit{is bounded in } L^1_{loc}(\mathrm{d}t; L^1(M\mathrm{d}v\,\mathrm{d}x)), \tag{7.8}$$

and that for every $[0,T] \subset [0,\infty)$

$$\lim_{\eta\to 0}\int_0^T\int_{\mathbb{T}^D} \sup_{\langle \mathbf{1}_S\rangle<\eta} \langle \mathbf{1}_S\, a\gamma_\epsilon{}^2\rangle\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \textit{uniformly in } \epsilon, \tag{7.9}$$

where the supremum is taken over all measurable $S \subset \mathbb{R}^D\times\mathbb{T}^D\times[0,T]$.

Proof. Let $[0,T]\subset[0,\infty)$. We employ the decomposition $\gamma_\epsilon = \mathcal{P}\gamma_\epsilon + \mathcal{P}^\perp\gamma_\epsilon$ in order to exploit the facts that $\mathcal{P}\gamma_\epsilon$ is well-behaved in v and that $\mathcal{P}^\perp\gamma_\epsilon$ is small. Then for every measurable $S\subset\mathbb{R}^D\times\mathbb{T}^D\times[0,T]$ this decomposition yields

$$\big\langle \mathbf{1}_S\, a\gamma_\epsilon{}^2\big\rangle = \big\langle \mathbf{1}_S\, a\gamma_\epsilon\mathcal{P}\gamma_\epsilon\big\rangle + \big\langle \mathbf{1}_S\, a\gamma_\epsilon\mathcal{P}^\perp\gamma_\epsilon\big\rangle. \tag{7.10}$$

The first term on the right-hand side above can be bounded as

$$\big\langle \mathbf{1}_S\, a|\gamma_\epsilon\mathcal{P}\gamma_\epsilon|\big\rangle \leqq \big\langle \gamma_\epsilon{}^2\big\rangle^{\frac{1}{2}}\big\langle \mathbf{1}_S\, a^2(\mathcal{P}\gamma_\epsilon)^2\big\rangle^{\frac{1}{2}}.$$

Now let $\{\zeta_i\}_{i=0}^{D+1}$ be the orthonormal basis for $\mathrm{Null}(\mathcal{L}) = \mathrm{span}\{1, v_1,\ldots,v_D,|v|^2\}$ in $L^2(M\mathrm{d}v)$ given by $\zeta_0=1$, $\zeta_i = v_i$ for $i=1,\ldots,D$, and $\zeta_{D+1} = \frac{1}{\sqrt{2D}}(|v|^2-D)$. Then

$$(\mathcal{P}\gamma_\epsilon)^2 = \left(\sum_{i=0}^{D+1}\zeta_i\langle\zeta_i\gamma_\epsilon\rangle\right)^2 \leqq \left(\sum_{i=0}^{D+1}\zeta_i{}^2\right)\left(\sum_{i=0}^{D+1}\langle\zeta_i\gamma_\epsilon\rangle^2\right) \leqq D(1+|v|^4)\big\langle\gamma_\epsilon{}^2\big\rangle.$$

Upon placing this estimate into the previous estimate, we bound the first term on the right-hand side of (7.10) as

$$\big\langle \mathbf{1}_S\, a|\gamma_\epsilon\mathcal{P}\gamma_\epsilon|\big\rangle \leqq \big\langle D\mathbf{1}_S\, a^2(1+|v|^4)\big\rangle^{\frac{1}{2}}\big\langle\gamma_\epsilon{}^2\big\rangle. \tag{7.11}$$

Now let us consider how we might control the second term on the right-hand side of (7.10). An application of Cauchy–Schwarz gives

$$\big\langle \mathbf{1}_S\, a|\gamma_\epsilon\mathcal{P}^\perp\gamma_\epsilon|\big\rangle \leqq \big\langle \mathbf{1}_S\, a\gamma_\epsilon{}^2\big\rangle^{\frac{1}{2}}\big\langle a(\mathcal{P}^\perp\gamma_\epsilon)^2\big\rangle^{\frac{1}{2}}. \tag{7.12}$$

At first glance, this estimate might look unproductive because the first factor on the right-hand side is exactly the object that we are trying to control. However, the idea is that because the second factor is small, we can employ a rough control of the first factor.

We begin by bounding the second factor on the right-hand side of (7.12). We do this with a coercivity bound. For every $\delta > 0$ we introduce a saturated collision kernel $b_\delta(\omega, v_1 - v)$ by

$$b_\delta(\omega, v_1 - v) = \frac{b(\omega, v_1 - v)}{1 + \delta\, \dfrac{\bar{b}(v_1 - v)}{1 + |v_1 - v|^2}}. \tag{7.13}$$

We let $\mathcal{Q}_\delta$ and $\mathcal{L}_\delta$ denote the associated operators given by

$$\begin{aligned} \mathcal{Q}_\delta(G, G) &= \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \big(G_1' G' - G_1 G\big)\, b_\delta(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1, \\ \mathcal{L}_\delta g &= \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \big(g + g_1 - g' - g_1'\big)\, b_\delta(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1. \end{aligned} \tag{7.14}$$

Because $b_\delta(\omega, v_1 - v)$ has the form (7.13), the operators $\mathcal{Q}_\delta$ and $\mathcal{L}_\delta$ share most of the formal properties satisfied by $\mathcal{Q}$ and $\mathcal{L}$. In particular, we show in Lemma 7.1 that for every sufficiently small δ there exists a $\ell_\delta > 0$ such that $\mathcal{L}_\delta$ satisfies the coercivity bound

$$\ell_\delta \langle a(\mathcal{P}^\perp \tilde{g})^2 \rangle \leqq \langle \tilde{g} \mathcal{L}_\delta \tilde{g} \rangle \quad \text{for every } \tilde{g} \in L^2(aM\mathrm{d}v). \tag{7.15}$$

We now fix δ small enough that this bound holds for some $\ell_\delta > 0$.

When the bound (7.15) is combined with the fact $\sqrt{G_\epsilon} = 1 + \epsilon\, \gamma_\epsilon$ and the identity

$$\frac{1}{\epsilon}\, \mathcal{L}_\delta \gamma_\epsilon = \mathcal{Q}_\delta(\gamma_\epsilon, \gamma_\epsilon) - \frac{1}{\epsilon^2}\, \mathcal{Q}_\delta\Big(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\Big),$$

we obtain

$$\begin{aligned} \frac{\ell_\delta}{\epsilon} \big\langle a(\mathcal{P}^\perp \gamma_\epsilon)^2 \big\rangle &\leqq \big|\big\langle (\mathcal{P}^\perp \gamma_\epsilon)\, \mathcal{Q}_\delta(\gamma_\epsilon, \gamma_\epsilon) \big\rangle\big| \\ &\quad + \frac{1}{\epsilon^2} \Big|\Big\langle (\mathcal{P}^\perp \gamma_\epsilon)\, \mathcal{Q}_\delta\Big(\sqrt{G_\epsilon}, \sqrt{G_\epsilon}\Big) \Big\rangle\Big|. \end{aligned} \tag{7.16}$$

To bound the first term on the right-hand side of (7.16), we use saturated kernel assumption (3.28) to see

$$\begin{aligned} \bar{b}_\delta(v_1 - v) &= \int_{\mathbb{S}^{D-1}} b_\delta(\omega, v_1 - v)\, \mathrm{d}\omega \\ &= \frac{\bar{b}(v_1 - v)}{1 + \delta\, \dfrac{\bar{b}(v_1 - v)}{1 + |v_1 - v|^2}} \leqq C_\delta \big(1 + a(v_1)\big)\big(1 + a(v)\big). \end{aligned}$$

We thereby obtain the bound

$$\begin{aligned} \big|\big\langle (\mathcal{P}^\perp \gamma_\epsilon)\, \mathcal{Q}_\delta(\gamma_\epsilon, \gamma_\epsilon) \big\rangle\big| &= \left|\left\langle\!\!\left\langle \frac{\bar{b}_\delta}{\bar{b}}\, (\mathcal{P}^\perp \gamma_\epsilon)\, \big(\gamma_{\epsilon 1}' \gamma_\epsilon' - \gamma_{\epsilon 1} \gamma_\epsilon\big) \right\rangle\!\!\right\rangle\right| \\ &\leqq 2 \big\langle\!\big\langle (\mathcal{P}^\perp \gamma_\epsilon)^2 \big\rangle\!\big\rangle^{\frac{1}{2}} \left\langle\!\!\left\langle \frac{\bar{b}_\delta}{\bar{b}}\, \gamma_{\epsilon 1}^2 \gamma_\epsilon^2 \right\rangle\!\!\right\rangle^{\frac{1}{2}} \\ &\leqq 2 C_\delta \big\langle a(\mathcal{P}^\perp \gamma_\epsilon)^2 \big\rangle^{\frac{1}{2}} \big\langle (1 + a) \gamma_\epsilon^2 \big\rangle. \end{aligned}$$

We bound the second term on the right-hand side of (7.16) as

$$\begin{aligned}\left|\left\langle(\mathcal{P}^\perp\gamma_\epsilon)\,\mathcal{Q}_\delta\Big(\sqrt{G_\epsilon},\sqrt{G_\epsilon}\Big)\right\rangle\right| &= \left|\left\langle\!\!\left\langle\frac{\bar{b}_\delta}{\bar{b}}\,(\mathcal{P}^\perp\gamma_\epsilon)\Big(\sqrt{G'_{\epsilon 1}G'_\epsilon}-\sqrt{G_{\epsilon 1}G_\epsilon}\Big)\right\rangle\!\!\right\rangle\right| \\ &\leqq \left\langle\!\!\left\langle(\mathcal{P}^\perp\gamma_\epsilon)^2\right\rangle\!\!\right\rangle^{\frac12}\left\langle\!\!\left\langle\frac{\bar{b}_\delta}{\bar{b}}\Big(\sqrt{G'_{\epsilon 1}G'_\epsilon}-\sqrt{G_{\epsilon 1}G_\epsilon}\Big)^2\right\rangle\!\!\right\rangle^{\frac12} \\ &\leqq \left\langle a(\mathcal{P}^\perp\gamma_\epsilon)^2\right\rangle^{\frac12}\left\langle\!\!\left\langle\Big(\sqrt{G'_{\epsilon 1}G'_\epsilon}-\sqrt{G_{\epsilon 1}G_\epsilon}\Big)^2\right\rangle\!\!\right\rangle^{\frac12}.\end{aligned}$$

Upon collecting the above estimates, we obtain the bound

$$\frac{\ell_\delta}{\epsilon}\left\langle a(\mathcal{P}^\perp\gamma_\epsilon)^2\right\rangle^{\frac12} \leqq 2C_\delta\Big(\langle\gamma_\epsilon{}^2\rangle+\langle a\gamma_\epsilon{}^2\rangle\Big)+\frac{1}{\epsilon^2}\left\langle\!\!\left\langle\Big(\sqrt{G'_{\epsilon 1}G'_\epsilon}-\sqrt{G_{\epsilon 1}G_\epsilon}\Big)^2\right\rangle\!\!\right\rangle^{\frac12}. \tag{7.17}$$

Because $\big(\sqrt{Y}-\sqrt{X}\big)^2 \leqq \frac14(Y-X)\log(Y/X)$ for every $X, Y > 0$, it follows from the entropy bound (6.1) that the sequence

$$\frac{1}{\epsilon^4}\Big(\sqrt{G'_{\epsilon 1}G'_\epsilon}-\sqrt{G_{\epsilon 1}G_\epsilon}\Big)^2 \quad \text{is bounded in } L^1(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t). \tag{7.18}$$

Because the sequence $\langle\gamma_\epsilon{}^2\rangle$ is bounded in $L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x))$, it follows from the bound (7.17) that the second factor on the right-hand side of (7.12) is ϵ times a sequence that is at best bounded in $L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x))$.

We now bound the first factor on the right-hand side of (7.12). First, because $z^2 \leqq |2z+z^2|$ for $z \geqq -1$, and because $g_\epsilon = 2\gamma_\epsilon + \epsilon\,\gamma_\epsilon{}^2$, we have the bound

$$\epsilon\,\gamma_\epsilon{}^2 \leqq |g_\epsilon|.$$

Second, for every $\alpha > 0$ Young's inequality yields

$$a|g_\epsilon| \leqq \frac{\alpha}{\epsilon}\left[h^*\!\Big(\frac{a}{\alpha}\Big)+h(\epsilon\,g_\epsilon)\right]. \tag{7.19}$$

By combining the above bounds we obtain the rough bound

$$\left\langle \mathbf{1}_S\,a\gamma_\epsilon{}^2\right\rangle \leqq \frac{\alpha}{\epsilon^2}\left[\left\langle\mathbf{1}_S\,h^*\Big(\frac{a}{\alpha}\Big)\right\rangle+\left\langle h(\epsilon\,g_\epsilon)\right\rangle\right]. \tag{7.20}$$

Hence, the first factor on the right-hand side of (7.12) times ϵ is bounded in $L^\infty(\mathrm{d}t; L^2(\mathrm{d}x))$.

Given the bound (7.17) on the second factor of the right-hand side of (7.12), the rough bound (7.20) on the first factor is not enough to bound the right-hand side of (7.12) in $L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x))$. This problem can be overcome by using a relative entropy cutoff. Let $T > 0$. For every $\lambda > 0$ define the sets

$$\begin{aligned}\Omega^\lambda_\epsilon &= \big\{(x,t)\in\mathbb{T}^D\times[0,T] \,:\, \langle h(\epsilon g_\epsilon)\rangle \leqq \lambda^2\big\},\\ \widetilde{\Omega}^\lambda_\epsilon &= \big\{(x,t)\in\mathbb{T}^D\times[0,T] \,:\, \langle h(\epsilon g_\epsilon)\rangle > \lambda^2\big\},\end{aligned} \tag{7.21}$$

and introduce the decomposition

$$\left\langle \mathbf{1}_S\, a{\gamma_\epsilon}^2\right\rangle = \mathbf{1}_{\widetilde{\Omega}_\epsilon^\lambda}\left\langle \mathbf{1}_S\, a{\gamma_\epsilon}^2\right\rangle + \mathbf{1}_{\Omega_\epsilon^\lambda}\left\langle \mathbf{1}_S\, a{\gamma_\epsilon}^2\right\rangle. \tag{7.22}$$

We use the rough bound (7.20) to bound the first term on the right-hand side of (7.22) as

$$\mathbf{1}_{\widetilde{\Omega}_\epsilon^\lambda}\left\langle \mathbf{1}_S\, a{\gamma_\epsilon}^2\right\rangle \leqq \frac{\alpha}{\epsilon^2}\left[\mathbf{1}_{\widetilde{\Omega}_\epsilon^\lambda}\left\langle \mathbf{1}_S\, h^*\!\left(\frac{a}{\alpha}\right)\right\rangle + \left\langle h(\epsilon\, g_\epsilon)\right\rangle\right]. \tag{7.23}$$

The Chebychev inequality and the entropy bound (6.1) shows that

$$\operatorname{meas}\left(\widetilde{\Omega}_\epsilon^\lambda\right) = \int_0^T\!\!\int_{\mathbb{T}^D} \mathbf{1}_{\widetilde{\Omega}_\epsilon^\lambda}\,\mathrm{d}x\,\mathrm{d}t \leqq \int_0^T\!\!\int_{\mathbb{T}^D} \frac{\langle h(\epsilon g_\epsilon)\rangle}{\lambda^2}\,\mathrm{d}x\,\mathrm{d}t \leqq \frac{\epsilon^2 C^{in} T}{\lambda^2}. \tag{7.24}$$

