

Well-posedness for compressible Euler with physical vacuum singularity

Juhi Jang and Nader Masmoudi

May 28, 2008

Abstract

An important problem in the theory of compressible gas flows is to understand the singular behavior of vacuum states. The main difficulty lies in the fact that the system becomes degenerate at the vacuum boundary, where the characteristics coincide and have unbounded derivative. In this paper, we overcome this difficulty by presenting a new formulation and new energy spaces. We establish the local in time well-posedness of one-dimensional compressible Euler equations for isentropic flows with the physical vacuum singularity in some spaces adapted to the singularity.

1 Introduction

One-dimensional compressible Euler equations for the isentropic flow with damping in Eulerian coordinates read as follows:

$$\begin{aligned}\rho_t + (\rho u)_x &= 0, \\ \rho u_t + \rho u u_x + p_x &= -\rho u,\end{aligned}\tag{1}$$

with initial data $\rho(0, x) = \rho_0(x) \geq 0$ and $u(0, x) = u_0(x)$ prescribed. Here ρ , u , and p denote respectively the density, velocity, and pressure. We consider the polytropic gases: the equation of state is given by $p = A\rho^\gamma$, where A is an entropy constant and $\gamma > 1$ is the adiabatic gas exponent. When the initial density function contains vacuum, the vacuum boundary Γ is defined as

$$\Gamma = cl\{(t, x) : \rho(t, x) > 0\} \cap cl\{(t, x) : \rho(t, x) = 0\}$$

where cl denotes the closure. A vacuum boundary is called **physical** if

$$0 < \left| \frac{\partial c^2}{\partial x} \right| < \infty\tag{2}$$

in a small neighborhood of the boundary, where $c = \sqrt{\frac{d}{d\rho} p(\rho)}$ is the sound speed. This physical vacuum behavior can be realized by some self-similar solutions and stationary solutions for different physical systems such as Euler equations with damping, Navier-Stokes equations or Euler-Poisson equations for gaseous stars. For more details and the physical background regarding this concept of the physical vacuum boundary, we refer to [7, 10, 19].

Despite its physical importance, even the local existence theory of smooth solutions featuring the physical vacuum boundary has not been completed yet. This is because the hyperbolic system becomes degenerate at the vacuum boundary and in particular if the physical

vacuum boundary condition (2) is assumed, the classical theory of hyperbolic systems can not be applied [10]: the characteristic speeds of Euler equations are $u \pm c$, thus they become singular with infinite spatial derivatives at the vacuum boundary and this singularity creates a severe analytical difficulty. To our knowledge, there has been no satisfactory theory to treat this kind of singularity. The purpose of this article is to investigate the local existence and uniqueness theory of regular solutions (in a sense that will be made precise later and which is adapted to the singularity of the problem) to compressible Euler equations featuring this physical vacuum boundary.

Before we formulate our problem, we briefly review some existence theories of compressible flows with vacuum states from various aspects. We will not attempt to address exhaustive references in this paper. First, in the absence of vacuum, namely if the density is bounded below from zero, then one can use the theory of symmetric hyperbolic systems; for instance, see [12]. In particular, the author in [17] gave a sufficient condition for non-global existence when the density is bounded below from zero. When the data is compactly supported, there are two ways of looking at the problem. The first consists in solving the Euler equations in the whole space and requiring that the equations hold for all x and $t \in (0, T)$. The second way is to require the Euler equations to hold on the set $\{(t, x) : \rho(t, x) > 0\}$ and write an equation for the vacuum boundary Γ which is a free boundary.

Of course in the first way, there is no need of knowing exactly the position of the vacuum boundary. The authors in [15] wrote the system in a symmetric hyperbolic form which allows the density to vanish. The system they get is not equivalent to the Euler equations when the density vanishes. This was also used for the Euler-Poisson system. As noted by the authors [13, 14], the requirement that $\rho^{\frac{\gamma-1}{2}}$ is continuously differentiable excludes many interesting solutions such as the stationary solutions of the Euler-Poisson which have a behavior of the type $\rho \sim |x|^{\frac{1}{\gamma-1}}$ at the vacuum boundary. This formulation was also used in [2] to prove the local existence of regular solutions in the sense that $\rho^{\frac{\gamma-1}{2}}, u \in C([0, T]; H^m(\mathbb{R}^d))$ for some $m > 1 + d/2$ and d is the space dimension (see also [16], for some global existence result of classical solutions under some special conditions on the initial data, by extracting a dispersive effect after some invariant transformation, and [3]).

For the second way and when the singularity is mild, some existence results of smooth solutions are available, based on the adaptation of the theory of symmetric hyperbolic system. In [9], the local in time solutions to Euler equations with damping (1) were constructed when c^α , $0 < \alpha \leq 1$ is smooth across Γ , by using the energy method and characteristic method. They also prove that C^1 solutions cross Γ can not be global. However, with or without damping, the methods developed therein are not applicable to the local well-posedness theory of the physical vacuum boundary. We only mention a result in [18] for perturbation of a planar wave. For other interesting aspects of vacuum states and related problems, we refer to [10, 19].

As the above results indicate, there is an interesting distinction between flows with damping and without damping, when the long time behavior is considered. Indeed, without damping, it was shown in [8] that the shock waves vanish at the vacuum and the singular behavior is similar to the behavior of the centered rarefaction waves corresponding to the case when c is regular [10]. On the other hand, with damping, it was conjectured in [7] that time asymptotically, Euler equations with damping (1) should behave like the porous media equation, where the canonical boundary is characterized by the physical vacuum boundary condition (2). This conjecture was established in [4] in the entropy solution framework where the method of compensated compactness yields a global weak solution in L^∞ . We point out

that the difficulty coming from the resonance due to vacuum in there is very different from the difficulty that we are facing, since we want to have some regularity so that the vacuum boundary is well-defined and the evolution of the vacuum boundary can be realized.

In order to understand the physical vacuum boundary behavior, the study of regular solutions is very important and fundamental; the evolution of the vacuum boundary should be considered as the free boundary/interface generated by vacuum. In the presence of viscosity, there are some existence theories available with the physical vacuum boundary: The vacuum interface behavior as well as the regularity to one-dimensional Navier-Stokes free boundary problems were investigated in [11]. And the local in time well-posedness of Navier-Stokes-Poisson equations in three dimensions with radial symmetry featuring the physical vacuum boundary was established in [5]. On the other hand, the free surface boundary problem was studied in [6] in the motion of a compressible liquid with vacuum by using Nash-Moser iteration; the physical boundary was treated in a sense that the pressure vanishes on the boundary and the pressure gradient is bounded away from zero, but the density has to be bounded away from vacuum, and thus the analysis is not applicable for the motion of gas with vacuum.

In the next section, we formulate the problem and we state the main result: The vacuum free boundary problem is studied in Lagrangian coordinates so that the free boundary becomes fixed. By change of variables, the equations can be written as the first order system with non-degenerate propagation speeds that have different behaviors inside the domain and on the vacuum boundary. In order to cope with these nonlinear coefficients, which give rise to the main analytical difficulty, the new operators V, V^* are introduced. Our theorem is stated in V, V^* framework.