By combining (7.23) with $\alpha = 1$, (7.24), and the entropy bound (6.1) we see that

$$\begin{aligned}\int_0^T\!\!\int_{\mathbb{T}^D} \mathbf{1}_{\widetilde{\Omega}_\epsilon^\lambda}\left\langle a{\gamma_\epsilon}^2\right\rangle \mathrm{d}x\,\mathrm{d}t &\leqq \frac{1}{\epsilon^2}\left[\operatorname{meas}\left(\widetilde{\Omega}_\epsilon^\lambda\right)\left\langle h^*(a)\right\rangle + \int_0^T\!\!\int_{\mathbb{T}^D}\left\langle h(\epsilon\, g_\epsilon)\right\rangle \mathrm{d}x\,\mathrm{d}t\right]\\ &\leqq \frac{C^{in}T}{\lambda^2}\left\langle h^*(a)\right\rangle + C^{in}T.\end{aligned}$$

It follows that for every λ the sequence

$$\mathbf{1}_{\widetilde{\Omega}_\epsilon^\lambda}\, a{\gamma_\epsilon}^2 \quad \text{is bounded in } L^1_{loc}(\mathrm{d}t; L^1(M\mathrm{d}v\,\mathrm{d}x)). \tag{7.25}$$

We use (7.10), (7.11), and (7.12) to bound the second term on the right-hand side of (7.22) as

$$\mathbf{1}_{\Omega_\epsilon^\lambda}\left\langle \mathbf{1}_S\, a{\gamma_\epsilon}^2\right\rangle \leqq \mathbf{1}_{\Omega_\epsilon^\lambda}\left\langle D\mathbf{1}_S\, a^2(1+|v|^4)\right\rangle^{\frac12}\left\langle{\gamma_\epsilon}^2\right\rangle + \mathbf{1}_{\Omega_\epsilon^\lambda}\left\langle \mathbf{1}_S\, a{\gamma_\epsilon}^2\right\rangle^{\frac12}\left\langle a(\mathcal{P}^\perp\gamma_\epsilon)^2\right\rangle^{\frac12}. \tag{7.26}$$

For every $\lambda \leqq 1$ we can use the rough bound (7.20) with $\alpha = 1/\lambda$ and the superquadratic property of h^* to bound the first two factors of the last term on the right-hand side above as

$$\mathbf{1}_{\Omega_\epsilon^\lambda}\left\langle \mathbf{1}_S\, a{\gamma_\epsilon}^2\right\rangle \leqq \mathbf{1}_{\Omega_\epsilon^\lambda}\,\frac{1}{\lambda\epsilon^2}\left[\langle \mathbf{1}_S\, h^*(\lambda a)\rangle + \lambda^2\right] \leqq \mathbf{1}_{\Omega_\epsilon^\lambda}\,\frac{\lambda}{\epsilon^2}\left[\langle \mathbf{1}_S\, h^*(a)\rangle + 1\right].$$

The last factor of the last term on the right-hand side of (7.26) satisfies the inequality (7.17). We thereby obtain the bound

$$\begin{aligned}\mathbf{1}_{\Omega_\epsilon^\lambda}\left\langle \mathbf{1}_S\, a{\gamma_\epsilon}^2\right\rangle &\leqq \mathbf{1}_{\Omega_\epsilon^\lambda}\left\langle D\mathbf{1}_S\, a^2(1+|v|^4)\right\rangle^{\frac12}\left\langle{\gamma_\epsilon}^2\right\rangle\\ &\quad + \mathbf{1}_{\Omega_\epsilon^\lambda}\,\frac{\lambda^{\frac12}}{\ell_\delta}\left[\langle \mathbf{1}_S\, h^*(a)\rangle + 1\right]^{\frac12}\\ &\quad \times\left[2C_\delta\left(\left\langle{\gamma_\epsilon}^2\right\rangle + \left\langle a{\gamma_\epsilon}^2\right\rangle\right) + \frac{1}{\epsilon^2}\left\langle\!\!\left\langle\left(\sqrt{G'_{\epsilon 1}G'_\epsilon} - \sqrt{G_{\epsilon 1}G_\epsilon}\right)^2\right\rangle\!\!\right\rangle^{\frac12}\right].\end{aligned} \tag{7.27}$$

This implies that for every λ that satisfies

$$\lambda^{\frac{1}{2}} \frac{2C_\delta}{\ell_\delta} \left[\langle h^*(a)\rangle + 1\right]^{\frac{1}{2}} \leqq \frac{1}{2}, \quad \lambda \leqq 1, \tag{7.28}$$

we obtain the bound

$$\begin{aligned} \mathbf{1}_{\Omega_\epsilon^\lambda} \langle a\gamma_\epsilon{}^2\rangle &\leqq \left[2\big\langle D\, a^2(1+|v|^4)\big\rangle + 1\right]^{\frac{1}{2}} \langle \gamma_\epsilon{}^2\rangle \\ &+ \frac{1}{2C_\delta}\frac{1}{\epsilon^2} \left\langle\!\!\left\langle \left(\sqrt{G'_{\epsilon 1}G'_\epsilon} - \sqrt{G_{\epsilon 1}G_\epsilon}\right)^2 \right\rangle\!\!\right\rangle^{\frac{1}{2}}. \end{aligned} \tag{7.29}$$

This bound shows that for every λ that satisfies the bounds (7.28) the sequence

$$\mathbf{1}_{\Omega_\epsilon^\lambda}\, a\gamma_\epsilon{}^2 \quad \text{is bounded in } L^1_{loc}(\mathrm{d}t; L^1(M\mathrm{d}v\,\mathrm{d}x)). \tag{7.30}$$

This result combined with (7.25) implies that assertion (7.8) of the Proposition holds.

Now let $\eta > 0$ and let $S \subset \mathbb{R}^D \times \mathbb{T}^D \times [0, T]$ be any measurable set such that $\langle \mathbf{1}_S\rangle < \eta$. Because $h^*(a/\alpha)$ and $a^2(1+|v|^4)$ are in $L^2(M\mathrm{d}v)$, the Cauchy-Schwarz inequality yields

$$\left\langle \mathbf{1}_S\, h^*\!\left(\frac{a}{\alpha}\right)\right\rangle \leqq \langle \mathbf{1}_S\rangle^{\frac{1}{2}} \left\langle h^*\!\left(\frac{a}{\alpha}\right)^2\right\rangle^{\frac{1}{2}}, \quad \big\langle \mathbf{1}_S\, Da^2(1+|v|^4)\big\rangle \leqq \langle \mathbf{1}_S\rangle^{\frac{1}{2}} \big\langle D^2a^4(1+|v|^4)^2\big\rangle^{\frac{1}{2}}.$$

Upon combining these inequalities, the bounds (7.22) (7.23), (7.27), (7.29), and the fact that $\langle \mathbf{1}_S\rangle < \eta$, for some positive constant C we obtain the bound

$$\begin{aligned} \langle \mathbf{1}_S\, a\gamma_\epsilon{}^2\rangle \leqq \mathbf{1}_{\tilde{\Omega}_\epsilon^\lambda}\, \eta^{\frac{1}{2}} \frac{\alpha}{\epsilon^2} \left\langle h^*\!\left(\frac{a}{\alpha}\right)^2\right\rangle^{\frac{1}{2}} + \frac{\alpha}{\epsilon^2} \langle h(\epsilon\, g_\epsilon)\rangle + \eta^{\frac{1}{4}} \big\langle D^2a^4(1+|v|^4)^2\big\rangle^{\frac{1}{4}} \\ + \lambda^{\frac{1}{2}} C\left[\langle \gamma_\epsilon{}^2\rangle + \frac{1}{\epsilon^2}\left\langle\!\!\left\langle \left(\sqrt{G'_{\epsilon 1}G'_\epsilon} - \sqrt{G_{\epsilon 1}G_\epsilon}\right)^2 \right\rangle\!\!\right\rangle^{\frac{1}{2}}\right]. \end{aligned}$$

It follows from the bounds (7.24) and (6.1) that

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^D} \sup_{\langle \mathbf{1}_S\rangle < \eta} \langle \mathbf{1}_S\, a\gamma_\epsilon{}^2\rangle\, \mathrm{d}x\, \mathrm{d}t \leqq \left[\frac{\eta^{\frac{1}{2}}}{\lambda^2} \left\langle h^*\!\left(\frac{a}{\alpha}\right)^2\right\rangle^{\frac{1}{2}} + 1\right] \alpha C^{in} T \\ + \eta^{\frac{1}{4}} \big\langle D^2a^4(1+|v|^4)^2\big\rangle^{\frac{1}{4}} T + \lambda^{\frac{1}{2}} C\big[\tfrac{4}{3}C^{in}T + C^{in}T^{\frac{1}{2}}\big], \end{aligned}$$

where the supremum is taken over all measurable $S \subset \mathbb{R}^D \times \mathbb{T}^D \times [0, T]$. By exploiting the arbitrariness of α and λ one can deduce assertion (7.9) of the Proposition from this inequality, thereby completing the proof. □

7.2. Coercivity bound lemma

We now establish the coercivity bound (7.15).

Lemma 7.1. *Let b satisfy the assumptions of Sect. 3. Let b_δ and $\mathcal{L}_\delta$ be defined by (7.13) and (7.14) respectively. Then for sufficiently small δ there exists $\ell_\delta > 0$ such that $\mathcal{L}_\delta$ satisfies the coercivity bound*

$$\ell_\delta \langle a(\mathcal{P}^\perp \tilde{g})^2\rangle \leqq \langle \tilde{g}\mathcal{L}_\delta \tilde{g}\rangle \quad \text{for every } \tilde{g} \in L^2(aM\mathrm{d}v). \tag{7.31}$$

Proof. We have

$$\frac{1}{a}\mathcal{L} = \mathcal{I} + \mathcal{K}^- - 2\mathcal{K}^+, \quad \frac{1}{a}\mathcal{L}_\delta = \frac{a_\delta}{a}\mathcal{I} + \mathcal{K}_\delta^- - 2\mathcal{K}_\delta^+, \tag{7.32}$$

where the integral operators $\mathcal{K}^-$ and $\mathcal{K}^+$ are defined by (3.12) and (3.13), while the integral operators $\mathcal{K}_\delta^-$ and $\mathcal{K}_\delta^+$ are defined for every $\tilde{h} \in L^2(aM\mathrm{d}v)$ by

$$\mathcal{K}_\delta^- \tilde{h} = \frac{1}{a}\int_{\mathbb{R}^D} \tilde{h}_1\, \bar{b}_\delta(v_1 - v)\, M_1 \mathrm{d}v_1, \tag{7.33}$$

$$\mathcal{K}_\delta^+ \tilde{h} = \frac{1}{2a}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \left(\tilde{h}' + \tilde{h}_1'\right) b_\delta(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1, \tag{7.34}$$

and the attenuation coefficient $a_\delta(v)$ is defined by

$$a_\delta(v) = \int_{\mathbb{R}^D} \bar{b}_\delta(v_1 - v)\, M_1 \mathrm{d}v_1. \tag{7.35}$$

One can show for every $\tilde{g}, \tilde{h} \in L^2(aM\mathrm{d}v)$ that

$$\left|\left\langle \tilde{g}\, a\left(\mathcal{K}^- - \mathcal{K}_\delta^-\right)\tilde{h}\right\rangle\right| = \left|\left\langle\!\left\langle \frac{\bar{b} - \bar{b}_\delta}{\bar{b}}\, \tilde{g}\, \tilde{h}_1\right\rangle\!\right\rangle\right| \leqq N_\delta \langle a\tilde{g}^2\rangle^{\frac{1}{2}}\langle a\tilde{h}^2\rangle^{\frac{1}{2}},$$

$$\left|\left\langle \tilde{g}\, a\left(\mathcal{K}^+ - \mathcal{K}_\delta^+\right)\tilde{h}\right\rangle\right| = \left|\frac{1}{2}\left\langle\!\left\langle \frac{\bar{b} - \bar{b}_\delta}{\bar{b}}\, \tilde{g}\left(\tilde{h}_1' + \tilde{h}'\right)\right\rangle\!\right\rangle\right| \leqq N_\delta \langle a\tilde{g}^2\rangle^{\frac{1}{2}}\langle a\tilde{h}^2\rangle^{\frac{1}{2}},$$

where

$$N_\delta = \sup\left\{\frac{a(v) - a_\delta(v)}{a(v)} \;:\; v \in \mathbb{R}^D\right\} \leqq 1. \tag{7.36}$$

It follows that $\|\mathcal{K}^- - \mathcal{K}_\delta^-\| \leqq N_\delta$ and $\|\mathcal{K}^+ - \mathcal{K}_\delta^+\| \leqq N_\delta$, whereby (7.32) implies $\|\frac{1}{a}\mathcal{L} - \frac{1}{a}\mathcal{L}_\delta\| \leqq 4N_\delta$. Whenever $N_\delta < \frac{1}{4}\ell$ one can derive the coercivity bound (7.31) for $\mathcal{L}_\delta$ with $\ell_\delta = \ell - 4N_\delta$ from the coercivity bound (3.33) for $\mathcal{L}$. The result will follow upon showing that $N_\delta \to 0$ as $\delta \to 0$.

Let $r(v_1 - v) = \bar{b}(v_1 - v)/(1 + |v_1 - v|^2)$. We then use the bound (3.15) to obtain the pointwise bound

$$\begin{aligned}\frac{a(v) - a_\delta(v)}{a(v)} &= \int_{\mathbb{R}^D} \frac{\delta\, r(v_1 - v)}{1 + \delta\, r(v_1 - v)} \frac{\bar{b}(v_1 - v)}{a(v)\, a(v_1)}\, a_1 M_1 \mathrm{d}v_1 \\ &\leqq \left(\int_{\mathbb{R}^D} \left| \frac{\delta\, r(v_1 - v)}{1 + \delta\, r(v_1 - v)} \right|^{s^*} a_1 M_1 \mathrm{d}v_1 \right)^{\frac{1}{s^*}} \\ &\quad \times \left(\int_{\mathbb{R}^D} \left| \frac{\bar{b}(v_1 - v)}{a(v)\, a(v_1)} \right|^{s} a_1 M_1 \mathrm{d}v_1 \right)^{\frac{1}{s}} \\ &\leqq \left(\int_{\mathbb{R}^D} \delta\, r(v_1 - v)\, a_1 M_1 \mathrm{d}v_1 \right)^{\frac{1}{s^*}} C_b. \end{aligned} \tag{7.37}$$

By again using the bound (3.15), we see that

$$\begin{aligned}\int_{\mathbb{R}^D} r(v_1 - v)\, a_1 M_1 \mathrm{d}v_1 &= \int_{\mathbb{R}^D} \frac{a(v)\, a(v_1)}{1 + |v_1 - v|^2} \frac{\bar{b}(v_1 - v)}{a(v)\, a(v_1)}\, a_1 M_1 \mathrm{d}v_1 \\ &\leqq \left(\int_{\mathbb{R}^D} \left| \frac{a(v)\, a(v_1)}{1 + |v_1 - v|^2} \right|^{s^*} a_1 M_1 \mathrm{d}v_1 \right)^{\frac{1}{s^*}} \\ &\quad \times \left(\int_{\mathbb{R}^D} \left| \frac{\bar{b}(v_1 - v)}{a(v)\, a(v_1)} \right|^{s} a_1 M_1 \mathrm{d}v_1 \right)^{\frac{1}{s}} \\ &\leqq a(v) \left(\int_{\mathbb{R}^D} \left| \frac{a(v_1)}{1 + |v_1 - v|^2} \right|^{s^*} a_1 M_1 \mathrm{d}v_1 \right)^{\frac{1}{s^*}} C_b. \end{aligned}$$

It follows from (3.7) that right-hand side above is uniformly bounded. It then follows from the bound (7.37) and definition (7.36) that $N_\delta \to 0$ as $\delta \to 0$, thereby completing the proof. □

7.3. Relative compactness proposition

The relative compactness of the sequence g_ϵ^2/n_ϵ for bounded velocities asserted by (7.7) is established by the following proposition.