2 Formulation and Main result

We study the initial boundary value problem to one-dimensional Euler equations with or without damping for isentropic flows (1). First, we impose the fixed boundary condition on one boundary $x = b : u(t, b) = 0$. The class of the initial data ρ_0, u_0 of our interest is characterized as follows: for $a \leq x \leq b$, where $-\infty < a < b \leq \infty$

$$\begin{aligned} (i) \quad & \rho_0(a) = 0, \quad 0 < \frac{d}{dx} \rho_0^{\gamma-1} \Big|_{x=a} < \infty; \\ (ii) \quad & \rho_0(x) > 0 \text{ for } a < x \leq b; \\ (iii) \quad & \int_a^b \rho_0(x) dx < \infty; (iv) \quad u_0(b) = 0. \end{aligned}$$

The condition (i) implies that the initial vacuum is physical, (ii) means that $x = a$ is the only vacuum, (iii) represents the finite total mass of gas, (iv) is the compatibility condition with the boundary condition at $x = b$. We seek $\rho(t, x)$, $u(t, x)$, and $a(t)$ for $t \in [0, T]$, $T > 0$ and $x \in [a(t), b]$, so that for such t and x ,

$$\begin{aligned} & \rho(t, x) \text{ and } u(t, x) \text{ satisfy (1);} \\ & \rho(t, a(t)) = 0; \quad u(t, b) = 0; \\ & 0 < \frac{\partial}{\partial x} \rho^{\gamma-1} \Big|_{x=a(t)} < \infty. \end{aligned}$$

For regular solutions, the vacuum boundary $a(t)$ is the particle path through $x = a$. In one-dimensional gas dynamics, there is a natural Lagrangian coordinates transformation where all the particle paths are straight lines:

$$y \equiv \int_{a(t)}^x \rho(t, z) dz, \quad a(t) \leq x \leq b.$$

Note that $0 \leq y \leq M$, where M is the total mass of the gas. Under this transformation, the vacuum free boundary $x = a(t)$ corresponds to $y = 0$, and $x = b$ to $y = M$; thus both boundaries are fixed in (t, y) . By this change of variables,

$$\partial_t = \partial_t - u \partial_y, \quad \partial_x = \rho \partial_y$$

the system (1) takes the following form in Lagrangian coordinates (t, y) : for $t \geq 0$ and $0 \leq y \leq M$,

$$\begin{aligned} \rho_t + \rho^2 u_y &= 0 \\ u_t + p_y &= -u \end{aligned} \tag{3}$$

where $p = A\rho^\gamma$ with $\gamma > 1$. The boundary conditions are given by $\rho(t, 0) = 0$ and $u(t, M) = 0$. The physical singularity (2) in Eulerian coordinates corresponds to $0 < |p_y| < \infty$ in Lagrangian coordinates and thus the physical vacuum boundary condition at $y = 0$ can be realized as

$$\rho \sim y^{\frac{1}{\gamma}} \quad \text{for } y \sim 0.$$

Euler equations (3) can be rewritten as a symmetric hyperbolic system

$$\begin{aligned} \phi_t + \mu u_y &= 0, \\ u_t + \mu \phi_y &= -u, \end{aligned} \tag{4}$$

in the variables

$$\phi = \frac{2\sqrt{A\gamma}}{\gamma-1} \rho^{\frac{\gamma-1}{2}} \quad \text{and} \quad \mu = \sqrt{A\gamma} \rho^{\frac{\gamma+1}{2}}.$$

Note that the propagation speed μ becomes degenerate and the degeneracy for the physical singularity is given by $\mu \sim y^{\frac{\gamma+1}{2\gamma}}$. In order to get around this difficulty, we introduce the following change of variables:

$$\xi \equiv \frac{2\gamma}{\gamma-1} y^{\frac{\gamma-1}{2\gamma}} \quad \text{such that} \quad \partial_y = y^{-\frac{\gamma+1}{2\gamma}} \partial_\xi.$$

We normalize A and M appropriately such that the equations (4) take the form:

$$\begin{aligned} \phi_t + \left(\frac{\phi}{\xi}\right)^{\frac{\gamma+1}{\gamma-1}} u_\xi &= 0, \\ u_t + \left(\frac{\phi}{\xi}\right)^{\frac{\gamma+1}{\gamma-1}} \phi_\xi &= -u, \end{aligned}$$

for $t \geq 0$ and $0 \leq \xi \leq 1$. The physical singularity condition $0 < |p_y| < \infty$ is written as $0 < |\phi_\xi| < \infty$. Thus we expect ϕ to be more or less ξ for short time near 0. Since the damping is not important for the local theory, for simplicity, we consider the pure Euler

equations. Letting

$$k = k_\gamma \equiv \frac{1}{2} \frac{\gamma + 1}{\gamma - 1},$$

the Euler equations read in (t, ξ) as follows: for $t \geq 0$ and $0 \leq \xi \leq 1$,

$$\begin{aligned} \phi_t + \left(\frac{\phi}{\xi}\right)^{2k} u_\xi &= 0, \\ u_t + \left(\frac{\phi}{\xi}\right)^{2k} \phi_\xi &= 0. \end{aligned} \tag{5}$$

The range of k is $\frac{1}{2} < k < \infty$, since $\gamma > 1$. When $\gamma \rightarrow 1$, $k \rightarrow \infty$ and when $\gamma = 3$, we get $k = 1$. Note that the propagation speed is now non-degenerate. However, its behavior is quite different in the interior and on the boundary, since $\lim_{\xi \searrow 0} \frac{\phi}{\xi} = \phi_\xi(0)$, but $\phi \leq \frac{\phi}{\xi} \leq c\phi$ if $\xi \geq \frac{1}{c} > 0$. This makes it hard to apply any standard energy method to construct solutions in the current formulation.

We will propose a new formulation to (5) such that some energy estimates can be closed in the appropriate energy space. As a preparation, we first define the operators V and V^* associated to (5) as follows:

$$V(f) \equiv \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^k} f \right], \quad V^*(g) \equiv -\frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{1}{\xi^k} g \right].$$

for $f, g \in L_\xi^2$, where we have denoted $L_\xi^2[0, 1]$ by L_ξ^2 . We can think of V and V^* as modified first order spatial derivatives. We also incorporate the boundary condition at $\xi = 1$ in the domain of V and V^* , namely $\mathcal{D}(V)$ and $\mathcal{D}(V^*)$, are given as follows:

$$\begin{aligned} \mathcal{D}(V) &= \{f \in L_\xi^2 : V(f) \in L_\xi^2\} \\ \mathcal{D}(V^*) &= \{g \in L_\xi^2 : V^*(g) \in L_\xi^2, g(\xi = 1) = 0\} \end{aligned} \tag{6}$$

We also introduce the higher order operators $(V)^i$ and $(V^*)^i$: for $f \in \mathcal{D}(V)$ and $g \in \mathcal{D}(V^*)$,

$$(V)^i(f) \equiv \begin{cases} (V^*V)^j(f) & \text{if } i = 2j \\ V(V^*V)^j(f) & \text{if } i = 2j + 1 \end{cases} \tag{7}$$

$$(V^*)^i(g) \equiv \begin{cases} (VV^*)^j(g) & \text{if } i = 2j \\ V^*(VV^*)^j(g) & \text{if } i = 2j + 1 \end{cases} \tag{8}$$

and the associated function spaces $X^{k,s}$ and $Y^{k,s}$ for s a given nonnegative integer:

$$\begin{aligned} X^{k,s} &\equiv \{f \in L_\xi^2 : (V)^i(f) \in L_\xi^2, 0 \leq i \leq s\} \\ Y^{k,s} &\equiv \{g \in L_\xi^2 : (V^*)^i(g) \in L_\xi^2, 0 \leq i \leq s\} \end{aligned} \tag{9}$$

equipped with the following norms

$$\|f\|_{X^{k,s}}^2 \equiv \sum_{i=0}^s \|(V)^i(f)\|_{L_\xi^2}^2 \quad \text{and} \quad \|g\|_{Y^{k,s}}^2 \equiv \sum_{i=0}^s \|(V^*)^i(g)\|_{L_\xi^2}^2.$$

In order to emphasize the dependence of k , equivalently γ , we keep k in the above definitions.

In terms of V and V^* , the Euler equations (5) can be rewritten as follows:

$$\begin{aligned}\partial_t(\xi^k \phi) - V^*(\xi^k u) &= 0, \\ \partial_t(\xi^k u) + \frac{1}{2k+1} V(\xi^k \phi) &= 0,\end{aligned}\tag{10}$$

with the boundary conditions

$$\phi(t, 0) = 0 \quad \text{and} \quad u(t, 1) = 0.\tag{11}$$

In this new V, V^* formulation, the system is akin to the symmetric hyperbolic system with respect to V, V^* . In particular, the zeroth energy estimates assert that this V, V^* formulation retains the energy conservation property, which is equivalent to the physical energy in Eulerian coordinates: It is well known that the energy of Euler equations without damping is conserved for regular solutions:

$$\frac{d}{dt} \left\{ \int_{a(t)}^b \frac{1}{2} \rho u^2 + \frac{p}{\gamma-1} dx \right\} = 0,$$

and in turn, since $dy = \rho dx$, it is written as, in y variable,

$$\frac{d}{dt} \left\{ \int_0^M \frac{1}{2} u^2 + \frac{A}{\gamma-1} \rho^{\gamma-1} dy \right\} = 0.$$

It is routine to check that this is equivalent to

$$\frac{1}{2} \frac{d}{dt} \left\{ \int_0^1 \xi^{\frac{\gamma+1}{\gamma-1}} u^2 + \frac{\gamma-1}{2\gamma} \xi^{\frac{\gamma+1}{\gamma-1}} \phi^2 d\xi \right\} = 0,$$

which is exactly the zeroth energy estimates of (10) with respect to V, V^* :

$$\frac{1}{2} \frac{d}{dt} \left\{ \int_0^1 \frac{1}{2k+1} |\xi^k \phi|^2 + |\xi^k u|^2 d\xi \right\} = 0,\tag{12}$$

This verifies that this V, V^* formulation does not destroy the underlying structure of the Euler equations. Indeed, it also captures the precise structure of the singularity caused by the physical vacuum boundary, and furthermore, it enables us to perform V, V^* energy estimates yielding the a priori estimates and the well-posedness of the system.

In order to state the main result of this article precisely, we now define the energy functional $\mathcal{E}^{k,s}(\phi, u)$ by

$$\begin{aligned}\mathcal{E}^{k,s}(\phi, u) &\equiv \frac{1}{2k+1} \|\xi^k \phi\|_{L_\xi^2}^2 + \|\xi^k u\|_{L_\xi^2}^2 \\ &\quad + \frac{1}{(2k+1)^2} \|V(\xi^k \phi)\|_{Y^{k,s-1}}^2 + \|V^*(\xi^k u)\|_{X^{k,s-1}}^2.\end{aligned}\tag{13}$$

To close the energy estimates, s will be chosen as $s = [k] + 3$, where $[k]$ is a ceiling function, namely $[k] = \min\{n \in \mathbb{Z} : k \leq n\}$. We are now ready to state the main results.

Theorem 2.1. *Fix k , where $\frac{1}{2} < k < \infty$. Suppose initial data ϕ_0 and u_0 satisfy the following*

conditions:

$$(i) \mathcal{E}^{k, [k]+3}(\phi_0, u_0) < \infty; (ii) \frac{1}{C_0} \leq \frac{\phi_0}{\xi} \leq C_0 \text{ for some } C_0 > 1$$

There exist a time $T > 0$ only depending on $\mathcal{E}^{k, [k]+3}(\phi_0, u_0)$ and C_0 , and a unique solution (ϕ, u) to the reformulated Euler equations (10) with the boundary conditions (11) on the time interval $[0, T]$ satisfying

$$\mathcal{E}^{k, [k]+3}(\phi, u) \leq 2\mathcal{E}^{k, [k]+3}(\phi_0, u_0),$$

and moreover, the vacuum boundary behavior of ϕ is preserved on that time interval:

$$\frac{1}{2C_0} \leq \frac{\phi}{\xi} \leq 2C_0.$$

Remark 2.2. The evolution of the vacuum boundary $x = a(t)$ is given by $\dot{a}(t) = u(t, \xi = 0)$. By Theorem 2.1, one can derive that $u_\xi(t, \xi)$ is bounded and continuous in $(t, x) \in [0, T] \times [0, 1]$, and since $u(t, 0) = \int_1^0 u_\xi(t, \xi) d\xi$, we deduce that the vacuum interface is well-defined and within short time $t \leq T$, the vacuum boundary at time t stays close to the initial position with $|a(t) - a| \leq CT$ for some constant C depending on the initial energy in Theorem 2.1.

Remark 2.3. Different constants in the energy functional (13) are due to the nonlinearity of (5) or (10). The structure of the equations are to be systematic after applying V, V^* to (10) as in (40) and thereafter. Since we are dealing with the local existence theory, one may work with the energy functional $\mathcal{E}^{k, s} = \|\xi^k \phi\|_{X^{k, s}}^2 + \|\xi^k u\|_{Y^{k, s}}^2$, which is equivalent to (13).

Our work is a fundamental step in understanding the long time behavior of regular solutions and the vacuum boundary of Euler equations with the physical singularity rigorously. With or without damping case, it would be interesting to study whether our solution exists globally in time. In particular, for the damping case, the physical singularity is expected to be canonical as in the porous media equation, it would be also interesting to investigate the asymptotic relationship between our solution and regular solutions to the porous media equation.

Parallel to the recent progress in free surface boundary problems with geometry involved, we expect that our result can be generalized to multidimensional case, since the difficulty of the physical singularity lies in how the solution behaves with respect to the normal direction to the boundary. We leave this for future study. The physical vacuum boundary also naturally appears in Euler-Poisson equations for gaseous stars. It would be very interesting to study the behavior of solutions and the vacuum boundary under the influence of the gravitation.

The method of the proof is based on a careful study of V, V^* operators and V, V^* energy estimates. The first key ingredient is to establish the relationship between the multiplication with $\frac{1}{\xi}$, a common operation embedded in the equations (5) or (10), and V, V^* by using the underlying functional analytic properties of V, V^* which can be obtained from a thorough speculation on the behavior of V, V^* near the vacuum boundary $\xi = 0$. Another essential idea is to find the right form of spatial derivatives of $\partial_t \phi$, which is critical to cope with strong nonlinearity, in particular of the second and third order equations and to close the energy estimates in the end. For large k , this can be done by introducing the representation formula of $(V^*)^i(\xi^k u)$. In the similar vein, due to the strong nonlinearity, the approximate scheme starts from the third order equation and lower order terms including ϕ and u are recovered

by integrals and boundary conditions. Lastly, we point out that it is not trivial to build the well-posedness of linear approximate systems and the duality argument is employed for that as in [1].

The rest of the paper is organized as follows: In Section 3, we study the operators V and V^* built in the reformulated Euler equations (5) or (10). In Section 4, we establish the a priori estimates in V, V^* formulation. Based on the a priori estimates, in Section 5, we implement the approximate scheme and prove that each approximate system is well-posed. In Section 6, we finish the proof of Theorem 2.1. In Section 7, the duality argument is proved.

3 Preliminaries

Throughout this section, we assume that ϕ is a given nonnegative, smooth function of t and ξ , and moreover, $\frac{\phi}{\xi}$ and $\partial_\xi \phi$ are bounded from below and above near $\xi = 0$.

3.1 Basic properties of V and V^*

In this subsection, we study the operators V and V^* . Let us denote $C_0^\infty((0, 1))$ (respectively $C_0^\infty((0, 1])$) the set of C^∞ functions with compact support in $(0, 1)$ (respectively $(0, 1]$).