Proposition 7.3. *Under the hypotheses of Theorem 5.1, for every $R > 0$ the sequence*

$$\mathbf{1}_{\{|v| \leqq R\}} \frac{g_\epsilon^2}{n_\epsilon} \quad \textit{is relatively compact in } L^1_{loc}(\mathrm{d}t; L^1(M \mathrm{d}v\, \mathrm{d}x)). \tag{7.38}$$

Proof. Let $R > 0$. Let χ be a $C^\infty([0, \infty))$ cutoff function that satisfies $0 \leqq \chi(s) \leqq 1$ for $s \geqq 0$, $\chi(s) = 1$ for $0 \leqq s \leqq 1$, and $\chi(s) = 0$ for $s \geqq 2$. Let $h(z) = (1 + z)\log(1 + z) - z$. For every $\lambda > \frac{3}{2}$ one can show that

$$\frac{z^2}{1 + \frac{1}{3}z} \leqq \frac{\lambda}{h(\frac{1}{3}\lambda)} h(z), \quad \text{for every } z \text{ such that } \quad \frac{z^2}{1 + \frac{1}{3}z} \geqq \lambda.$$

It then follows from the entropy bound (6.1) that for every $\lambda > \frac{3}{2}$ we have

$$\begin{aligned}\int_{\mathbb{T}^D} \left\langle \frac{g_\epsilon^2}{n_\epsilon} - \frac{g_\epsilon^2}{n_\epsilon}\,\chi\!\left(\frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon}\right)\right\rangle \mathrm{d}x &\leqq \int_{\mathbb{T}^D} \left\langle \mathbf{1}_{\left\{\frac{\epsilon^2 g_\epsilon^2}{n_\epsilon} \geqq \lambda\right\}} \frac{g_\epsilon^2}{n_\epsilon}\right\rangle \mathrm{d}x \\ &\leqq \frac{\lambda}{h(\frac{1}{3}\lambda)}\,\frac{1}{\epsilon^2}\int_{\mathbb{T}^D} \langle h(\epsilon\, g_\epsilon)\rangle\,\mathrm{d}x \\ &= \frac{\lambda}{h(\frac{1}{3}\lambda)}\,\frac{1}{\epsilon^2}\,H(G_\epsilon) \leqq \frac{\lambda}{h(\frac{1}{3}\lambda)}\,C^{in}.\end{aligned}$$

We therefore see that

$$\left\| \frac{g_\epsilon^2}{n_\epsilon} - \frac{g_\epsilon^2}{n_\epsilon}\,\chi\!\left(\frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon}\right)\right\|_{L^\infty(\mathrm{d}t;L^1(M\mathrm{d}v\,\mathrm{d}x))} \leqq \frac{C}{\log(\lambda)} \quad \text{as } \lambda \to \infty.$$

The relative compactness result will therefore follow if for every $\lambda > \frac{3}{2}$ we can show that

$$\mathbf{1}_{\{|v|\leqq R\}} \frac{g_\epsilon^2}{n_\epsilon}\,\chi\!\left(\frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon}\right) \quad \text{is relatively compact in } L^1_{loc}(\mathrm{d}t;\, L^1(M\mathrm{d}v\,\mathrm{d}x)). \tag{7.39}$$

Because for every $\lambda > \frac{3}{2}$ we have

$$\epsilon\, \partial_t \phi_\epsilon^\lambda + v\cdot\nabla_x \phi_\epsilon^\lambda = \psi_\epsilon^\lambda, \tag{7.40}$$

where

$$\begin{aligned}\phi_\epsilon^\lambda &= \frac{g_\epsilon^2}{n_\epsilon}\,\chi\!\left(\frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon}\right), \\ \psi_\epsilon^\lambda &= \frac{1}{\epsilon^2}\,\mathcal{Q}(G_\epsilon, G_\epsilon)\,\frac{g_\epsilon}{n_\epsilon}\,\frac{1+n_\epsilon}{n_\epsilon}\left[\chi\!\left(\frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon}\right) + \frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon}\,\chi'\!\left(\frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon}\right)\right],\end{aligned} \tag{7.41}$$

the relative compactness (7.39) will follow from the L^1 velocity averaging theory (see [16]) once we show that $\mathbf{1}_{\{|v|\leqq R\}}\phi_\epsilon^\lambda$ is equi-integrable in v and that ψ_ϵ^λ is bounded in $L^1_{loc}(\mathrm{d}t;\, L^1(M\mathrm{d}v\,\mathrm{d}x))$.

Because a is bounded below by a positive constant over $\{|v| \leqq R\}$, the fact that $\mathbf{1}_{\{|v|\leqq R\}}\gamma_\epsilon^2$ is equi-integrable in v follows directly from Proposition 7.2. Because (7.41) implies that ϕ_ϵ^λ is bounded by g_ϵ^2/n_ϵ while (7.2) implies that g_ϵ^2/n_ϵ is bounded by γ_ϵ^2, it thereby follows that $\mathbf{1}_{\{|v|\leqq R\}}\phi_\epsilon^\lambda$ is equi-integrable in v.

To show that the sequence ψ_ϵ^λ is bounded in $L^1_{loc}(\mathrm{d}t;\, L^1(M\mathrm{d}v\,\mathrm{d}x))$, we use the fact that

$$G_{\epsilon 1}' G_\epsilon' - G_{\epsilon 1} G_\epsilon = \left(\sqrt{G_{\epsilon 1}' G_\epsilon'} - \sqrt{G_{\epsilon 1} G_\epsilon}\right)^2 + 2\left(\sqrt{G_{\epsilon 1}' G_\epsilon'} - \sqrt{G_{\epsilon 1} G_\epsilon}\right)\sqrt{G_{\epsilon 1} G_\epsilon},$$

and the fact that $\sqrt{G_{\epsilon 1}} = 1 + \epsilon\, \gamma_{\epsilon 1}$ to deduce that

$$\psi_\epsilon^\lambda = \frac{1}{\epsilon^2} \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right)^2 b\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\, \frac{g_\epsilon}{n_\epsilon}\, \chi_\epsilon^\lambda$$
$$+ \frac{2}{\epsilon^2} \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right) b\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1 \frac{\sqrt{G_\epsilon}}{\sqrt{n_\epsilon}}\, \frac{g_\epsilon}{\sqrt{n_\epsilon}}\, \chi_\epsilon^\lambda$$
$$+ \frac{2}{\epsilon^2} \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right) \gamma_{\epsilon 1}\, b\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\, \frac{\sqrt{G_\epsilon}}{\sqrt{n_\epsilon}}\, \frac{\epsilon\, g_\epsilon}{\sqrt{n_\epsilon}}\, \chi_\epsilon^\lambda, \tag{7.42}$$

where χ_ϵ^λ is defined by

$$\chi_\epsilon^\lambda = \frac{1 + n_\epsilon}{n_\epsilon} \left[\chi\!\left(\frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon} \right) + \frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon}\, \chi'\!\left(\frac{\epsilon^2 g_\epsilon^2}{\lambda\, n_\epsilon} \right) \right].$$

To bound the three terms on the right-hand side of (7.42) we use the fact that the sequences

$$\frac{\epsilon\, g_\epsilon}{n_\epsilon}, \quad \frac{\sqrt{G_\epsilon}}{\sqrt{n_\epsilon}}, \quad \chi_\epsilon^\lambda, \quad \frac{\epsilon\, g_\epsilon}{\sqrt{n_\epsilon}}\, \chi_\epsilon^\lambda \quad \text{are bound in } L^\infty(M \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}t),$$

where the last bound above follows because χ_ϵ^λ is supported within the set where $\epsilon^2 g_\epsilon^2 / n_\epsilon \leqq 2\lambda$. We also use the fact that, by Proposition 7.2 and bound (7.2), the sequences

$$a\, \frac{g_\epsilon^2}{n_\epsilon}, \quad a\gamma_\epsilon^2 \quad \text{are bound in } L^1_{loc}(\mathrm{d}t;\, L^1(M \mathrm{d}v\, \mathrm{d}x)).$$

Finally, we use the fact (7.18) that the sequence

$$\frac{1}{\epsilon^2} \left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right) \quad \text{is bound in } L^2(\mathrm{d}\mu\, \mathrm{d}x\, \mathrm{d}t).$$

The first term on the right-hand side of (7.42) is bounded in $L^1_{loc}(\mathrm{d}t;\, L^1(M \mathrm{d}v\, \mathrm{d}x))$ by the above bounds on $\left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right)$, $\epsilon\, g_\epsilon / n_\epsilon$, and χ_ϵ^λ. The second term on the right-hand side of (7.42) is bounded in $L^1_{loc}(\mathrm{d}t;\, L^1(M \mathrm{d}v\, \mathrm{d}x))$ by the above bounds on $\left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right)$, $a g_\epsilon^2 / n_\epsilon$, $\sqrt{G_\epsilon} / \sqrt{n_\epsilon}$, and χ_ϵ^λ. The third term on the right-hand side of (7.42) is bounded in $L^1_{loc}(\mathrm{d}t;\, L^1(M \mathrm{d}v\, \mathrm{d}x))$ by the above bounds on $\left(\sqrt{G'_{\epsilon 1} G'_\epsilon} - \sqrt{G_{\epsilon 1} G_\epsilon} \right)$, $a\gamma_\epsilon^2$, $\sqrt{G_\epsilon} / \sqrt{n_\epsilon}$, and $\left(\epsilon\, g_\epsilon / \sqrt{n_\epsilon} \right) \chi_\epsilon^\lambda$. Therefore, for every $\lambda > 0$ the sequence ψ_ϵ^λ is bounded in $L^1_{loc}(\mathrm{d}t;\, L^1(M \mathrm{d}v\, \mathrm{d}x))$, thereby completing the proof. □

Remark. The fact that the L^1 velocity averaging theory of [16] applies to the torus $\mathbb{T}^D$ was pointed out in section 7 of that paper. This is the setting in which we apply it above.

8. Removal of the conservation defects

8.1. Conservation defect theorem

The conservation defects have the form

$$\frac{1}{\epsilon}\langle\!\langle \zeta\, \Gamma'_\epsilon(G_\epsilon)\, q_\epsilon \rangle\!\rangle = \frac{1}{\epsilon}\left\langle\!\!\left\langle \zeta \left(\frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}\right) q_\epsilon \right\rangle\!\!\right\rangle,$$

where $\zeta \in \operatorname{span}\{1, v_1, \ldots, v_D, |v|^2\}$ and $N_\epsilon = 1 + \epsilon^2 g_\epsilon^2$. In order to establish momentum and energy conservation laws from the scaled Boltzmann equation we must show that these defects vanish as $\epsilon \to 0$. This is done with the following proposition.

Proposition 8.1. *Let b be a collision kernel that satisfies the assumptions of Sect. 3. Let $G_\epsilon \geqq 0$ be a family of functions in $C([0,\infty); w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x))$ that satisfies the entropy bound (6.1). Let g_ϵ and q_ϵ be given by (5.1) and (6.3). Let $N_\epsilon = 1+\epsilon^2 g_\epsilon^2$. Assume that the family g_ϵ satisfies*

$$\frac{g_\epsilon^2}{\sqrt{N_\epsilon}} \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(aM\mathrm{d}v\,\mathrm{d}x)). \tag{8.1}$$

Then for $n = 1$ and $n = 2$ and for every $\zeta \in \operatorname{span}\{1, v_1, \ldots, v_D, |v|^2\}$ one has

$$\frac{1}{\epsilon}\left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_\epsilon^n} \right\rangle\!\!\right\rangle \to 0 \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)) \text{ as } \epsilon \to 0. \tag{8.2}$$

Proof. The case $n = 1$ is treated first. The proof simply exploits the collisional symmetries (2.14) and the fact ζ is a collision invariant to decompose the defect into three parts, each of which is dominated by a function that is then shown to vanish as $\epsilon \to 0$. The case $n = 2$ proceeds similarly, with each part being dominated by the same function that dominates the corresponding part for the $n = 1$ case. The estimates on these dominating functions are obtained from the entropy inequality (6.1) through the bound on the dissipation rate and from the compactness hypothesis (8.1). For the case $n = 1$, begin with the elementary decomposition

$$\left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \zeta\, \frac{\epsilon^2 g_{\epsilon 1}^2 q_\epsilon}{N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle. \tag{8.3}$$

Because ζ is a collision invariant, the collisional symmetries (2.14) can be used to rewrite the second term on the right-hand side of (8.3) as

$$\left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle = \tfrac{1}{2}\left\langle\!\!\left\langle (\zeta + \zeta_1)\, \frac{q_\epsilon}{N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle = \tfrac{1}{4}\left\langle\!\!\left\langle (\zeta + \zeta_1)\, \frac{N'_{\epsilon 1} N'_\epsilon - N_{\epsilon 1} N_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}\, q_\epsilon \right\rangle\!\!\right\rangle. \tag{8.4}$$

We now observe that

$$\begin{aligned} N'_{\epsilon 1} N'_\epsilon - N_{\epsilon 1} N_\epsilon &= \epsilon^2 \big(g'^2_{\epsilon 1} + g'^2_\epsilon - g^2_{\epsilon 1} - g^2_\epsilon\big) + \epsilon^4 \big(g'^2_{\epsilon 1} g'^2_\epsilon - g^2_{\epsilon 1} g^2_\epsilon\big) \\ &= \epsilon^2 \big((g'_{\epsilon 1} + g'_\epsilon)^2 - (g_{\epsilon 1} + g_\epsilon)^2\big) \\ &\quad - 2\epsilon^2 \big(g'_{\epsilon 1} g'_\epsilon - g_{\epsilon 1} g_\epsilon\big) + \epsilon^4 \big(g'^2_{\epsilon 1} g'^2_\epsilon - g^2_{\epsilon 1} g^2_\epsilon\big) \\ &= \epsilon^3 q_\epsilon \big(g'_{\epsilon 1} + g'_\epsilon + g_{\epsilon 1} + g_\epsilon\big) - \epsilon^2 \big(g'_{\epsilon 1} g'_\epsilon - g_{\epsilon 1} g_\epsilon\big)\, J_\epsilon, \end{aligned} \tag{8.5}$$

where J_ϵ is given by

$$J_\epsilon = 2 + \epsilon\,(g'_{\epsilon 1} + g'_\epsilon + g_{\epsilon 1} + g_\epsilon) - \epsilon^2(g'_{\epsilon 1}g'_\epsilon + g_{\epsilon 1}g_\epsilon). \tag{8.6}$$

Upon placing (8.5) into (8.4), using collisional symmetries and the fact ζ is a collision invariant, and placing the result into (8.3), we obtain the decomposition

$$\frac{1}{\epsilon}\left\langle\!\!\left\langle \zeta\,\frac{q_\epsilon}{N_\epsilon}\right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \zeta\,\frac{\epsilon\,g_{\epsilon 1}^2 q_\epsilon}{N_{\epsilon 1}N_\epsilon}\right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta\,\frac{\epsilon^2(g_{\epsilon 1}+g_\epsilon)\,q_\epsilon^2}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}\right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \zeta'\,\frac{\epsilon\,g'_{\epsilon 1}g'_\epsilon\,q_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}\,J_\epsilon\right\rangle\!\!\right\rangle. \tag{8.7}$$

This decomposition is derived in the same spirit as was decomposition (9.12) in [12]. The difference in the two arises because the role played by $N_\epsilon = 1 + \epsilon^2 g_\epsilon^2$ here was played by $n_\epsilon = 1 + \frac{1}{3}\epsilon\, g_\epsilon$ there.