Lemma 3.1.

$$\begin{aligned} \overline{C_0^\infty((0, 1])}^{X^{k,1}} &= \mathcal{D}(V) = X^{k,1}; \\ \overline{C_0^\infty((0, 1))}^{Y^{k,1}} &= \mathcal{D}(V^*) = Y^{k,1}; \end{aligned} \tag{14}$$

Proof. Take $f \in X^{k,1}$. Hence,

$$\|f\|_{X^{k,1}}^2 = \int_0^1 \frac{1}{\xi^{2k}} |\partial_\xi (\frac{\phi^{2k}}{\xi^k} f)|^2 + |f|^2 d\xi < \infty.$$

We make the change of variable $y = \xi^{2k+1}$ and $F(y) = \frac{\phi^{2k}}{\xi^k} f$. Hence

$$\|f\|_{X^{k,1}}^2 = \int_0^1 (2k+1) |\partial_y F|^2 + \frac{1}{2k+1} \frac{\xi^{4k}}{\phi^{4k}} \frac{1}{y^{\frac{4k}{2k+1}}} |F|^2 dy < \infty.$$

Since, $2k > 1$, we deduce that $\frac{4k}{2k+1} > 1$ and hence, necessary $F(0) = 0$. Applying Hardy inequality to F , we deduce that

$$\int_0^1 \frac{F^2}{y^2} dy \leq C \int_0^1 |\partial_y F|^2.$$

Hence, going back to the original variables, we deduce that for $f \in X^{k,1}$, we have

$$\int_0^1 \frac{f^2}{\xi^2} d\xi \leq C \|\frac{\xi}{\phi}\|_{L^\infty}^{4k} \|V(f)\|_{L_\xi^2}^2. \tag{15}$$

Now, consider a cut-off function $\chi \in C^\infty(\mathbb{R})$ given by $\chi(\xi) = 0$ for $\xi \leq 1/2$ and $\chi(\xi) = 1$ for $\xi \geq 1$. We also define $\chi_n(\xi) = \chi(n\xi)$. For $f \in X^{k,1}$, we define $f_n(\xi) = \chi_n(\xi)f(\xi)$. Hence,

$f_n \in X^{k,1}$ and it is clear that f_n goes to f in L^2 . Moreover,

$$V(f - f_n) = (1 - \chi_n)V(f) - n\chi'(n\xi)\frac{\phi^{2k}}{\xi^{2k}}f.$$

The first term on the right hand side goes to zero in L^2_ξ when n goes to infinity. For the second term, we use the fact that

$$\int_0^1 |n\chi'(n\xi)\frac{\phi^{2k}}{\xi^{2k}}f|^2 d\xi \leq \|\frac{\phi}{\xi}\|_{L^\infty}^{4k} \int_0^1 \frac{f^2}{\xi^2} 1_{\frac{1}{2n} \leq \xi \leq \frac{1}{n}} d\xi$$

which goes to zero when n goes to infinity in view of (15). Hence, we deduce that f_n goes to f in $X^{k,1}$. Now, it is clear that f_n can be approximated in $X^{k,1}$ by functions which are in $C_0^\infty((0, 1])$. Indeed, one can just convolute f_n by some mollifier. Hence, the first equality of (14) holds.

To prove a similar result for V^* , we take $g \in Y^{k,1}$ and fix some $c \in (0, 1]$. For $\xi \in (0, c]$, we have

$$\begin{aligned} \left| \frac{g(\xi)}{\xi^k} \right| &= \left| - \int_\xi^c \partial_\xi \left(\frac{g(\xi')}{\xi'^k} \right) d\xi' + \frac{g(c)}{c^k} \right| \\ &\leq \left(\int_\xi^c \frac{\phi(\xi')^{4k}}{|\xi'|^{2k}} |\partial_\xi \left(\frac{g(\xi')}{\xi'^k} \right)|^2 d\xi' \int_\xi^c \frac{|\xi'|^{2k}}{\phi(\xi')^{4k}} d\xi' \right)^{1/2} + \left| \frac{g(c)}{c^k} \right| \\ &\leq \epsilon \xi^{\frac{1-2k}{2}} + \left| \frac{g(c)}{c^k} \right| \end{aligned}$$

where $\epsilon = C \int_0^c \frac{\phi(\xi')^{4k}}{|\xi'|^{2k}} |\partial_\xi \left(\frac{g(\xi')}{\xi'^k} \right)|^2 d\xi'$. By choosing c small enough, we can make ϵ small.

Hence, we deduce that for all $\epsilon > 0$, there exists a constant $C_\epsilon(g) = \left| \frac{g(c)}{c^k} \right|$ such that for all $\xi \in (0, 1]$, we have

$$|g(\xi)| \leq \epsilon \sqrt{\xi} + C_\epsilon \xi^k.$$

We denote $g_n = \chi_n g$, hence it is clear that g_n goes to g in L^2_ξ . Moreover,

$$V^*(g - g_n) = (1 - \chi_n)V^*(g) - n\chi'(n\xi)\frac{\phi^{2k}}{\xi^{2k}}g.$$

The second term on the right hand side satisfies

$$\int_0^1 |n\chi'(n\xi)\frac{\phi^{2k}}{\xi^{2k}}g|^2 d\xi \leq C \int_0^1 \left(\frac{\epsilon}{\xi} + C_\epsilon \xi^{2k-1} \right) 1_{\frac{1}{2n} \leq \xi \leq \frac{1}{n}} d\xi$$

Hence, since $\epsilon > 0$ is arbitrary, it goes to zero when n goes to infinity. Therefore, we deduce that g_n goes to g in $Y^{k,1}$. Now, it is clear that g_n can be approximated by functions which are in $C_0^\infty((0, 1])$ and the second equality of (14) follows. \square

Lemma 3.2. (1) For $f \in X^{k,1}$, $g \in Y^{k,1}$

$$\int V(f) \cdot g d\xi = \int f \cdot V^*(g) d\xi$$

- (2) $\ker V = \{0\}$ and $\ker V^* = \{0\}$
(3) V_t and V_t^* are commutators of V, V^* and ∂_t :

$$\partial_t V(f) = V(\partial_t f) + V_t(f), \quad \partial_t V^*(g) = V^*(\partial_t g) + V_t^*(g)$$

where

$$V_t(f) \equiv 2k \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{2k-1} \partial_t \phi}{\xi^k} f \right], \quad V_t^*(g) \equiv -2k \frac{\phi^{2k-1} \partial_t \phi}{\xi^k} \partial_\xi \left[\frac{1}{\xi^k} g \right]$$

In addition,

$$V_t^*(g) = -2k \frac{\partial_t \phi}{\phi} V^*(g) \quad \text{and} \quad V V_t^*(g) = V_t V^*(g).$$

Proof. The proof of this lemma directly follows from the density property proved in the previous lemma. It will be used frequently during the energy estimates. \square

The following lemma displays a key ingredient of the main estimates. Dividing by ξ is a common operation embedded in the equations (5) and (10). The lemma claims that the operation, which acts like derivatives when ξ approach 0, is completely controlled by modified derivatives V and V^* .

Lemma 3.3. (1) If $f \in X^{k,1}$ and $g \in Y^{k,1}$, then $\frac{f}{\xi}$ and $\frac{g}{\xi}$ are bounded in L_ξ^2 and we obtain the following inequalities:

$$\left\| \frac{f}{\xi} \right\|_{L_\xi^2} \leq C \left\| \frac{\xi}{\phi} \right\|_{L_\xi^\infty}^{2k} \|Vf\|_{L_\xi^2}, \quad \left\| \frac{g}{\xi} \right\|_{L_\xi^2} \leq C \left\{ \left\| \frac{\xi}{\phi} \right\|_{L_\xi^\infty}^{2k} \|V^*g\|_{L_\xi^2} \right\} \quad (16)$$

(2) More generally, if $f \in L_\xi^2$ satisfies $\frac{V(f)}{\xi^{m-1}} \in L_\xi^2$ for some nonnegative real number m , $m + k > \frac{3}{2}$, then

$$\left\| \frac{f}{\xi^m} \right\|_{L_\xi^2} \leq C \left\| \frac{\xi}{\phi} \right\|_{L_\xi^\infty}^{2k} \left\| \frac{Vf}{\xi^{m-1}} \right\|_{L_\xi^2}.$$

(3) Also, if g satisfies $\frac{g}{\xi^{m-1}} \in L_\xi^2$ and $\frac{V^*(g)}{\xi^{m-1}} \in L_\xi^2$ for some nonnegative real number m , $m < k + \frac{1}{2}$, then

$$\left\| \frac{g}{\xi^m} \right\|_{L_\xi^2} \leq C \left\| \frac{\xi}{\phi} \right\|_{L_\xi^\infty}^{2k} \left\| \frac{V^*g}{\xi^{m-1}} \right\|_{L_\xi^2} + \left\| \frac{g}{\xi^{m-1}} \right\|_{L_\xi^2}. \quad (17)$$

Here, m is not necessarily an integer. In practice, m will be chosen as $\frac{1}{2} < m \leq k$.