We now dominate the integrands of the three terms on the right-hand side of (8.7). Because for every $\zeta \in \operatorname{span}\{1, v_1, \ldots, v_D, |v|^2\}$ there exists a constant $C < \infty$ such that $|\zeta| \leqq C\sigma$ where $\sigma(v) \equiv 1 + |v|^2$, the integrand of the first term is dominated by

$$\sigma\,\frac{\epsilon\,g_{\epsilon 1}^2|q_\epsilon|}{N_{\epsilon 1}N_\epsilon}. \tag{8.8}$$

Because

$$\frac{\epsilon\,|g_{\epsilon 1}+g_\epsilon|}{\sqrt{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}} \leqq 2,$$

the integrand of the second term is dominated by

$$\sigma\,\frac{\epsilon\,q_\epsilon^2}{\sqrt{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}}. \tag{8.9}$$

Finally, because

$$\frac{|J_\epsilon|}{\sqrt{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}} \leqq 8,$$

the integrand of the third term is dominated by

$$\sigma'\,\frac{\epsilon\,|g'_{\epsilon 1}g'_\epsilon|\,|q_\epsilon|}{\sqrt{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}}. \tag{8.10}$$

Hence, the result (8.2) for the case $n = 1$ will follow once we establish that the terms (8.8), (8.9), and (8.10) vanish as $\epsilon \to 0$.

The term (8.9) can be treated easily. Lemma 9.3 of [12] implies that

$$\sigma\,\frac{\epsilon\,q_\epsilon^2}{n'_{\epsilon 1}n'_\epsilon n_{\epsilon 1}n_\epsilon} = O\big(\epsilon\,|\log(\epsilon)|\big) \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \text{ as } \epsilon \to 0.$$

By the elementary inequality $n'_{\epsilon 1} n'_{\epsilon} n_{\epsilon 1} n_{\epsilon} \leqq 2\sqrt{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_{\epsilon}}$, the above estimate implies that

$$\sigma \frac{\epsilon\, q_\epsilon^2}{\sqrt{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_{\epsilon}}} = O\big(\epsilon\,|\log(\epsilon)|\big) \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \text{ as } \epsilon \to 0.$$

The terms (8.8) and (8.10) require much more work. Lemmas 8.2 and 8.3 respectively will yield the limits

$$\sigma \frac{\epsilon\, g_{\epsilon 1}^2 q_\epsilon}{N_{\epsilon 1} N_\epsilon} \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \text{ as } \epsilon \to 0, \tag{8.11}$$

$$\sigma' \frac{\epsilon\, g'_{\epsilon 1} g'_{\epsilon}\, q_\epsilon}{\sqrt{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_{\epsilon}}} \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \text{ as } \epsilon \to 0, \tag{8.12}$$

These lemmas are stated and proved in the next subsection, thereby establishing the result (8.2) for the case $n = 1$.

The case $n = 2$ follows similarly. Begin with the elementary decomposition

$$\left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_\epsilon^2} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \zeta\, \frac{\epsilon^2 g_{\epsilon 1}^2 q_\epsilon}{N_{\epsilon 1} N_\epsilon} \left(1 + \frac{1}{N_{\epsilon 1}}\right) \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_{\epsilon 1}^2 N_\epsilon^2} \right\rangle\!\!\right\rangle. \tag{8.13}$$

Because ζ is a collision invariant, the collisional symmetries (2.14) can be used to rewrite the second term on the right-hand side of (8.13) as

$$\begin{aligned}
\left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_{\epsilon 1}^2 N_\epsilon^2} \right\rangle\!\!\right\rangle &= \tfrac{1}{2} \left\langle\!\!\left\langle (\zeta + \zeta_1)\, \frac{q_\epsilon}{N_{\epsilon 1}^2 N_\epsilon^2} \right\rangle\!\!\right\rangle \\
&= \tfrac{1}{4} \left\langle\!\!\left\langle (\zeta + \zeta_1)\, \frac{N'_{\epsilon 1} N'_{\epsilon} - N_{\epsilon 1} N_\epsilon}{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_\epsilon}\, \frac{N'_{\epsilon 1} N'_{\epsilon} + N_{\epsilon 1} N_\epsilon}{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_\epsilon}\, q_\epsilon \right\rangle\!\!\right\rangle.
\end{aligned}$$

Upon placing (8.5) into the above, using collisional symmetries and the fact ζ is a collision invariant, and placing the result into (8.13), we obtain the decomposition

$$\begin{aligned}
\frac{1}{\epsilon} \left\langle\!\!\left\langle \zeta\, \frac{q_\epsilon}{N_\epsilon^2} \right\rangle\!\!\right\rangle = & \left\langle\!\!\left\langle \zeta\, \frac{\epsilon\, g_{\epsilon 1}^2 q_\epsilon}{N_{\epsilon 1} N_\epsilon} \left(1 + \frac{1}{N_{\epsilon 1}}\right) \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta\, \frac{\epsilon^2 (g_{\epsilon 1} + g_\epsilon)\, q_\epsilon^2}{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_\epsilon} \left(\frac{1}{N'_{\epsilon 1} N'_{\epsilon}} + \frac{1}{N_{\epsilon 1} N_\epsilon}\right) \right\rangle\!\!\right\rangle \\
& - \left\langle\!\!\left\langle \zeta'\, \frac{\epsilon\, g'_{\epsilon 1} g'_{\epsilon}\, q_\epsilon}{N'_{\epsilon 1} N'_{\epsilon} N_{\epsilon 1} N_\epsilon}\, J_\epsilon \left(\frac{1}{N'_{\epsilon 1} N'_{\epsilon}} + \frac{1}{N_{\epsilon 1} N_\epsilon}\right) \right\rangle\!\!\right\rangle,
\end{aligned} \tag{8.14}$$

where J_ϵ is given by (8.6). Because the factors in parentheses above are each bounded by 2, by arguing as was done for the case $n = 1$, the result for the case $n = 2$ will also follow upon establishing (8.11) and (8.12). The proof of Proposition 8.1 will therefore be complete upon proving Lemmas 8.2 and 8.3. $\square$

8.2. Conservation defect lemmas

The proofs of Lemmas 8.2 and 8.3 use the compactness hypothesis (8.1) of Proposition 8.1 through the following lemma.

Lemma 8.1. *Let b, g_ϵ, and N_ϵ be as in Proposition 8.1. Let $s^* \in [1,\infty)$ be given by $\frac{1}{s} + \frac{1}{s^*} = 1$ where $s \in (1,\infty]$ is from the assumed bound (3.15) on b. Then for every $w \in L^{s^*}(aM\mathrm{d}v)$ one has that*

$$\frac{w(v_1)\, g_\epsilon^2}{\sqrt{N_\epsilon}} \quad \textit{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x)). \tag{8.15}$$

Proof. By hypothesis (8.1) we can pass to a subsequence such that

$$\frac{g_\epsilon^2}{\sqrt{N_\epsilon}} \quad \text{is convergent in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(aM\mathrm{d}v\,\mathrm{d}x)).$$

It is then straightforward to show that the corresponding subsequence

$$\frac{w(v_1)\, g_\epsilon^2}{\sqrt{N_\epsilon}} \quad \text{is convergent in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x)).$$

Indeed, one simply uses the fact that for every $Y \in L^\infty_{loc}(\mathrm{d}t;\, L^\infty(\mathrm{d}\mu\,\mathrm{d}x))$

$$\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} Y\, \frac{w_1\, g_\epsilon^2}{\sqrt{N_\epsilon}}\,\mathrm{d}\mu\,\mathrm{d}x = \int_{\mathbb{R}^D} y\, \frac{g_\epsilon^2}{\sqrt{N_\epsilon}}\, aM\mathrm{d}v,$$

where $y \in L^\infty_{loc}(\mathrm{d}t;\, L^\infty(aM\mathrm{d}v\,\mathrm{d}x))$ is given almost everywhere by

$$y(v,x,t) = \frac{1}{a(v)} \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} Y(\omega, v_1, v, x, t)\, w(v_1)\, b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1\mathrm{d}v_1.$$

The L^∞ bound on y follows because for almost every (v,x,t) one sees from the Hölder inequality and from the bound (3.15) on b that

$$\begin{aligned}
|y(v,x,t)| &\leqq \|Y\|_\infty \int_{\mathbb{R}^D} w(v_1)\, \frac{\bar{b}(v_1 - v)}{a(v_1)a(v)}\, a(v_1)\, M_1\mathrm{d}v_1 \\
&\leqq \|Y\|_\infty \left(\int_{\mathbb{R}^D} |w(v_1)|^{s^*} a(v_1)\, M_1\mathrm{d}v_1 \right)^{\frac{1}{s^*}} \\
&\qquad \times \left(\int_{\mathbb{R}^D} \left| \frac{\bar{b}(v_1 - v)}{a(v_1)a(v)} \right|^s a(v_1)\, M_1\mathrm{d}v_1 \right)^{\frac{1}{s}} \\
&\leqq \|Y\|_\infty \|w\|_{L^{s^*}(aM\mathrm{d}v)} C_b.
\end{aligned}$$

The compactness assertion (8.15) then follows. □

The proofs of Lemmas 8.2 and 8.3 also crucially use the fact that the entropy inequality (6.1) implies that the dissipation rate R satisfies the bound

$$\frac{1}{\epsilon^4}\int_0^\infty R(G_\epsilon)\,\mathrm{d}t \leqq C^{in}.$$

More specifically, following [3], these proofs use the definitions of R and q_ϵ, given by (2.20) and (6.3) respectively, to re-express this bound as

$$\frac{1}{\epsilon^4}\int_0^\infty\int_{\mathbb{T}^D}\left\langle\!\!\left\langle \frac{1}{4}\,r\left(\frac{\epsilon^2 q_\epsilon}{G_{\epsilon 1}G_\epsilon}\right)G_{\epsilon 1}G_\epsilon\right\rangle\!\!\right\rangle \mathrm{d}x\,\mathrm{d}t \leqq C^{in}, \tag{8.16}$$

where the function r is defined over $z > -1$ by $r(z) = z\log(1+z)$ and is strictly convex.

The proofs of Lemmas 8.2 and 8.3 are each based on a delicate use of the classical Young inequality satisfied by r and its Legendre dual, r^*—namely, the inequality

$$pz \leqq r^*(p) + r(z), \quad \text{for every } p \in \mathbb{R} \text{ and } z > -1.$$

Upon setting $p = \epsilon^2 y/\alpha$ and $z = \epsilon^2|q_\epsilon|/(G_{\epsilon 1}G_\epsilon)$ above, and noticing that $r(|z|) \leqq r(z)$ for every $z > -1$, for every positive α and y one obtains

$$y|q_\epsilon| \leqq \frac{\alpha}{\epsilon^4}\,r^*\left(\frac{\epsilon^2 y}{\alpha}\right)G_{\epsilon 1}G_\epsilon + \frac{\alpha}{\epsilon^4}\,r\left(\frac{\epsilon^2 q_\epsilon}{G_{\epsilon 1}G_\epsilon}\right)G_{\epsilon 1}G_\epsilon. \tag{8.17}$$

This inequality is the starting point for the proofs of Lemmas 8.2 and 8.3. These proofs also use the facts, recalled from [3], that r^* is superquadratic in the sense

$$r^*(\lambda p) \leqq \lambda^2 r^*(p), \quad \text{for every } p > 0 \text{ and } \lambda \in [0,1], \tag{8.18}$$

and that r^* has the exponential asymptotics $r^*(p) \sim \exp(p)$ as $p \to \infty$.

Lemma 8.2. *Let b, g_ϵ, q_ϵ, and N_ϵ be as in Proposition 8.1. Then*

$$\sigma\,\frac{\epsilon\, g_{\epsilon 1}^2\, q_\epsilon}{N_{\epsilon 1}N_\epsilon} \longrightarrow 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \text{ as } \epsilon \to 0.$$

Proof. For the proof of this lemma we use inequality (8.17) with

$$y = \frac{\sigma}{4s^*}\,\frac{\epsilon\, g_{\epsilon 1}^2}{N_{\epsilon 1}N_\epsilon}.$$

where $s^* \in [1,\infty)$ is as in Lemma 8.1. We then use the superquadratic property (8.18) with

$$\lambda = \frac{\epsilon^3 g_{\epsilon 1}^2}{\alpha\, N_{\epsilon 1}N_\epsilon} \quad \text{and} \quad p = \frac{\sigma}{4s^*},$$

where we note that $\lambda \leqq 1$ whenever $\epsilon \leqq \alpha$. This leads to

$$\frac{\sigma}{4s^*}\,\frac{\epsilon\, g_{\epsilon 1}^2\,|q_\epsilon|}{N_{\epsilon 1}N_\epsilon} \leqq \frac{1}{\alpha}\,\frac{\epsilon^2 g_{\epsilon 1}^4}{N_{\epsilon 1}^2 N_\epsilon^2}\,r^*\left(\frac{\sigma}{4s^*}\right)G_{\epsilon 1}G_\epsilon + \frac{\alpha}{\epsilon^4}\,r\left(\frac{\epsilon^2 q_\epsilon}{G_{\epsilon 1}G_\epsilon}\right)G_{\epsilon 1}G_\epsilon. \tag{8.19}$$

Because $G_{\epsilon 1} G_\epsilon \leqq 2\sqrt{N_{\epsilon 1} N_\epsilon}$, we can bound the first term on the right-hand side above by

$$\frac{2\,\epsilon^2 g_{\epsilon 1}^2}{\alpha\, N_{\epsilon 1}} \frac{g_{\epsilon 1}^2}{\sqrt{N_{\epsilon 1}}} r^*\left(\frac{\sigma}{4s^*}\right).$$

The first factor above is bounded by $2/\alpha$ and tends to zero almost everywhere as $\epsilon \to 0$. Because $r^*(p) \sim \exp(p)$ as $p \to \infty$ one sees that $r^*(\sigma/4s^*) \in L^{s^*}(aM\mathrm{d}v)$. Hence, we can thereby apply Lemma 8.1 with $w = r^*(\sigma/4s^*)$ to see that the remaining factors satisfy

$$\frac{g_{\epsilon 1}^2}{\sqrt{N_{\epsilon 1}}} r^*\left(\frac{\sigma}{4s^*}\right) \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x)).$$

The first term on the right-hand side of (8.19) thereby converges to zero in $L^1_{loc}(\mathrm{d}t;$ $L^1(\mathrm{d}\mu\,\mathrm{d}x))$ as $\epsilon \to 0$ by the Product Limit Theorem of [3]. On the other hand, the dissipation bound (8.16) implies that the integral of the second term on the right-hand side of (8.19) is bounded by $4\alpha\, C^{in}$. The Lemma therefore follows from the arbitrariness of α. □

Lemma 8.3. *Let b, g_ϵ, q_ϵ, and N_ϵ be as in Proposition 8.1. Then*

$$\sigma \frac{\epsilon\, g_{\epsilon 1} g_\epsilon\, q_\epsilon}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \longrightarrow 0 \quad \textit{in } L^1_{loc}(\mathrm{d}t;\, L^1(\mathrm{d}\mu\,\mathrm{d}x)) \textit{ as } \epsilon \to 0.$$