Proof. The point (1) was already proved for f in the previous lemma by Hardy inequality (see (15)). For the second inequality, consider first $g \in C_0^\infty((0, 1))$, hence

$$\int V^*g \cdot \frac{\xi^{2k}}{\phi^{2k}} \frac{g}{\xi} d\xi = - \int \partial_\xi \left(\frac{g}{\xi^k} \right) \cdot \xi^k \frac{g}{\xi} d\xi = \frac{2k-1}{2} \int \left| \frac{g}{\xi} \right|^2 d\xi + \frac{1}{2} |g(1)|^2.$$

and $g(1) = 0$. Since $k \neq \frac{1}{2}$ and

$$\int V^*g \cdot \frac{\xi^{2k}}{\phi^{2k}} \frac{g}{\xi} d\xi \leq \left\| \frac{\xi}{\phi} \right\|_{L_\xi^\infty}^{2k} \|V^*g\|_{L_\xi^2} \left\| \frac{g}{\xi} \right\|_{L_\xi^2},$$

we obtain the desired result. The case where $g \in Y^{k,1}$ follows by density.

Now, we concentrate on (2). For $\tilde{\phi}$ satisfying the same type of bounds as ϕ , we define $\tilde{V}_\alpha(f) = \frac{1}{\xi^\alpha} \partial_\xi (\frac{\tilde{\phi}^{2\alpha}}{\xi^\alpha} f)$. For f as in (2), we use that

$$\frac{V(f)}{\xi^{m-1}} = \frac{1}{\xi^{k+m-1}} \partial_\xi \left(\frac{\phi^{2k}}{\xi^{k-m+1}} \frac{f}{\xi^{m-1}} \right) = \tilde{V}_{k+m-1} \left(\frac{f}{\xi^{m-1}} \right)$$

where $\tilde{\phi}$ is given by $\tilde{\phi}^{2(k+m-1)} = \phi^{2k} \xi^{2(m-1)}$. Since, $k+m-1 > \frac{1}{2}$, we can apply the estimate (15) for V replaced by \tilde{V}_{k+m-1} and f replaced by $\frac{f}{\xi^{m-1}}$. This gives the desired bound.

For (3), we write

$$\frac{V^*(g)}{\xi^{m-1}} = -\frac{\phi^{2k}}{\xi^{2k}} \tilde{V}_{m-1-k} \left(\frac{g}{\xi^{m-1}} \right)$$

with $\tilde{\phi} = \xi$. Since, $m-1-k < -\frac{1}{2}$, \tilde{V}_{m-1-k} satisfies the same estimates as V^* , in particular the second estimate of (16) holds for \tilde{V}_{m-1-k} , hence (17) holds. \square

Remark 3.4. We note that the boundary conditions on f and g at $\xi = 0$ in Lemma 3.3 are imbedded in $X^{k,s}$ and $Y^{k,1}$, namely L_ξ^2 boundedness of Vf and V^*g forces f and g to vanish at $\xi = 0$. Actually, it forces them to be less than $\xi^{1/2}$.

As a direct result of Lemma 3.3, we obtain the following L_ξ^2 estimates for $\frac{f}{\xi^m}$ and $\frac{g}{\xi^m}$:

Corollary 3.5. For given nonnegative real number m , $m < k + \frac{3}{2}$, if $f \in X^{k,[m]}$, then there exists C_1 only depending on $\|\frac{\xi}{\phi}\|_{L_\xi^\infty}$ so that

$$\left\| \frac{f}{\xi^m} \right\|_{L_\xi^2} \leq C_1 \|f\|_{X^{k,[m]}}.$$

Also, for given nonnegative real number m , $m < k + \frac{1}{2}$, if $g \in Y^{k,[m]}$, then there exists C_2 only depending on $\|\frac{\xi}{\phi}\|_{L_\xi^\infty}$ so that

$$\left\| \frac{g}{\xi^m} \right\|_{L_\xi^2} \leq C_2 \|g\|_{Y^{k,[m]}}.$$

Proof. We apply (2) and (3) of Lemma 3.3 alternately until negative powers of ξ disappear. Note that the conditions on m come from the one in (3) of Lemma 3.3. \square

We next prove the Sobolev imbedding inequalities of V, V^* version, which will be useful tools to control nonlinear terms.

Lemma 3.6. If $f \in X^{k,[k]+1}$ and $g \in Y^{k,[k]+1}$, then there exist constants C_3 and C_4 only depending on $\|\frac{\xi}{\phi}\|_{L_\xi^\infty}$ so that

$$\left\| \frac{f}{\xi^k} \right\|_{L_\xi^\infty} \leq C_3 \|f\|_{X^{k,[k]+1}} \quad \text{and} \quad \left\| \frac{g}{\xi^k} \right\|_{L_\xi^\infty} \leq C_4 \|g\|_{Y^{k,[k]+1}}.$$

Proof. We start with g part. By the definition of V and V^* , one finds that

$$\partial_\xi \left[\frac{g}{\xi^k} \right] = -\frac{\xi^{2k}}{\phi^{2k}} \frac{V^*g}{\xi^k}$$

Thus, by Sobolev embedding theorem in one dimension, it suffices to show that

$$\frac{g}{\xi^k}, \frac{V^*g}{\xi^k} \in L_\xi^2.$$

This follows from Corollary 3.5. Hence, we conclude that L_ξ^∞ bound of $\frac{g}{\xi^k}$ is controlled by $\|g\|_{Y^{k, [k]+1}}$. For f part, we show that $\|\frac{\phi^{2k}}{\xi^{2k}} \frac{f}{\xi^k}\|_{L_\xi^\infty}$ is bounded by $\|f\|_{X^{k, [k]+1}}$. Note that

$$\partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \frac{f}{\xi^k} \right] = \frac{Vf}{\xi^k} - 2k \frac{\phi^{2k}}{\xi^{2k}} \frac{f}{\xi^{k+1}}.$$

Thus, by applying Corollary 3.5, we obtain the desired conclusion. \square

More generally, we obtain the following:

Lemma 3.7. *Let $0 \leq j < k - \frac{1}{2}$ be a given nonnegative number. If $f \in X^{k, [j]+1}$ and $g \in Y^{k, [j]+1}$, then there exist constants C_5 and C_6 only depending on $\|\frac{\xi}{\phi}\|_{L_\xi^\infty}$ so that*

$$\|\frac{f}{\xi^j}\|_{L_\xi^\infty} \leq C_5 \|f\|_{X^{k, [j]+1}} \quad \text{and} \quad \|\frac{g}{\xi^j}\|_{L_\xi^\infty} \leq C_6 \|g\|_{Y^{k, [j]+1}}.$$

Proof. We only treat $\frac{g}{\xi^j}$. Note that

$$\partial_\xi \left[\frac{g}{\xi^j} \right] = -\frac{\xi^{2k}}{\phi^{2k}} \frac{V^*g}{\xi^j} + (k-j) \frac{g}{\xi^{j+1}}$$

Hence, by the Sobolev embedding theorem, it suffices to show that $\frac{g}{\xi^j}$, $\frac{V^*g}{\xi^j}$ and $\frac{g}{\xi^{j+1}}$ are in L_ξ^2 . This follows from Corollary 3.5. Note that j has to be less than $k - \frac{1}{2}$. \square

Next we present the product rule for the operators V, V^* .

Lemma 3.8. *Let $f, g \in \mathcal{D}V \cap \mathcal{D}V^*$ be given. Let h be a given smooth function. The following identities hold:*

- $V^*f = -Vf + 2k \frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \frac{f}{\phi} = -Vf + \frac{2k}{2k+1} \frac{V(\xi^k \phi)}{\xi^k} \frac{f}{\phi}$
- $Vg = -V^*g + 2k \frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \frac{g}{\phi} = -V^*g + \frac{2k}{2k+1} \frac{V(\xi^k \phi)}{\xi^k} \frac{g}{\phi}$
- $V(fh) = V(f)h + f \frac{\phi^{2k}}{\xi^{2k}} \partial_\xi h$, $V^*(gh) = V^*(g)h - g \frac{\phi^{2k}}{\xi^{2k}} \partial_\xi h$
- $\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi h = Vh + k \frac{\phi^{2k}}{\xi^{2k}} \left(\frac{1}{\xi} - 2 \frac{\partial_\xi \phi}{\phi} \right) h = -V^*h + k \frac{\phi^{2k}}{\xi^{2k}} \frac{h}{\xi}$
- $\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi [fg] = V(f)g + \frac{\phi^{4k}}{\xi^{3k}} f \partial_\xi \left[\frac{\xi^k}{\phi^{2k}} g \right] = V(f)g - fV^*(g) + 2k \frac{\phi^{2k}}{\xi^{2k}} \left(\frac{1}{\xi} - \frac{\partial_\xi \phi}{\phi} \right) fg$

The lemma tells that when V or V^* act on a function h , depending on that function, they can yield $\frac{h}{\xi}$ or $\frac{V(\xi^k \phi)}{\xi^k} \frac{h}{\phi}$ besides Vh or V^*h .