Proof. For the proof of this lemma we use inequality (8.17) with

$$y = \frac{\sigma'}{4s^*} \frac{\epsilon\, |g'_{\epsilon 1} g'_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}}.$$

where $s^* \in [1, \infty)$ is as in Lemma 8.1. We then use the superquadratic property (8.18) with

$$\lambda = \frac{\epsilon^3 |g'_{\epsilon 1} g'_\epsilon|}{\alpha \sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}}, \quad \text{and} \quad p = \frac{\sigma'}{4s^*}.$$

where we note that $\lambda \leqq 1$ whenever $\epsilon \leqq \alpha$. This leads to

$$\begin{aligned} \frac{\sigma'}{4s^*} \frac{\epsilon\, |g'_{\epsilon 1} g'_\epsilon|\, |q_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} &\leqq \frac{1}{\alpha} \frac{\epsilon^2 g'^2_{\epsilon 1} g'^2_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} r^*\left(\frac{\sigma'}{4s^*}\right) G_{\epsilon 1} G_\epsilon \\ &\quad + \frac{\alpha}{\epsilon^4} r\left(\frac{\epsilon^2 q_\epsilon}{G_{\epsilon 1} G_\epsilon}\right) G_{\epsilon 1} G_\epsilon. \end{aligned} \tag{8.20}$$

Because $G_{\epsilon 1} G_\epsilon \leqq 2\sqrt{N_{\epsilon 1} N_\epsilon}$, we can bound the first term on the right-hand side above by

$$\frac{2\,\epsilon^2 g'^2_\epsilon}{\alpha\, N'_\epsilon} \frac{g'^2_{\epsilon 1}}{N'_{\epsilon 1}} r^*\left(\frac{\sigma'}{4s^*}\right)$$

The first factor above is bounded by $2/\alpha$ and tends to zero almost everywhere as $\epsilon \to 0$. We can again apply Lemma 8.1 with $w = r^*(\sigma/4s^*)$ to see that the remaining factors satisfy

$$\frac{g'^2_{\epsilon 1}}{\sqrt{N'_{\epsilon 1}}} r^* \left(\frac{\sigma'}{4s^*} \right) \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t;\, w\text{-}L^1(\mathrm{d}\mu\, \mathrm{d}x)).$$

The first term on the right-hand side of (8.20) thereby converges to zero in $L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\, \mathrm{d}x))$ as $\epsilon \to 0$ by the Product Limit Theorem of [3]. On the other hand, the dissipation bound (8.16) implies that the integral of the second term on the right-hand side of (8.20) is bounded by $4\alpha\, C^{in}$. The Lemma therefore follows from the arbitrariness of α. □

9. Bilinear estimates

Key tools in our work are the following two lemmas dedicated to controlling terms that are quadratic in $\tilde{g}_\epsilon$. The first lemma provides a direct L^1 bound on such terms.

Lemma 9.1. *Let the collision kernel b satisfy assumption (3.15) for some $C_b < \infty$ and $s \in (1, \infty]$. Let $p = 2 + \frac{1}{s-1}$, so $p = 2$ when $s = \infty$.*

Let $\Xi = \Xi(\omega, v_1, v)$ be in $L^p(\mathrm{d}\mu)$ and let $\tilde{g}$ and $\tilde{h}$ be in $L^2(aM\mathrm{d}v)$. Then $\Xi\, \tilde{g}_1\, \tilde{h}$ is in $L^1(\mathrm{d}\mu)$ and satisfies the L^1 bound

$$\langle\!\langle |\Xi\, \tilde{g}_1\, \tilde{h}| \rangle\!\rangle \leqq C_b^{\frac{1}{p^*}} \langle\!\langle |\Xi|^p \rangle\!\rangle^{\frac{1}{p}} \langle a\, \tilde{g}^2 \rangle^{\frac{1}{2}}\, \langle a\, \tilde{h}^2 \rangle^{\frac{1}{2}}, \tag{9.1}$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\tilde{g}_1$ denotes $\tilde{g}(v_1)$.

Proof. The Hölder inequality yields

$$\langle\!\langle \left|\Xi\, \tilde{g}_1\, \tilde{h}\right| \rangle\!\rangle \leqq \langle\!\langle |\Xi|^p \rangle\!\rangle^{\frac{1}{p}} \langle\!\langle \left|\tilde{g}_1\, \tilde{h}\right|^{p^*} \rangle\!\rangle^{\frac{1}{p^*}}. \tag{9.2}$$

In order to bound the last factor on the left-hand side above, we first observe that

$$\begin{aligned} \langle\!\langle \left|\tilde{g}_1\, \tilde{h}\right|^{p^*} \rangle\!\rangle &= \iint_{\mathbb{R}^D \times \mathbb{R}^D} \left|\tilde{g}_1\, \tilde{h}\right|^{p^*} \bar{b}(v_1 - v)\, M_1 \mathrm{d}v_1\, M\mathrm{d}v \\ &= \iint_{\mathbb{R}^D \times \mathbb{R}^D} K^-(v_1, v)\, |\tilde{g}_1|^{p^*} |\tilde{h}|^{p^*}\, a_1 M_1 \mathrm{d}v_1\, aM\mathrm{d}v \\ &= \left\langle a \left(\mathcal{K}^- |\tilde{g}|^{p^*}\right) |\tilde{h}|^{p^*} \right\rangle, \end{aligned} \tag{9.3}$$

where the integral operator $\mathcal{K}^-$ and its kernel K^- are given by (3.17).

Next, let $r = \frac{2}{p^*} \in (1, 2)$ and $\frac{1}{r} + \frac{1}{r^*} = 1$. Observe that because $\frac{1}{r} + \frac{1}{s} = 1 + \frac{1}{r^*}$, by (3.23) the operator $\mathcal{K}^- : L^r(aM\mathrm{d}v) \to L^{r^*}(aM\mathrm{d}v)$ is bounded with $\|\mathcal{K}^-\| \leqq C_b$. Use this fact after another application of the Hölder inequality to find

$$\begin{aligned} \left\langle a \left(\mathcal{K}^- |\tilde{g}|^{p^*}\right) |\tilde{h}|^{p^*} \right\rangle = \left\langle a \left(\mathcal{K}^- |\tilde{g}|^{\frac{2}{r}}\right) |\tilde{h}|^{\frac{2}{r}} \right\rangle &\leqq \left\| \mathcal{K}^- |\tilde{g}|^{\frac{2}{r}} \right\|_{L^{r^*}(aM\mathrm{d}v)} \left\| |\tilde{h}|^{\frac{2}{r}} \right\|_{L^r(aM\mathrm{d}v)} \\ &\leqq C_b \langle a\, \tilde{g}^2 \rangle^{\frac{1}{r}}\, \langle a\, \tilde{h}^2 \rangle^{\frac{1}{r}}. \end{aligned}$$

When the above bound is combined with (9.3) we obtain the key bound

$$\langle\!\langle |\tilde{g}_1\,\tilde{h}|^{p^*} \rangle\!\rangle \leqq C_b \langle a\,\tilde{g}^2\rangle^{\frac{p^*}{2}} \langle a\,\tilde{h}^2\rangle^{\frac{p^*}{2}} . \tag{9.4}$$

The L^1 bound (9.1) then follows when the above inequality is applied to the last factor on the left-hand side of (9.2). □

The next lemma provides w-L^1 compactness of certain terms quadratic in fluctuations, provided those fluctuations satisfy a weaker compactness hypothesis.

Lemma 9.2. *Let the collision kernel b satisfy assumption (3.15) for some $C_b < \infty$ and $s \in (1, \infty]$. Let $p = 2 + \frac{1}{s-1}$, so $p = 2$ when $s = \infty$.*

Let $\Xi = \Xi(\omega, v_1, v)$ be in $L^p(\mathrm{d}\mu)$ and let $\tilde{g}_\epsilon = \tilde{g}_\epsilon(v, x, t)$ and $\tilde{h}_\epsilon = \tilde{h}_\epsilon(v, x, t)$ be families that are bounded in $L^2_{loc}(\mathrm{d}t; L^2(aM\mathrm{d}v\,\mathrm{d}x))$. If the family

$$\langle a\tilde{g}_\epsilon^2\rangle \quad \textit{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)), \tag{9.5}$$

then the family

$$\Xi\,\tilde{g}_{\epsilon 1}\,\tilde{h}_\epsilon \quad \textit{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x)). \tag{9.6}$$

Here $\tilde{g}_{\epsilon 1}$ denotes $\tilde{g}_\epsilon(v_1, x, t)$.

Proof. To establish the w-L^1 compactness assertion (9.6) we must show that the family $\Xi\,\tilde{g}_{\epsilon 1}\,\tilde{h}_\epsilon$ is equi-integrable. Begin with the classical Young's inequality, which for every $\eta > 0$ yields

$$\left|\Xi\,\tilde{g}_{\epsilon 1}\,\tilde{h}_\epsilon\right| \leqq \frac{\eta^p}{p}\,|\Xi|^p + \frac{1}{p^*\,\eta^{p^*}}\,\left|\tilde{g}_{\epsilon 1}\tilde{h}_\epsilon\right|^{p^*}.$$

Now let $\alpha > 0$ be arbitrary and set $\eta = \langle \tilde{g}_\epsilon^2\rangle^{\frac{1}{p}}/\alpha$ above to obtain

$$\left|\Xi\,\tilde{g}_{\epsilon 1}\,\tilde{h}_\epsilon\right| \leqq \frac{1}{p\,\alpha^p}\,|\Xi|^p\,\langle a\,\tilde{g}_\epsilon^2\rangle + \frac{\alpha^{p^*}}{p^*}\,\frac{\left|\tilde{g}_{\epsilon 1}\tilde{h}_\epsilon\right|^{p^*}}{\langle \tilde{g}_\epsilon^2\rangle^{\frac{p^*}{p}}}. \tag{9.7}$$

The last term on the right-hand side above is a bounded family in $L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x))$ because by the key bound (9.4) of Lemma 9.1 one has

$$\begin{aligned}\frac{\langle\!\langle \left|\tilde{g}_{\epsilon 1}\tilde{h}_\epsilon\right|^{p^*} \rangle\!\rangle}{\langle \tilde{g}_\epsilon^2\rangle^{\frac{p^*}{p}}} &\leqq C_b \langle a\,\tilde{g}_\epsilon^2\rangle^{\frac{p^*}{2}-\frac{p^*}{p}} \langle a\,\tilde{h}_\epsilon^2\rangle^{\frac{p^*}{2}} \\ &= C_b \langle a\,\tilde{g}_\epsilon^2\rangle^{\frac{1}{r^*}} \langle a\,\tilde{h}_\epsilon^2\rangle^{\frac{1}{r}} \leqq C_b\left[\frac{1}{r^*}\langle a\,\tilde{g}_\epsilon^2\rangle + \frac{1}{r}\langle a\,\tilde{h}_\epsilon^2\rangle\right].\end{aligned} \tag{9.8}$$

Because $\tilde{g}_\epsilon$ and $\tilde{h}_\epsilon$ are bounded families in $L^2_{loc}(\mathrm{d}t; L^2(aM\mathrm{d}v\,\mathrm{d}x))$, the last expression above is clearly bounded in $L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x))$.

Next, we integrate inequality (9.7) over an arbitrary measurable set $\Omega \subset \mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{T}^D \times [0, T]$ and use (9.8) to obtain

$$\iiint_\Omega \left| \Xi \, \tilde{g}_{\epsilon 1} \tilde{g}_\epsilon \right| \mathrm{d}\mu \, \mathrm{d}x \, \mathrm{d}t \leqq \frac{1}{p\,\alpha^p} \iiint_\Omega |\Xi|^p \, \langle a \, \tilde{g}_\epsilon^2 \rangle \, \mathrm{d}\mu \, \mathrm{d}x \, \mathrm{d}t + \frac{\alpha^{p^*}}{p^*} \, C_b \int_0^T \int_{\mathbb{T}^D} \left[\frac{1}{r^*} \langle a \, \tilde{g}_\epsilon^2 \rangle + \frac{1}{r} \langle a \, \tilde{h}_\epsilon^2 \rangle \right] \mathrm{d}x \, \mathrm{d}t.$$

We now use this inequality to argue that the left-hand side above can be made arbitrarily small uniformly in ϵ by picking the measure of Ω sufficiently small. To begin, because $\tilde{g}_\epsilon$ and $\tilde{h}_\epsilon$ are bounded families in $L^2_{loc}(\mathrm{d}t; L^2(aM\mathrm{d}v\,\mathrm{d}x))$, the terms on the second line above can be made arbitrarily small uniformly in ϵ by a suitable choice of α. Next, by hypothesis (9.5), $\langle a \, \tilde{g}_\epsilon^2 \rangle$ is equi-integrable with respect to $\mathrm{d}x \, \mathrm{d}t$ over $\mathbb{T}^D \times [0, T]$ while, because $\Xi \in L^p(\mathrm{d}\mu)$, $|\Xi|^p$ is integrable with respect to $\mathrm{d}\mu$ over $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, one thereby sees that $|\Xi|^p \langle a \, \tilde{g}_\epsilon^2 \rangle$ is equi-integrable with respect to $\mathrm{d}\mu \, \mathrm{d}x \, \mathrm{d}t$ over $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D \times \mathbb{T}^D \times [0, T]$. The first term on the right-hand side above can therefore be made arbitrarily small uniformly in ϵ by picking the measure of Ω sufficiently small. We conclude that the family $\Xi \, \tilde{g}_{\epsilon 1} \, \tilde{h}_\epsilon$ is equi-integrable with respect to $\mathrm{d}\mu \, \mathrm{d}x \, \mathrm{d}t$, whereby the w-L^1 compactness assertion (9.6) is established. □

10. Removal of the flux remainders

The flux remainders have the form

$$\langle\!\langle \hat{\xi} \, T_\epsilon \rangle\!\rangle,$$

where $\hat{\xi}$ is an entry of either $\widehat{A}$ or $\widehat{B}$ and where T_ϵ is defined by

$$T_\epsilon = \frac{q_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} - \frac{1}{\epsilon} \left(\tilde{g}'_{\epsilon 1} + \tilde{g}'_\epsilon - \tilde{g}_{\epsilon 1} - \tilde{g}_\epsilon \right) - \left(\tilde{g}'_{\epsilon 1} \tilde{g}'_\epsilon - \tilde{g}_{\epsilon 1} \tilde{g}_\epsilon \right). \quad (10.1)$$

In order to establish momentum and energy conservation laws from the scaled Boltzmann equation we must show that these remainders vanish as $\epsilon \to 0$. This is done with the following proposition.