3.2 Homogeneous operators \bar{V}, \bar{V}^*

Next we introduce homogeneous linear operators \bar{V} and \bar{V}^* of V and V^* as follows:

$$\bar{V}(f) \equiv \frac{1}{\xi^k} \partial_\xi [\xi^k f], \quad \bar{V}^*(g) \equiv -\xi^k \partial_\xi \left[\frac{g}{\xi^k} \right] \text{ and } g(\xi = 1) = 0. \quad (18)$$

These homogeneous operators are special cases of V and V^* for which ϕ is simply taken as ξ . Function spaces $\bar{X}^{k,s}$ and $\bar{Y}^{k,s}$ for s a given nonnegative integer are given as follows:

$$\begin{aligned} \bar{X}^{k,s} &\equiv \{f \in L_\xi^2 : (\bar{V})^i(f) \in L_\xi^2, 0 \leq i \leq s\} \\ \bar{Y}^{k,s} &\equiv \{g \in L_\xi^2 : (\bar{V}^*)^i(g) \in L_\xi^2, 0 \leq i \leq s\} \end{aligned} \quad (19)$$

where $(\bar{V})^i$ and $(\bar{V}^*)^i$ are defined as in (7) and (8). These spaces are equipped with the following norms

$$\|f\|_{\bar{X}^{k,s}}^2 \equiv \sum_{i=0}^s \|(\bar{V})^i(f)\|_{L_\xi^2}^2 \quad \text{and} \quad \|g\|_{\bar{Y}^{k,s}}^2 \equiv \sum_{i=0}^s \|(\bar{V}^*)^i(g)\|_{L_\xi^2}^2.$$

Indeed, \bar{V}, \bar{V}^* and V, V^* share many good properties; we summarize the analog of Lemma 3.3, 3.6, 3.7 and 3.8 for \bar{V}, \bar{V}^* in the following.

Lemma 3.9. (1) *If $f \in L_\xi^2$ satisfies $\frac{\bar{V}f}{\xi^{m-1}} \in L_\xi^2$ for some real number $m, m+k > \frac{3}{2}$, then*

$$\left\| \frac{f}{\xi^m} \right\|_{L_\xi^2}^2 \leq \left\| \frac{\bar{V}f}{\xi^{m-1}} \right\|_{L_\xi^2}^2$$

*Also, if g satisfies $\frac{g}{\xi^{m-1}} \in L_\xi^2$ and $\frac{\bar{V}^*g}{\xi^{m-1}} \in L_\xi^2$ for some real number $m, m < k + \frac{1}{2}$, then*

$$\left\| \frac{g}{\xi^m} \right\|_{L_\xi^2}^2 \leq \left\| \frac{\bar{V}^*g}{\xi^{m-1}} \right\|_{L_\xi^2}^2 + \left\| \frac{g}{\xi^{m-1}} \right\|_{L_\xi^2}^2.$$

(2) *If $f \in \bar{X}^{k, [k]+1}$ and $g \in \bar{Y}^{k, [k]+1}$, then we obtain*

$$\left\| \frac{f}{\xi^k} \right\|_{L_\xi^\infty} \leq C \|f\|_{\bar{X}^{k, [k]+1}} \quad \text{and} \quad \left\| \frac{g}{\xi^k} \right\|_{L_\xi^\infty} \leq C \|g\|_{\bar{Y}^{k, [k]+1}}.$$

(3) *Let $0 \leq j < k - \frac{1}{2}$ be a given nonnegative number. If $f \in \bar{X}^{k, [j]+1}$ and $g \in \bar{Y}^{k, [j]+1}$, then we obtain*

$$\left\| \frac{f}{\xi^j} \right\|_{L_\xi^\infty} \leq C \|f\|_{\bar{X}^{k, [j]+1}} \quad \text{and} \quad \left\| \frac{g}{\xi^j} \right\|_{L_\xi^\infty} \leq C \|g\|_{\bar{Y}^{k, [j]+1}}.$$

(4) *Product rule for \bar{V}, \bar{V}^* :*

$$\bar{V}^* f = -\bar{V} f + 2k \frac{f}{\xi}, \quad \bar{V}(fh) = \bar{V}(f)h + f \partial_\xi h, \quad \bar{V}^*(gh) = \bar{V}^*(g)h - g \partial_\xi h.$$

Now we show that two norms induced by V, V^* and \bar{V}, \bar{V}^* are equivalent.

Proposition 3.10. *Let ϕ be given so that $\|\xi^k \phi\|_{X^{k, [k]+2}} \leq A$ and $\frac{1}{C} < \frac{\xi}{\phi} < C$ for positive constants A, C . Then for any $f \in X^{k, [k]}$ and $g \in Y^{k, [k]}$, there exists a constant M depending only on A, C so that*

$$\frac{1}{M} \|f\|_{\bar{X}^{k, [k]}}^2 \leq \|f\|_{X^{k, [k]}}^2 \leq M \|f\|_{\bar{X}^{k, [k]}}^2, \quad \frac{1}{M} \|g\|_{\bar{Y}^{k, [k]}}^2 \leq \|g\|_{Y^{k, [k]}}^2 \leq M \|g\|_{\bar{Y}^{k, [k]}}^2. \quad (20)$$

Proof. We will only prove the second inequality (\leq). The other one can be shown in the same way. Let us start with Vf and V^*g .

$$\begin{aligned} Vf &= \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^k} f \right] = \frac{\phi^{2k}}{\xi^{2k}} \bar{V}f + \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \right] f = \frac{\phi^{2k}}{\xi^{2k}} \bar{V}f + \left\{ \frac{2k}{2k+1} \frac{V(\xi^k \phi)}{\xi^k \phi} - 2k \frac{\phi^{2k}}{\xi^{2k+1}} \right\} f \\ V^*g &= -\frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{g}{\xi^k} \right] = \frac{\phi^{2k}}{\xi^{2k}} \bar{V}^*g \end{aligned}$$

Next, note that by the product rule of Lemma 3.8 and 3.9, the higher order terms $(V)^i f$ and $(V^*)^i g$ can be expanded into the following form in terms of $(\bar{V})^j f$ and $(\bar{V}^*)^j g$ for $j \leq i$:

$$(V)^i f = \sum_{j=0}^i \Psi_j^1(\xi^k \phi) \cdot (\bar{V})^{i-j} f, \quad (V^*)^i g = \sum_{j=0}^{i-1} \Psi_j^2(\xi^k \phi) \cdot (\bar{V}^*)^{i-j} g$$

where for $s = 1$ or 2

$$\Psi_j^s(\xi^k \phi) \equiv \sum_{r=0}^j \left\{ C_{rs} \frac{1}{\xi^r} \cdot \sum_{\substack{l_1 + \dots + l_p = j-r \\ l_1, \dots, l_p \geq 1}} C_{l_1 \dots l_p s} \prod_{q=1}^p \frac{(V)^{l_q}(\xi^k \phi)}{\xi^k \phi} \right\}$$

for some functions $C_{rs}, C_{l_1 \dots l_p s}$ which may only depend on $k, \frac{\phi}{\xi}$ and $\frac{\xi}{\phi}$ and therefore $C_{rs}, C_{l_1 \dots l_p s}$ are bounded by some power function of C . In order to show (20), first we rewrite $(V)^i f$ and $(V^*)^i g$ as

$$(V)^i f = \sum_{j=0}^i \xi^j \Psi_j^1(\xi^k \phi) \cdot \frac{(\bar{V})^{i-j} f}{\xi^j}, \quad (V^*)^i g = \sum_{j=0}^{i-1} \xi^j \Psi_j^2(\xi^k \phi) \cdot \frac{(\bar{V}^*)^{i-j} g}{\xi^j}$$

Since $\|\frac{(\bar{V})^{i-j} f}{\xi^j}\|_{L_\xi^2}$ and $\|\frac{(\bar{V}^*)^{i-j} g}{\xi^j}\|_{L_\xi^2}$ are bounded by $\|f\|_{\bar{X}^{k, j}}$ and $\|g\|_{\bar{Y}^{k, j}}$ respectively and moreover, by Lemma 3.7, $\|\xi^j \Psi_j^s\|_{L_\xi^\infty}$ is bounded by $\|\xi^k \phi\|_{X^{k, j+2}}$ for $j \leq [k]$, by adding all the inequalities over $i \leq [k]$ we obtain the desired inequality as well as the desired bound M as a function of A and C . \square

The above equivalence dictates the linear character of higher order energy. Note that we cannot deduce the same result for the full energy $\mathcal{E}^{k, [k]+3}$ due to the nonlinearity.

4 V, V^* a priori energy estimates

This section is devoted to V, V^* energy estimates to get the following a priori estimates, a key to construct strong solutions.

Proposition 4.1. *Suppose ϕ and u solve (10) and (11) with $\mathcal{E}^{k, \lceil k \rceil + 3}(\phi, u) < \infty$. If we further assume that*

$$\frac{1}{C} \leq \frac{\phi}{\xi} \leq C \text{ for some } C > 1, \quad (21)$$

we obtain the following a priori estimates:

$$\frac{d}{dt} \mathcal{E}^{k, \lceil k \rceil + 3}(\phi, u) \leq \mathcal{C}(\mathcal{E}^{k, \lceil k \rceil + 3}(\phi, u))$$

where $\mathcal{C}(\mathcal{E}^{k, \lceil k \rceil + 3}(\phi, u))$ is a continuous function of $\mathcal{E}^{k, \lceil k \rceil + 3}(\phi, u)$ and C . Moreover, the a priori assumption (21) can be justified: the boundedness of $\mathcal{E}^{k, \lceil k \rceil + 3}(\phi, u)$ imply the boundedness of $\frac{\phi}{\xi}$.

In order to illustrate the idea of the proof, we start with the simplest case when $k = 1$ of which corresponding γ is 3. In the next subsections, we generalize it to arbitrary $k > \frac{1}{2}$.

4.1 A priori estimates for $k = 1$ ($\gamma = 3$)

When $k = 1$, the Euler equations read as follows:

$$\begin{aligned} \phi_t + \left(\frac{\phi}{\xi}\right)^2 u_\xi &= 0 \\ u_t + \left(\frac{\phi}{\xi}\right)^2 \phi_\xi &= 0 \end{aligned} \quad (22)$$

The operators V and V^* take the form:

$$V(f) \equiv \frac{1}{\xi} \partial_\xi \left(\frac{\phi^2}{\xi} f \right), \quad V^*(g) \equiv -\frac{\phi^2}{\xi} \partial_\xi \left(\frac{1}{\xi} g \right)$$

In terms of V and V^* , (22) can be rewritten as follows:

$$\begin{aligned} \partial_t(\xi\phi) - V^*(\xi u) &= 0 \\ \partial_t(\xi u) + \frac{1}{3}V(\xi\phi) &= 0 \end{aligned} \quad (23)$$

The energy functional $\mathcal{E}^{1,4}(\phi, u)$ reads as the following:

$$\mathcal{E}^{1,4}(\phi, u) \equiv \int \frac{1}{3} |\xi\phi|^2 + |\xi u|^2 d\xi + \sum_{i=1}^4 \int \frac{1}{9} |(V)^i(\xi\phi)|^2 + |(V^*)^i(\xi u)|^2 d\xi \quad (24)$$

Before carrying out the energy estimates, we verify the assumption (21). In order to do so, we examine each term in the energy functional (24). Let us start with V , V^*V , VV^*V ,

V^*VV^*V of $\xi\phi$.

- $V(\xi\phi) = \frac{1}{\xi}\partial_\xi[\phi^3] = 3\frac{\phi^2}{\xi}\partial_\xi\phi$
- $V^*V(\xi\phi) = -3\frac{\phi^2}{\xi}\partial_\xi[\frac{\phi^2}{\xi^2}\partial_\xi\phi]$
- $VV^*V(\xi\phi) = -\frac{3}{\xi}\partial_\xi[\frac{\phi^4}{\xi^2}\partial_\xi[\frac{\phi^2}{\xi^2}\partial_\xi\phi]]$
- $V^*VV^*V(\xi\phi) = 3\frac{\phi^2}{\xi}\partial_\xi[\frac{1}{\xi^2}\partial_\xi[\frac{\phi^4}{\xi^2}\partial_\xi[\frac{\phi^2}{\xi^2}\partial_\xi\phi]]]$

One advantage of V and V^* formulation is that L_ξ^∞ control of ϕ is cheap to get: since

$$\int \xi\phi \cdot V(\xi\phi)d\xi = \int \phi\partial_\xi[\phi^3]d\xi = \frac{3}{4}\phi^4,$$

we deduce that

$$\|\phi\|_{L_\xi^\infty}^4 \leq \|\xi\phi\|_{L_\xi^2}^2 + \|V(\xi\phi)\|_{L_\xi^2}^2 \leq \mathcal{E}^{1,4}(\phi, u) \quad (25)$$

The boundedness of $\partial_\xi\phi$ also follows from the boundedness of $\mathcal{E}^{1,4}(\phi, u)$: first note that

$$\frac{\phi^2}{\xi^2}\partial_\xi\phi = \frac{1}{3}\frac{V(\xi\phi)}{\xi}$$

and then by applying Lemma 3.6 to $g = V(\xi\phi)$ when $k = 1$, we deduce that $\frac{\phi^2}{\xi^2}\partial_\xi\phi$ is bounded and continuous, and in turn $\partial_\xi\phi$ is in L_ξ^∞ under the assumption (21). In the same way, by applying Lemma 3.6 to $f = V^*V(\xi\phi)$ when $k = 1$, we can derive that $\partial_\xi[\frac{\phi^2}{\xi^2}\partial_\xi\phi]$ is bounded. However, we remark that this does not imply that $\partial_\xi^2\phi$ is bounded in our energy space, since it is not clear how to control $\partial_\xi[\frac{\phi}{\xi}]$. Thus we keep the form as they are rather than try to go back to the standard Sobolev space.

Next we turn to u variable. We list out V^* , VV^* , V^*VV^* , VV^*VV^* of ξu .

- $V^*(\xi u) = -\frac{\phi^2}{\xi}\partial_\xi u$
- $VV^*(\xi u) = -\frac{1}{\xi}\partial_\xi[\frac{\phi^4}{\xi^2}\partial_\xi u]$
- $V^*VV^*(\xi u) = \frac{\phi^2}{\xi}\partial_\xi[\frac{1}{\xi^2}\partial_\xi[\frac{\phi^4}{\xi^2}\partial_\xi u]]$
- $VV^*VV^*(\xi u) = \frac{1}{\xi}\partial_\xi[\frac{\phi^4}{\xi^2}\partial_\xi[\frac{1}{\xi^2}\partial_\xi[\frac{\phi^4}{\xi^2}\partial_\xi u]]]$

Note that $\frac{\partial_t\phi}{\xi}$ can be estimated in terms of $\partial_\xi u$ via the equation (22):

$$\partial_t\phi = -\frac{\phi^2}{\xi^2}\partial_\xi u = \frac{V^*(\xi u)}{\xi}$$

Apply Lemma 3.6 to deduce that $\frac{VV^*(\xi u)}{\xi} = \frac{1}{\xi^2}\partial_\xi[\phi^2\partial_t\phi]$ is bounded and continuous if

$\mathcal{E}^{1,4}(\phi, u)$ is bounded. Letting h be $\frac{1}{\xi^2}\partial_\xi[\phi^2\partial_t\phi]$, we can write $\phi^2\partial_t\phi = \int_0^\xi \xi'^2 h d\xi'$, and therefore we conclude $\frac{\partial_t\phi}{\xi}$ is also bounded by $\mathcal{E}^{1,4}(\phi, u)$ and C .