Proposition 10.1. *Let b be a collision kernel that satisfies the assumptions of Sect. 3. Let $s \in (1, \infty]$ be as in the assumed bound (3.15) on b. Let $p = 2 + 1/(s-1)$, so that $p = 2$ when $s = \infty$.*

Let $G_\epsilon \geqq 0$ be a family of functions in $C([0,\infty); w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x))$ that satisfies the entropy bound (6.1). Let g_ϵ and q_ϵ be given by (5.1) and (6.3). Let $N_\epsilon = 1 + \epsilon^2 g_\epsilon^2$, $\tilde{g}_\epsilon = g_\epsilon / N_\epsilon$, and T_ϵ be given by (10.1). Assume that the family g_ϵ satisfies

$$\left\langle a \, \frac{g_\epsilon^2}{N_\epsilon} \right\rangle \text{ is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)). \quad (10.2)$$

Then for every $\Xi \in L^p(\mathrm{d}\mu)$ one has that T_ϵ given by (10.1) satisfies

$$\Xi \, T_\epsilon \longrightarrow 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu\,\mathrm{d}x)) \quad \text{as } \epsilon \to 0. \quad (10.3)$$

Proof. The key to the argument is to find a decomposition of T_ϵ for which each component can be bounded by one of the sequences

$$\frac{|g_{\epsilon 1} g_\epsilon|}{\sqrt{N_{\epsilon 1} N_\epsilon}}, \quad \frac{|g'_{\epsilon 1} g'_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon}}, \quad \frac{|q_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}}, \tag{10.4}$$

times a bounded sequence that vanishes almost everywhere as $\epsilon \to 0$. Assertion (10.3) will then follow from the Product Limit Theorem of [3] upon showing that $|\Xi|$ times each of the sequences in (10.4) is relatively compact in $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))$. For the first two sequences in (10.4) this relative compactness follows from assertion (9.6) of Lemma 9.2 and the compactness hypothesis (10.2). For the last sequence in (10.4) this relative compactness follows directly from the fact that it is bounded in $L^2(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t)$ by the elementary inequality $n'_{\epsilon 1} n'_\epsilon n_{\epsilon 1} n_\epsilon \leqq 2\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}$ and the fact that Lemma 10.1 of [12] implies that the sequence

$$\frac{q_\epsilon}{n'_{\epsilon 1} n'_\epsilon n_{\epsilon 1} n_\epsilon} \quad \text{is bounded in } L^2(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t).$$

We begin by decomposing T_ϵ given by (10.1) as

$$\begin{aligned} T_\epsilon &= \frac{1}{\epsilon^2} \frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} - \frac{1}{\epsilon}\left(\frac{g'_{\epsilon 1}}{N'_{\epsilon 1}} + \frac{g'_\epsilon}{N'_\epsilon} - \frac{g_{\epsilon 1}}{N_{\epsilon 1}} - \frac{g_\epsilon}{N_\epsilon} \right) \\ &\quad - \left(\frac{g'_{\epsilon 1}}{N'_{\epsilon 1}} \frac{g'_\epsilon}{N'_\epsilon} - \frac{g_{\epsilon 1}}{N_{\epsilon 1}} \frac{g_\epsilon}{N_\epsilon} \right) \\ &= T_{1\epsilon} - T'_{1\epsilon} + T_{2\epsilon} - T'_{2\epsilon} + T_{3\epsilon} - T'_{3\epsilon}, \end{aligned} \tag{10.5}$$

where $T_{1\epsilon}$, $T_{2\epsilon}$, and $T_{3\epsilon}$ are defined by

$$\begin{aligned} T_{1\epsilon} &= \frac{1}{\epsilon}\left(\frac{g_{\epsilon 1}}{N_{\epsilon 1}} + \frac{g_\epsilon}{N_\epsilon} - \frac{g_{\epsilon 1} + g_\epsilon}{N_{\epsilon 1} N_\epsilon} \right), \\ T_{2\epsilon} &= \frac{1}{\epsilon}\left(\frac{g_{\epsilon 1} + g_\epsilon}{N_{\epsilon 1} N_\epsilon} - \frac{g_{\epsilon 1} + g_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \right), \\ T_{3\epsilon} &= \left(\frac{g_{\epsilon 1} g_\epsilon}{N_{\epsilon 1} N_\epsilon} - \frac{g_{\epsilon 1} g_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \right), \end{aligned}$$

and where $T'_{1\epsilon}$, $T'_{2\epsilon}$, and $T'_{3\epsilon}$ are defined by simply exchanging the roles of the primed and unprimed quantities in the respective definitions of $T_{1\epsilon}$, $T_{2\epsilon}$, and $T_{3\epsilon}$.

It is easy to obtain the desired bounds for $T_{1\epsilon}$, $T'_{1\epsilon}$, $T_{3\epsilon}$, and $T'_{3\epsilon}$. For $T_{1\epsilon}$ we have

$$|T_{1\epsilon}| = \frac{\epsilon\,|g_{\epsilon 1} + g_\epsilon|\,|g_{\epsilon 1} g_\epsilon|}{N_{\epsilon 1} N_\epsilon} = \frac{|g_{\epsilon 1} g_\epsilon|}{\sqrt{N_{\epsilon 1} N_\epsilon}} \, \frac{\epsilon\,|g_{\epsilon 1} + g_\epsilon|}{\sqrt{N_{\epsilon 1} N_\epsilon}}.$$

The last factor above is a sequence that is bounded by 2 and that vanishes almost everywhere as $\epsilon \to 0$. For $T_{3\epsilon}$ we have

$$|T_{3\epsilon}| = \frac{|g_{\epsilon 1} g_\epsilon|}{N_{\epsilon 1} N_\epsilon} \left(1 - \frac{1}{N'_{\epsilon 1} N'_\epsilon} \right).$$

The last factor above is a sequence that is bounded by 1 and that vanishes almost everywhere as $\epsilon \to 0$. The bounds for $T'_{1\epsilon}$ and $T'_{3\epsilon}$ are obtained by simply exchanging the roles of the primed and unprimed quantities in the respective bounds of $T_{1\epsilon}$ and $T_{3\epsilon}$.

To treat $T_{2\epsilon}$ and $T'_{2\epsilon}$ we need the further decompositions

$$T_{2\epsilon} = \frac{\epsilon\,(g_{\epsilon 1} + g_\epsilon)\left(g'^2_{\epsilon 1} + g'^2_\epsilon + \epsilon^2 (g'_{\epsilon 1} g'_\epsilon)^2\right)}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} = T_{4\epsilon} - T_{5\epsilon} + T_{6\epsilon}, \quad (10.6)$$

$$T'_{2\epsilon} = \frac{\epsilon\,(g'_{\epsilon 1} + g'_\epsilon)\left(g^2_{\epsilon 1} + g^2_\epsilon + \epsilon^2 (g_{\epsilon 1} g_\epsilon)^2\right)}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} = T'_{4\epsilon} - T'_{5\epsilon} + T'_{6\epsilon},$$

where $T_{4\epsilon}$, $T_{5\epsilon}$, and $T_{6\epsilon}$ are defined by

$$T_{4\epsilon} = \frac{\epsilon\,(g_{\epsilon 1} + g_\epsilon)\,(g'_{\epsilon 1} + g'_\epsilon)^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}, \quad T_{5\epsilon} = \frac{\epsilon\,(g_{\epsilon 1} + g_\epsilon)\,(2 g'_{\epsilon 1} g'_\epsilon)}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon},$$

$$T_{6\epsilon} = \frac{\epsilon^3 (g_{\epsilon 1} + g_\epsilon)\,(g'_{\epsilon 1} g'_\epsilon)^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon},$$

and where $T'_{4\epsilon}$, $T'_{5\epsilon}$, and $T'_{6\epsilon}$ are defined by simply exchanging the roles of the primed and unprimed quantities in the respective definitions of $T_{4\epsilon}$, $T_{5\epsilon}$, and $T_{6\epsilon}$.

It is easy to obtain the desired bounds for $T_{5\epsilon}$, $T'_{5\epsilon}$, $T_{6\epsilon}$, and $T'_{6\epsilon}$. For $T_{5\epsilon}$ we have

$$|T_{5\epsilon}| = \frac{\epsilon\,|g_{\epsilon 1} + g_\epsilon|\,|2 g'_{\epsilon 1} g'_\epsilon|}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} = \frac{|g'_{\epsilon 1} g'_\epsilon|}{N'_{\epsilon 1} N'_\epsilon}\,\frac{\epsilon\, 2\,|g_{\epsilon 1} + g_\epsilon|}{N_{\epsilon 1} N_\epsilon}.$$

The last factor above is a sequence that is bounded by 2 and that vanishes almost everywhere as $\epsilon \to 0$. For $T_{6\epsilon}$ we have

$$|T_{6\epsilon}| = \frac{|g'_{\epsilon 1} g'_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon}} \left(\frac{|g'_{\epsilon 1} g'_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon}}\,\frac{\epsilon\,|g_{\epsilon 1} + g_\epsilon|}{N_{\epsilon 1} N_\epsilon} \right).$$

The factor in parenthesis above is a sequence that is bounded by 1 and that vanishes almost everywhere as $\epsilon \to 0$. The bounds for $T'_{5\epsilon}$ and $T'_{6\epsilon}$ are obtained by simply exchanging the roles of the primed and unprimed quantities in the respective bounds of $T_{5\epsilon}$ and $T_{6\epsilon}$.

The trick now is to *not* treat $T_{4\epsilon}$ and $T'_{4\epsilon}$ separately. Rather, we use the decomposition

$$T_{4\epsilon} - T'_{4\epsilon} = \frac{\epsilon\,(g_{\epsilon 1} + g_\epsilon)\,(g'_{\epsilon 1} + g'_\epsilon)\,(g'_{\epsilon 1} + g'_\epsilon - g_{\epsilon 1} - g_\epsilon)}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} = T_{7\epsilon} + T_{8\epsilon} - T'_{8\epsilon}, \quad (10.7)$$

where $T_{7\epsilon}$, and $T_{8\epsilon}$ are defined by

$$T_{7\epsilon} = \frac{\epsilon^2 (g_{\epsilon 1} + g_\epsilon)\,(g'_{\epsilon 1} + g'_\epsilon)\,q_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}, \quad T_{8\epsilon} = \frac{\epsilon^2 (g_{\epsilon 1} + g_\epsilon)\,(g'_{\epsilon 1} + g'_\epsilon)\,g_{\epsilon 1} g_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon},$$

and where $T'_{8\epsilon}$ is defined by simply exchanging the roles of the primed and unprimed quantities in the definition of $T_{8\epsilon}$.

Finally, it is easy to obtain the desired bounds for $T_{7\epsilon}$, $T_{8\epsilon}$, and $T'_{8\epsilon}$. For $T_{7\epsilon}$ we have

$$|T_{7\epsilon}| = \frac{|q_\epsilon|}{\sqrt{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}} \, \frac{\epsilon^2 |g_{\epsilon 1} + g_\epsilon| \, |g'_{\epsilon 1} + g'_\epsilon|}{\sqrt{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}}.$$

The last factor above is a sequence that is bounded by 4 and that vanishes almost everywhere as $\epsilon \to 0$. For $T_{8\epsilon}$ we have

$$|T_{8\epsilon}| = \frac{|g_{\epsilon 1}g_\epsilon|}{\sqrt{N_{\epsilon 1}N_\epsilon}} \left(\frac{\epsilon \, |g_{\epsilon 1} + g_\epsilon|}{\sqrt{N_{\epsilon 1}N_\epsilon}} \, \frac{\epsilon \, |g'_{\epsilon 1} + g'_\epsilon|}{N'_{\epsilon 1}N'_\epsilon} \right).$$

The factor in parenthesis above is a sequence that is bounded by 2 and that vanishes almost everywhere as $\epsilon \to 0$. The bound for $T'_{8\epsilon}$ is obtained by simply exchanging the roles of the primed and unprimed quantities in the bound of $T_{8\epsilon}$.

We therefore obtain from (10.5), (10.6), and (10.7) the decomposition

$$T_\epsilon = T_{1\epsilon} - T'_{1\epsilon} + T_{3\epsilon} - T'_{3\epsilon} - T_{5\epsilon} + T'_{5\epsilon} + T_{6\epsilon} - T'_{6\epsilon} + T_{7\epsilon} + T_{8\epsilon} - T'_{8\epsilon},$$

with the desired bounds on each component. This proves the Proposition. □

11. Quadratic limits

In order to establish our main result, Theorem 5.1, we need to pass to the limit in certain of quadratic terms containing $\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon$ and $\tilde{u}_\epsilon \, \tilde{\theta}_\epsilon$. Recall that we have the weak limits

$$\left.\begin{aligned} \tilde{u}_\epsilon &\to u \\ \tilde{\theta}_\epsilon &\to \theta \end{aligned}\right\} \quad \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x)) \text{ as } \epsilon \to 0. \tag{11.1}$$

These limits have to be strengthened in order to pass to the limit in any quadratic term containing either $\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon$ or $\tilde{u}_\epsilon \, \tilde{\theta}_\epsilon$. We follow [29], which adapted to the kinetic setting an idea introduced in [27] to pass to an incompressible Navier–Stokes–Fourier limit from the compressible Navier–Stokes–Fourier system. The main result of this section is the following.

Proposition 11.1. *Under the hypotheses of Theorem 5.1, we have the limits*

$$\left.\begin{aligned} \lim_{\epsilon\to 0} \Pi \nabla_x \cdot \left(\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon\right) &= \Pi \nabla_x \cdot (u \otimes u) \\ \lim_{\epsilon\to 0} \nabla_x \cdot \left(\tilde{\theta}_\epsilon \, \tilde{u}_\epsilon\right) &= \nabla_x \cdot (\theta \, u) \end{aligned}\right\} \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)), \tag{11.2}$$

where Π *is the Leray projection onto divergence-free vector fields in* $L^2(\mathrm{d}x; \mathbb{R}^D)$.

Proof. We employ a mollifier over the periodic space variable. Recall that $\mathbb{T}^D = \mathbb{R}^D/\mathbb{L}^D$, where $\mathbb{L}^D \subset \mathbb{R}^D$ is some D-dimensional lattice. Let $j \in C^\infty(\mathbb{R}^D)$ be such that $j \geqq 0$, $\int_{\mathbb{R}^D} j(x)\,\mathrm{d}x = 1$, and $j(x) = 0$ for $|x| > 1$. We then define $j^\delta \in C^\infty(\mathbb{T}^D)$ by

$$j^\delta(x) = \frac{1}{\delta^D} \sum_{l \in \mathbb{L}^D} j\left(\frac{x+l}{\delta}\right).$$

In this section all convolutions are taken only in the x variable.

Define $\tilde{u}_\epsilon^\delta = j^\delta * \tilde{u}_\epsilon$ and $\tilde{\theta}_\epsilon^\delta = j^\delta * \tilde{\theta}_\epsilon$. It will follow from Proposition 11.2 that

$$\left.\begin{array}{l} \lim_{\delta\to 0} \tilde{u}_\epsilon^\delta = \tilde{u}_\epsilon \\ \lim_{\delta\to 0} \tilde{\theta}_\epsilon^\delta = \tilde{\theta}_\epsilon \end{array}\right\} \quad \text{in } L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x)) \text{ uniformly in } \epsilon. \tag{11.3}$$

It will follow from Proposition 11.3 that for every $\delta > 0$

$$\left.\begin{array}{ll} \lim_{\epsilon\to 0} \Pi\nabla_x \cdot \left(\tilde{u}_\epsilon^\delta \otimes \tilde{u}_\epsilon^\delta\right) = \Pi\nabla_x \cdot \left(u^\delta \otimes u^\delta\right) \\ \lim_{\epsilon\to 0} \nabla_x \cdot \left(\tilde{\theta}_\epsilon^\delta\, \tilde{u}_\epsilon^\delta\right) \qquad = \nabla_x \cdot \left(\theta^\delta\, u^\delta\right) \end{array}\right\} \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)), \tag{11.4}$$

where $u^\delta = j^\delta * u$ and $\theta^\delta = j^\delta * \theta$.

By first using the uniformity of the L^2 limits in (11.3) to commute limits and then using the limits (11.4), we obtain

$$\begin{aligned} \lim_{\epsilon\to 0} \Pi\nabla_x \cdot \left(\tilde{u}_\epsilon \otimes \tilde{u}_\epsilon\right) &= \lim_{\epsilon\to 0}\lim_{\delta\to 0} \Pi\nabla_x \cdot \left(\tilde{u}_\epsilon^\delta \otimes \tilde{u}_\epsilon^\delta\right) = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \Pi\nabla_x \cdot \left(\tilde{u}_\epsilon^\delta \otimes \tilde{u}_\epsilon^\delta\right) \\ &= \lim_{\delta\to 0} \Pi\nabla_x \cdot \left(u^\delta \otimes u^\delta\right) = \Pi\nabla_x \cdot (u \otimes u) \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)), \end{aligned}$$

and

$$\begin{aligned} \lim_{\epsilon\to 0} \nabla_x \cdot \left(\tilde{\theta}_\epsilon\, \tilde{u}_\epsilon\right) &= \lim_{\epsilon\to 0}\lim_{\delta\to 0} \nabla_x \cdot \left(\tilde{\theta}_\epsilon^\delta\, \tilde{u}_\epsilon^\delta\right) = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \nabla_x \cdot \left(\tilde{\theta}_\epsilon^\delta\, \tilde{u}_\epsilon^\delta\right) \\ &= \lim_{\delta\to 0} \nabla_x \cdot \left(\theta^\delta\, u^\delta\right) = \nabla_x \cdot (\theta\, u) \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)). \end{aligned}$$

This proves assertion (11.2) and thereby establishes the proposition modulo the proofs of Propositions 11.2 and 11.3, which are given in the subsequent subsections. □

11.1. Uniformity of mollification limits

Now define $\tilde{g}_\epsilon^\delta = j^\delta * \tilde{g}_\epsilon$ and $g^\delta = j^\delta * g$. By basic properties of mollifiers we know that for every $\xi \in L^2(M\mathrm{d}v)$ one has the L^2 limits

$$\left.\begin{array}{l} \lim_{\delta\to 0}\langle \xi\, \tilde{g}_\epsilon^\delta\rangle = \langle \xi\, \tilde{g}_\epsilon\rangle \\ \lim_{\delta\to 0}\langle \xi\, g^\delta\rangle = \langle \xi\, g\rangle \end{array}\right\} \quad \text{in } L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x)). \tag{11.5}$$

The main result of this subsection is the following assertion that for certain ξ the top limit above is uniform in ϵ.

Proposition 11.2. *For every* $\zeta \in \mathrm{span}\{1, v_1, \ldots, v_D, |v|^2\}$ *one has*

$$\lim_{\delta\to 0}\langle \zeta\, \tilde g_\epsilon^\delta\rangle = \langle \zeta\, \tilde g_\epsilon\rangle \quad \text{in } L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x)) \text{ uniformly in } \epsilon. \tag{11.6}$$

Proof. We see from (6.2) and (6.4) that the sequences $\zeta\, \tilde g_\epsilon$ and $\zeta\, \Gamma'(G_\epsilon)\, \mathcal{Q}(G_\epsilon, G_\epsilon)$ are relatively compact in w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1(M\mathrm{d}v\,\mathrm{d}x))$. An application of the L^1-velocity averaging result of [13] to the renormalized Boltzmann equation (6.9) then implies that for every $T \in (0,\infty)$ one has

$$\lim_{y\to 0}\iint_{\mathbb{T}^D\times[0,T]}\big|\langle \zeta\, \tilde g_\epsilon\rangle(x-y,t) - \langle \zeta\, \tilde g_\epsilon\rangle(x,t)\big|\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \text{uniformly in } \epsilon.$$

Because

$$\begin{aligned}&\iint_{\mathbb{T}^D\times[0,T]}\big|\langle \zeta\, \tilde g_\epsilon^\delta\rangle - \langle \zeta\, \tilde g_\epsilon\rangle\big|\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\leqq \iiint_{\mathbb{T}^D\times\mathbb{T}^D\times[0,T]}\big|\langle \zeta\, \tilde g_\epsilon\rangle(x-y,t) - \langle \zeta\, \tilde g_\epsilon\rangle(x,t)\big|\, j^\delta(y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

it therefore follows that

$$\lim_{\delta\to 0}\langle \zeta\, \tilde g_\epsilon^\delta\rangle = \langle \zeta\, \tilde g_\epsilon\rangle \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)) \text{ uniformly in } \epsilon. \tag{11.7}$$

This is the L^1 analog of assertion (11.6).

In order to replace L^1 by L^2 in (11.7) we now use the fact that $\langle a\, \tilde g_\epsilon^2\rangle$ is relatively compact in w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))$ to establish the fact that

$$\langle \zeta\, \tilde g_\epsilon^\delta\rangle^2 \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)). \tag{11.8}$$

Indeed, for almost every (x,t) one has the pointwise bound

$$\begin{aligned}\langle \zeta\, \tilde g_\epsilon^\delta\rangle^2(x,t) &= \iint_{\mathbb{T}^D\times\mathbb{T}^D}\langle \zeta\, \tilde g_\epsilon\rangle(y_1,t)\,\langle \zeta\, \tilde g_\epsilon\rangle(y_2,t)\, j^\delta(x-y_1)\, j^\delta(x-y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\\ &\leqq \iint_{\mathbb{T}^D\times\mathbb{T}^D}\tfrac{1}{2}\langle \zeta\, \tilde g_\epsilon\rangle^2(y_1,t)\, j^\delta(x-y_1)\, j^\delta(x-y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\\ &\quad + \iint_{\mathbb{T}^D\times\mathbb{T}^D}\tfrac{1}{2}\langle \zeta\, \tilde g_\epsilon\rangle^2(y_2,t)\, j^\delta(x-y_1)\, j^\delta(x-y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2\\ &= j^\delta * \big(\langle \zeta\, \tilde g_\epsilon\rangle^2\big)(x,t) \leqq \left\langle \frac{1}{a}\,\zeta^2\right\rangle j^\delta * \langle a\, \tilde g_\epsilon^2\rangle(x,t).\end{aligned}$$

Because the family $\langle a\, \tilde g_\epsilon^2\rangle$ is relatively compact in w-$L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))$, if follows easily that the doubly indexed family $j^\delta * \langle a\, \tilde g_\epsilon^2\rangle$ is as well, whereby the above inequality implies that (11.8) holds.

Assertion (11.6) follows from (11.7) and (11.8) upon applying the following lemma to the doubly indexed family $\langle \zeta\, \tilde g_\epsilon^\delta\rangle - \langle \zeta\, \tilde g_\epsilon\rangle$. □

The above proof of Proposition 11.2 required the following lemma.

Lemma 11.1. *Let ψ_ϵ^δ be a family in $L^2_{loc}(\mathrm{d}t; C^\infty(\mathbb{T}^D))$ doubly indexed by $\epsilon > 0$ and $\delta > 0$ such that*

$$(\psi_\epsilon^\delta)^2 \quad \textit{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)), \tag{11.9}$$

$$\lim_{\delta\to 0} \psi_\epsilon^\delta = 0 \quad \textit{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)) \textit{ uniformly in } \epsilon. \tag{11.10}$$

Then

$$\lim_{\delta\to 0} \psi_\epsilon^\delta = 0 \quad \textit{in } L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x)) \textit{ uniformly in } \epsilon. \tag{11.11}$$

Proof. Let $T \in (0,\infty)$. Because the family $(\psi_\epsilon^\delta)^2$ is relatively compact in $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))$ one has that

$$M = \sup\left\{\left(\iint_{\mathbb{T}^D\times[0,T]} |\psi_\epsilon^\delta|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\right\} < \infty.$$

For every $\lambda > 0$ define

$$\Omega_\epsilon^\delta(\lambda) = \left\{(x,t) \in \mathbb{T}^D \times [0,T] \,:\, |\psi_\epsilon^\delta(x,t)| > \lambda\right\}.$$

The Chebychev inequality then yields

$$\operatorname{meas}\left\{\Omega_\epsilon^\delta(\lambda)\right\} \leqq \frac{M^2}{\lambda^2}.$$

Now let $\eta > 0$ be arbitrary. Because $(\psi_\epsilon^\delta)^2$ is relatively compact in $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))$, by the above inequality we may pick λ large enough to insure that

$$\iint_{\Omega_\epsilon^\delta(\lambda)} |\psi_\epsilon^\delta|^2 \mathrm{d}x\,\mathrm{d}t < \frac{\eta}{2} \quad \text{for every } \epsilon \text{ and } \delta.$$

The assumed uniform L^1-limit (11.10) implies we may pick $\delta_o > 0$ small enough to insure that $\delta < \delta_o$ implies

$$\iint_{\mathbb{T}^D\times[0,T]} |\psi_\epsilon^\delta|\,\mathrm{d}x\,\mathrm{d}t < \frac{\eta}{2\lambda} \quad \text{for every } \epsilon.$$

The above two inequalities show that $\delta < \delta_o$ implies

$$\begin{aligned}\iint_{\mathbb{T}^D\times[0,T]} |\psi_\epsilon^\delta|^2 \mathrm{d}x\,\mathrm{d}t &\leqq \iint_{\Omega_\epsilon^\delta(\lambda)} |\psi_\epsilon^\delta|^2 \mathrm{d}x\,\mathrm{d}t + \lambda \iint_{\mathbb{T}^D\times[0,T]} |\psi_\epsilon^\delta|\,\mathrm{d}x\,\mathrm{d}t \\ &\leqq \frac{\eta}{2} + \lambda\frac{\eta}{2\lambda} = \eta \quad \text{for every } \epsilon.\end{aligned}$$

Because η was arbitrary while δ_o was independent of ϵ, the assertion (11.11) follows. □

11.2. Mollified quadratic limits

The main result of this section is the following proposition, which is adapted from [29] (also see [28,30]).

Proposition 11.3. *For every $\delta > 0$ one has*

$$\left.\begin{aligned} \lim_{\epsilon\to 0} \Pi\nabla_x\cdot\left(\tilde{u}^\delta_\epsilon\otimes\tilde{u}^\delta_\epsilon\right) &= \Pi\nabla_x\cdot\left(u^\delta\otimes u^\delta\right) \\ \lim_{\epsilon\to 0} \nabla_x\cdot\left(\tilde{\theta}^\delta_\epsilon\,\tilde{u}^\delta_\epsilon\right) &= \nabla_x\cdot\left(\theta^\delta\,u^\delta\right) \end{aligned}\right\} \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t;\mathcal{D}'(\mathbb{T}^D)). \tag{11.12}$$

Proof. Introduce the new fluid variables

$$\begin{aligned} \tilde{w}^\delta_\epsilon &= \Pi\tilde{u}^\delta_\epsilon, \quad \tilde{v}^\delta_\epsilon = \Pi^\perp\tilde{u}^\delta_\epsilon, \\ \tilde{\sigma}^\delta_\epsilon &= \tfrac{D}{2}\tilde{\theta}^\delta_\epsilon - \tilde{\rho}^\delta_\epsilon, \quad \tilde{\pi}^\delta_\epsilon = \tilde{\rho}^\delta_\epsilon + \tilde{\theta}^\delta_\epsilon, \end{aligned} \tag{11.13}$$

where Π is the Leray projection onto divergence-free vector-fields in $L^2(\mathrm{d}x;\mathbb{R}^D)$. Here $\tilde{w}^\delta_\epsilon$ and $\tilde{v}^\delta_\epsilon$ are respectively the divergence-free and gradient components of $\tilde{u}^\delta_\epsilon$, while $\tilde{\sigma}^\delta_\epsilon$ and $\tilde{\pi}^\delta_\epsilon$ are the infinitesimal entropy and pressure fluctuations associated with $\tilde{g}^\delta_\epsilon$.

Because for every $\zeta \in \operatorname{span}\{1, v_1, \ldots, v_D, |v|^2\}$ one has $\langle\zeta\,\tilde{g}_\epsilon\rangle \to \langle\zeta\,g\rangle$ in $w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x))$ as $\epsilon\to 0$, it can be easily shown that for every $s \geqq 0$ and $\delta > 0$ one has

$$\lim_{\epsilon\to 0}\langle\zeta\,\tilde{g}^\delta_\epsilon\rangle = \langle\zeta\,g^\delta\rangle \quad \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}H^s(\mathrm{d}x)), \tag{11.14}$$

where $H^s(\mathrm{d}x)$ denotes the s^{th} Sobolev space. In particular, for every $s > 0$ and $\delta > 0$ the families $\tilde{w}^\delta_\epsilon$, $\tilde{v}^\delta_\epsilon$, $\tilde{\sigma}^\delta_\epsilon$, and $\tilde{\pi}^\delta_\epsilon$ satisfy

$$\left.\begin{aligned} \lim_{\epsilon\to 0}\tilde{w}^\delta_\epsilon &= u^\delta, & \lim_{\epsilon\to 0}\tilde{v}^\delta_\epsilon &= 0, \\ \lim_{\epsilon\to 0}\tilde{\sigma}^\delta_\epsilon &= \tfrac{D+2}{2}\theta^\delta, & \lim_{\epsilon\to 0}\tilde{\pi}^\delta_\epsilon &= 0, \end{aligned}\right\} \text{ in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}H^s(\mathrm{d}x)), \tag{11.15}$$

where $u^\delta = j^\delta * u$ and $\theta^\delta = j^\delta * \theta$. Because $\tilde{u}^\delta_\epsilon$ and $\tilde{\theta}^\delta_\epsilon$ decompose as

$$\tilde{u}^\delta_\epsilon = \tilde{w}^\delta_\epsilon + \tilde{v}^\delta_\epsilon, \quad \tfrac{D+2}{2}\,\tilde{\theta}^\delta_\epsilon = \tilde{\sigma}^\delta_\epsilon + \tilde{\pi}^\delta_\epsilon, \tag{11.16}$$

the quadratic terms $\tilde{u}^\delta_\epsilon\otimes\tilde{u}^\delta_\epsilon$ and $\tilde{\theta}^\delta_\epsilon\tilde{u}^\delta_\epsilon$ decompose as

$$\begin{aligned} \tilde{u}^\delta_\epsilon\otimes\tilde{u}^\delta_\epsilon &= \tilde{w}^\delta_\epsilon\otimes\tilde{w}^\delta_\epsilon + \tilde{w}^\delta_\epsilon\otimes\tilde{v}^\delta_\epsilon + \tilde{v}^\delta_\epsilon\otimes\tilde{w}^\delta_\epsilon + \tilde{v}^\delta_\epsilon\otimes\tilde{v}^\delta_\epsilon, \\ \tfrac{D+2}{2}\,\tilde{\theta}^\delta_\epsilon\tilde{u}^\delta_\epsilon &= \tilde{\sigma}^\delta_\epsilon\tilde{w}^\delta_\epsilon + \tilde{\sigma}^\delta_\epsilon\tilde{v}^\delta_\epsilon + \tilde{\pi}^\delta_\epsilon\tilde{w}^\delta_\epsilon + \tilde{\pi}^\delta_\epsilon\tilde{v}^\delta_\epsilon. \end{aligned} \tag{11.17}$$

We will consider the limit of each term on the right-hand sides above as $\epsilon\to 0$.

It follows from (6.11) that $\tilde{w}^\delta_\epsilon$, $\tilde{v}^\delta_\epsilon$, $\tilde{\sigma}^\delta_\epsilon$, and $\tilde{\pi}^\delta_\epsilon$ satisfy the approximate conservation laws

$$\begin{aligned} \partial_t\tilde{w}^\delta_\epsilon &= \Pi\tilde{J}^\delta_\epsilon, \quad \partial_t\tilde{v}^\delta_\epsilon + \frac{1}{\epsilon}\nabla_x\tilde{\pi}^\delta_\epsilon = \Pi^\perp\tilde{J}^\delta_\epsilon, \\ \partial_t\tilde{\sigma}^\delta_\epsilon &= \tilde{K}^\delta_\epsilon, \quad \partial_t\tilde{\pi}^\delta_\epsilon + \frac{1}{\epsilon}\tfrac{D+2}{D}\nabla_x\cdot\tilde{\pi}^\delta_\epsilon = \tilde{I}^\delta_\epsilon, \end{aligned} \tag{11.18}$$

where $\tilde I_\epsilon^\delta$, $\tilde J_\epsilon^\delta$, and $\tilde K_\epsilon^\delta$ are defined by

$$\begin{aligned}
\tilde I_\epsilon^\delta &= \frac{1}{\epsilon}\, j^\delta * \Big(\big\langle\!\big\langle \tfrac{1}{D}|v|^2 \Gamma_\epsilon'(G_\epsilon)\, q_\epsilon \big\rangle\!\big\rangle - \tfrac{2}{D}\nabla_x\cdot \langle B\, \tilde g_\epsilon\rangle \Big),\\
\tilde J_\epsilon^\delta &= \frac{1}{\epsilon}\, j^\delta * \Big(\big\langle\!\big\langle v\, \Gamma_\epsilon'(G_\epsilon)\, q_\epsilon \big\rangle\!\big\rangle - \nabla_x\cdot \langle A\, \tilde g_\epsilon\rangle \Big),\\
\tilde K_\epsilon^\delta &= \frac{1}{\epsilon}\, j^\delta * \Big(\big\langle\!\big\langle (\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\, \Gamma_\epsilon'(G_\epsilon)\, q_\epsilon \big\rangle\!\big\rangle - \nabla_x\cdot \langle B\, \tilde g_\epsilon\rangle \Big).
\end{aligned} \tag{11.19}$$

Because $\tilde J_\epsilon^\delta$ and $\tilde K_\epsilon^\delta$ are relatively compact in $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}H^s(\mathrm{d}x))$, it follows from the first column of (11.18) that the families $\tilde w_\epsilon^\delta$ and $\tilde\sigma_\epsilon^\delta$ are equicontinuous in $C([0,\infty); w\text{-}L^2(\mathrm{d}x))$. Because these ϵ-families are also bounded in $L^2(\mathrm{d}x)$ at every $t \geqq 0$, the Arzela–Ascoli Theorem implies that they are relatively compact in $C([0,\infty); w\text{-}L^2(\mathrm{d}x))$. Because (11.15) holds for $s = 0$, it follows that

$$\lim_{\epsilon\to 0} \tilde w_\epsilon^\delta = u^\delta, \quad \lim_{\epsilon\to 0} \tilde\sigma_\epsilon^\delta = \tfrac{D+2}{2}\,\theta^\delta, \quad \text{in } C([0,\infty); w\text{-}L^2(\mathrm{d}x)). \tag{11.20}$$

Because for every $s > 0$ one has the continuous embedding

$$w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}H^s(\mathrm{d}x)) \cap C([0,\infty); w\text{-}L^2(\mathrm{d}x)) \to L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x)),$$

the limits (11.15) for $s > 0$ and (11.20) imply that the families $\tilde w_\epsilon^\delta$ and $\tilde\sigma_\epsilon^\delta$ satisfy the strong limits

$$\lim_{\epsilon\to 0} \tilde w_\epsilon^\delta = u^\delta, \quad \lim_{\epsilon\to 0} \tilde\sigma_\epsilon^\delta = \tfrac{D+2}{2}\,\theta^\delta, \quad \text{in } L^2_{loc}(\mathrm{d}t; L^2(\mathrm{d}x)). \tag{11.21}$$

When this result is combined with the weak limits for the families $\tilde v_\epsilon^\delta$ and $\tilde\pi_\epsilon^\delta$ found in (11.15), we obtain

$$\left.\begin{aligned}
&\lim\nolimits_{\epsilon\to 0} \tilde w_\epsilon^\delta \otimes \tilde w_\epsilon^\delta = u^\delta \otimes u^\delta,\\
&\lim\nolimits_{\epsilon\to 0} \tilde w_\epsilon^\delta \otimes \tilde v_\epsilon^\delta = \lim\nolimits_{\epsilon\to 0} \tilde v_\epsilon^\delta \otimes \tilde w_\epsilon^\delta = 0,\\
&\lim\nolimits_{\epsilon\to 0} \tilde\sigma_\epsilon^\delta\, \tilde w_\epsilon^\delta = \tfrac{D+2}{2}\,\theta^\delta\, u^\delta,\\
&\lim\nolimits_{\epsilon\to 0} \tilde\sigma_\epsilon^\delta\, \tilde v_\epsilon^\delta = \lim\nolimits_{\epsilon\to 0} \tilde\pi_\epsilon^\delta\, \tilde w_\epsilon^\delta = 0,
\end{aligned}\right\} \text{ in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)). \tag{11.22}$$

These limits treat all but the last term on the right-hand side of each decomposition in (11.17).

It follows from the second column of (11.18) that the families $\tilde v_\epsilon^\delta$ and $\tilde\pi_\epsilon^\delta$ satisfy the

$$\begin{aligned}
\nabla_x\cdot\big(\tilde v_\epsilon^\delta \otimes \tilde v_\epsilon^\delta\big) &= \tfrac{1}{2}\nabla_x\big|\tilde v_\epsilon^\delta\big|^2 - \tfrac{D}{D+2}\nabla_x\big(\tilde\pi_\epsilon^\delta\big)^2 - \epsilon\,\tfrac{D}{D+2}\,\partial_t\big(\tilde\pi_\epsilon^\delta\,\tilde v_\epsilon^\delta\big) + \epsilon\,\tfrac{D}{D+2}\\
&\qquad \times\big(\tilde\pi_\epsilon^\delta\,\Pi^\perp \tilde J_\epsilon^\delta + \tilde v_\epsilon^\delta\,\tilde I_\epsilon^\delta\big),\\
\nabla_x\cdot\big(\tilde\pi_\epsilon^\delta\,\tilde v_\epsilon^\delta\big) &= -\,\epsilon\,\partial_t\Big(\tfrac{1}{2}\big|\tilde v_\epsilon^\delta\big|^2 + \tfrac{1}{2}\tfrac{D}{D+2}\big(\tilde\pi_\epsilon^\delta\big)^2\Big) + \epsilon\,\big(\tilde v_\epsilon^\delta\,\Pi^\perp \tilde J_\epsilon^\delta + \tfrac{D}{D+2}\tilde\pi_\epsilon^\delta\,\tilde I_\epsilon^\delta\big),
\end{aligned} \tag{11.23}$$

Because the ϵ-families $\tilde v_\epsilon^\delta$ and $\tilde\pi_\epsilon^\delta$ are bounded in $L^\infty(\mathrm{d}t; L^2(\mathrm{d}x)) \cap C([0,\infty); H^s(\mathrm{d}x))$, while the ϵ-families $\tilde I_\epsilon^\delta$ and $\tilde J_\epsilon^\delta$ are bounded in $L^1_{loc}(\mathrm{d}t; L^2(\mathrm{d}x))$, and because $\Pi\nabla_x = 0$, it follows from the above relations that

$$\left.\begin{aligned}
&\lim\nolimits_{\epsilon\to 0} \Pi\nabla_x\cdot\big(\tilde v_\epsilon^\delta \otimes \tilde v_\epsilon^\delta\big) = 0\\
&\lim\nolimits_{\epsilon\to 0} \nabla_x\cdot\big(\tilde\pi_\epsilon^\delta\,\tilde v_\epsilon^\delta\big) = 0
\end{aligned}\right\} \text{ in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D)). \tag{11.24}$$

These limits treat the last term on the right-hand side of each decomposition in (11.17). Assertion (11.12) of the proposition follows by using decomposition (11.17) along with the limits in (11.22) and (11.24). □

Acknowledgements. C. David LEVERMORE was partially supported by the NSF under grant DMS-9803753. N.M. was partially supported by the NSF under grant DMS-0703145. The authors would like to thank F. GOLSE and L. SAINT-RAYMOND for many helpful discussions. They would also like to thank D. ARSENIO for carefully reading the paper.

References

1. BARDOS, C., GOLSE, F., LEVERMORE, D.: Sur les limites asymptotiques de la théorie cinétique conduisant à la dynamique des fluides incompressibles. *C.R. Acad. Sci. Paris Sr. I Math.* **309**, 727–732 (1989)
2. BARDOS, C., GOLSE, F., LEVERMORE, D.: Fluid dynamic limits of kinetic equations I: formal derivations. *J. Stat. Phys.* **63**, 323–344 (1991)
3. BARDOS, C., GOLSE, F., LEVERMORE, C.D.: Fluid dynamic limits of kinetic equations II: convergence proofs for the Boltzmann equation. *Commun. Pure Appl. Math.* **46**, 667–753 (1993)
4. BARDOS, C., GOLSE, F., LEVERMORE, C.D.: Acoustic and Stokes limits for the Boltzmann equation. *C.R. Acad. Sci. Paris* **327**, 323–328 (1999)
5. BARDOS, C., GOLSE, F., LEVERMORE, C.D.: The Acoustic limit for the Boltzmann equation. *Archive Rat. Mech. Anal.* **153**, 177–204 (2000)
6. CERCIGNANI, C.: *The Boltzmann Equation and its Applications*. Springer, New York, 1988
7. CERCIGNANI, C., ILLNER, R., PULVIRENTI, M.: *The Mathematical Theory of Dilute Gases, Applied Mathematical Sciences*, Vol. 106. Springer, New York, 1994
8. CONSTANTIN, P., FOIAS, C.: *Navier–Stokes Equations*. Chicago Lectures in Mathematics. The University of Chicago Press, Chicago, 1988
9. DIPERNA, R.J., LIONS, P.-L.: On the Cauchy problem for the Boltzmann equation: global existence and weak stability results. *Ann. Math.* **130**, 321–366 (1990)
10. GLASSEY, R.: *The Cauchy Problem in Kinetic Theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1996
11. GOLSE, F.: From kinetic to macroscopic models. *Kinetic Equations and Asymptotic Theory*, Vol. 4 (Eds. Perthame B. and Desvillettes L.) Series in Applied Mathematics. Gauthier-Villars, Paris, 41–126, 2000
12. GOLSE, F., LEVERMORE, C.D.: Stokes–Fourier and acoustic limits for the Boltsmann equation: convergence proofs. *Commun. Pure Appl. Math.* **55**, 336–393 (2002)
13. GOLSE, F., LIONS, P.-L., PERTHAME, B., SENTIS, R.: Regularity of the moments of the solution of a transport equation. *J. Funct. Anal.* **76**, 110–125 (1988)
14. GOLSE, F., POUPAUD, F.: Un résultat de compacité pour l'équation de Boltzmann avec potentiel mou. Application au problème de demi-espace. *C.R. Acad. Sci. Paris Sr. I Math.* **303**, 583–586 (1986)
15. GOLSE, F., SAINT-RAYMOND, L.: The Navier–Stokes limit for the Boltzmann equation. *C.R. Acad. Sci. Paris Sr. I Math.* **333**, 897–902 (2001)
16. GOLSE, F., SAINT-RAYMOND, L.: Velocity averaging in L^1 for the transport equation. *C.R. Acad. Sci. Paris Sr. I Math.* **334**, 557–562 (2002)
17. GOLSE, F., SAINT-RAYMOND, L.: The Navier–Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math.* **155**, 81–161 (2004)
18. GOLSE, F., SAINT-RAYMOND, L.: *The Incompressible Navier–Stokes Limit of the Boltzmann Equation for Hard Cutoff Potentials*, preprint 2008
19. GRAD, H.: Principles of the kinetic theory of gases. *Handbuch der Physik*, Vol. 12 (Ed. Flügge S.) Springer, Berlin, 205–294 (1958)

20. HILBERT, D.: Begründung der kinetischen Gastheorie. *Math. Annalen* **72**, 562–577 (1912); English: Foundations of the kinetic theory of gases. *Kinetic Theory*, Vol. 3 (Ed. Brush G.) Pergamon Press, Oxford, 89–101, 1972
21. JIANG, N., LEVERMORE, C.D., MASMOUDI, N.: Remarks on the acoustic limits for the Boltzmann equation. *Commun. P.D.E.*, submitted 2009
22. JIANG, N., MASMOUDI, N.: *From the Boltzmann Equation to the Navier–Stokes–Fourier System in a Bounded Domain*, preprint 2009
23. LERAY, J.: Sur le mouvement d'un fluide visqueux emplissant l'espace. *Acta Math.* **63**, 193–248 (1934)
24. LEVERMORE, C.D.: Entropic convergence and the linearized limit for the Boltzmann equation. *Commun. P.D.E.* **18**, 1231–1248 (1993)
25. LEVERMORE, C.D., SUN, W.: Compactness of the gain parts of the linearized Boltzmann operator with weakly cutoff kernels. *Commun. P.D.E.*, submitted 2009
26. LIONS, P.-L.: Compactness in Boltzmann's equation via Fourier integral operators and applications, I, II, & III. *J. Math. Kyoto Univ.* **34**, 391–427, 429–461, 539–584 (1994)
27. LIONS, P.-L., MASMOUDI, N.: Incompressible limit for a viscous compressible fluid. *J. Math. Pures Appl.* **77**, 585–627 (1998)
28. LIONS, P.-L., MASMOUDI, N.: Une approche locale de la limite incompressible. *C.R. Acad. Sci. Paris Sr. I Math.* **329**, 387–392 (1999)
29. LIONS, P.-L., MASMOUDI, N.: From the Boltzmann equations to the equations of incompressible fluid mechanics, I. *Archive Rat. Mech. Anal.* **158**, 173–193 (2001)
30. LIONS, P.-L., MASMOUDI, N.: From the Boltzmann equations to the equations of incompressible fluid mechanics II. *Archive Rat. Mech. Anal.* **158**, 195–211 (2001)
31. MASMOUDI, N.: Examples of singular limits in hydrodynamis. *Evolutionary Equations, Handb. Differ. Equ.*, Vol. 3. Elsevier/North-Holland, Amsterdam, 195–276, 2007
32. MASMOUDI, N., SAINT-RAYMOND, L.: From the Boltzmann equation to the Stokes–Fourier system in a bounded domain. *Commun. Pure Appl. Math* **56**, 1263–1293 (2003)
33. SAINT-RAYMOND, L.: Du modèl BGK de l'équation de Boltzmann aux équations d' Euler des fluides incompressibles. *Bull. Sci. Math.* **126**, 493–506 (2002)
34. SAINT-RAYMOND, L.: From the Boltzmann BGK equation to the Navier–Stokes system. *Ann. Sci.* École Norm. Sup. **36**, 271–317 (2003)
35. SAINT-RAYMOND, L.: Convergence of solutions to the Boltzmann equation in the incompressible Euler limit. *Archive Rat. Mech. Anal.* **166**, 47–80 (2003)
36. SUN, W.: *Fredholm Alternatives for Linearized Boltzmann Collision Operators with Weakly Cutoff Kernels*, (preprint 2005). Chapter I of *Mathematical Problems Arising when Connecting Kinetic to Fluid Regimes*. Ph.D. Dissertation, University of Maryland, 2008

Department of Mathematics and Institute for Physical
Science and Technology (IPST),
University of Maryland,
College Park, MD 20742-4015, USA.
e-mail: lvrmr@math.umd.edu

and

Courant Institute of Mathematical Sciences,
New York University,
251 Mercer Street,
New York, NY 10012, USA.
e-mail: masmoudi@cims.nyu.edu
e-mail: masmoudi@courant.nyu.edu

(*Received March 10, 2008 / Accepted June 16, 2009*)
Published online July 28, 2009 – © Springer-Verlag (*2009*)