Writing $\frac{\phi}{\xi}$ as

$$\frac{\phi}{\xi}(t, \xi) = \frac{\phi}{\xi}(0, \xi) + \int_0^t \frac{\partial_t\phi}{\xi}(\tau, \xi) d\tau,$$

we conclude that for a short time, the boundary behavior of $\frac{\phi}{\xi}$ is preserved and in particular, this justifies the assumption (21).

We now perform the energy estimates. The zeroth order energy energy is conserved as given by (12). Apply V and V^* to (23) and use $V_t(\xi\phi) = \frac{2}{3}\partial_t V(\xi\phi)$

$$\begin{aligned} \partial_t V(\xi\phi) - 3VV^*(\xi u) &= 0 \\ \partial_t V^*(\xi u) + \frac{1}{3}V^*V(\xi\phi) &= V_t^*(\xi u) \end{aligned} \quad (27)$$

Multiply by $\frac{1}{9}V(\xi\phi)$ and $V^*(\xi u)$ and integrate to get

$$\frac{1}{2} \frac{d}{dt} \int \frac{1}{9} |V(\xi\phi)|^2 + |V^*(\xi u)|^2 d\xi = \int V_t^*(\xi u) \cdot V^*(\xi u) d\xi \quad (28)$$

Apply V^* and V to (27) to get

$$\begin{aligned} \partial_t V^*V(\xi\phi) - 3V^*VV^*(\xi u) &= V_t^*V(\xi\phi) \\ \partial_t VV^*(\xi u) + \frac{1}{3}VV^*V(\xi\phi) &= VV_t^*(\xi u) + V_tV^*(\xi u) = 2V_tV^*(\xi u) \end{aligned} \quad (29)$$

Multiply by $\frac{1}{9}V^*V(\xi\phi)$ and $VV^*(\xi u)$ and integrate to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \frac{1}{9} |V^*V(\xi\phi)|^2 + |VV^*(\xi u)|^2 d\xi &= \int \frac{1}{9} V_t^*V(\xi\phi) \cdot V^*V(\xi\phi) d\xi \\ &\quad + \int 2V_tV^*(\xi u) \cdot VV^*(\xi u) d\xi \end{aligned} \quad (30)$$

Apply V^* and V to (29) to get

$$\begin{aligned} \partial_t VV^*V(\xi\phi) - 3VV^*VV^*(\xi u) &= VV_t^*V(\xi\phi) + V_tV^*V(\xi\phi) = 2V_tV^*V(\xi\phi) \\ \partial_t V^*VV^*(\xi u) + \frac{1}{3}V^*VV^*V(\xi\phi) &= 2V^*V_tV^*(\xi u) + V_t^*VV^*(\xi u) \end{aligned} \quad (31)$$

Multiply by $\frac{1}{9}VV^*V(\xi\phi)$ and $V^*VV^*(\xi u)$ and integrate to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \frac{1}{9} |VV^*V(\xi\phi)|^2 + |V^*VV^*(\xi u)|^2 d\xi &= \int \frac{2}{9} V_tV^*V(\xi\phi) \cdot VV^*V(\xi\phi) d\xi \\ &\quad + \int 2V^*V_tV^*(\xi u) \cdot V^*VV^*(\xi u) d\xi + \int V_t^*VV^*(\xi u) \cdot V^*VV^*(\xi u) d\xi \end{aligned} \quad (32)$$

Apply V^* and V to (31) to get

$$\begin{aligned} \partial_t V^* V V^* V(\xi\phi) - 3V^* V V^* V V^*(\xi u) &= 2V^* V_t V^* V(\xi\phi) + V_t^* V V^* V(\xi\phi) \\ \partial_t V V^* V V^*(\xi u) + \frac{1}{3} V V^* V V^* V(\xi\phi) &= 2V V^* V_t V^*(\xi u) + 2V_t V^* V V^*(\xi u) \end{aligned} \quad (33)$$

Multiply by $\frac{1}{9}V^*V V^*V(\xi\phi)$ and $V V^*V V^*(\xi u)$ and integrate to get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int \frac{1}{9} |V^* V V^* V(\xi\phi)|^2 + |V V^* V V^*(\xi u)|^2 d\xi \\ &= \int \frac{2}{9} V^* V_t V^* V(\xi\phi) \cdot V^* V V^* V(\xi\phi) d\xi + \int \frac{1}{9} V_t^* V V^* V(\xi\phi) \cdot V^* V V^* V(\xi\phi) d\xi \\ &\quad + \int 2V V^* V_t V^*(\xi u) \cdot V V^* V V^*(\xi u) d\xi + \int 2V_t V^* V V^*(\xi u) \cdot V V^* V V^*(\xi u) d\xi \end{aligned} \quad (34)$$

In order to prove Proposition 4.1, it now remains to estimate the following nonlinear terms coming from the energy estimates in terms of the energy functional (24):

$$\begin{aligned} &\int 2V_t V^*(\xi u) \cdot V V^*(\xi u) d\xi, \quad \int \frac{2}{9} V_t V^* V(\xi\phi) \cdot V V^* V(\xi\phi) d\xi, \\ &\int 2V^* V_t V^*(\xi u) \cdot V^* V V^*(\xi u) d\xi, \quad \int \frac{2}{9} V^* V_t V^* V(\xi\phi) \cdot V^* V V^* V(\xi\phi) d\xi, \\ &\int 2V V^* V_t V^*(\xi u) \cdot V V^* V V^*(\xi u) d\xi, \quad \int 2V_t V^* V V^*(\xi u) \cdot V V^* V V^*(\xi u) d\xi \end{aligned} \quad (35)$$

Our goal is to control these terms by our energy functional (24). All of them contain $\partial_t \phi$ and its derivatives with suitable weights. The estimates of $\partial_t \phi$ related terms can be obtained through the equation (22) by estimating V^* , $V V^*$, $V^* V V^*$, $V V^* V V^*$ of ξu in terms of $\partial_t \phi$. Before going any further, let us try to get the better understanding of them. First, we rewrite V^* , $V V^*$, $V^* V V^*$, $V V^* V V^*$ of ξu , namely (26), in terms of $\partial_t \phi$:

- $V^*(\xi u) = -\frac{\phi^2}{\xi} \partial_\xi u = \xi \partial_t \phi$
- $V V^*(\xi u) = \frac{1}{\xi} \partial_\xi [\phi^2 \partial_t \phi] = \frac{\phi^3}{\xi} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] + 3 \frac{\phi}{\xi} \partial_\xi \phi \partial_t \phi = \boxed{\frac{\phi^3}{\xi} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]} + V(\xi\phi) \frac{\partial_t \phi}{\phi}$
- $V^* V V^*(\xi u) = -\frac{\phi^2}{\xi} \partial_\xi \left[\frac{\phi^3}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] - 3 \frac{\phi^2}{\xi} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi} - 3 \frac{\phi^2}{\xi} \frac{\phi^2}{\xi^2} \partial_\xi \phi \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]$
 $= -\frac{1}{\xi \phi} \partial_\xi \left[\frac{\phi^6}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] - 3 \frac{\phi^2}{\xi} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi}$
 $= \boxed{-\frac{1}{\xi \phi} \partial_\xi \left[\frac{\phi^6}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right]} + V^* V(\xi\phi) \frac{\partial_t \phi}{\phi}$
- $V V^* V V^*(\xi u) = -\frac{1}{\xi} \partial_\xi \left[\frac{\phi}{\xi^2} \partial_\xi \left[\frac{\phi^6}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] \right] - 3 \frac{1}{\xi} \partial_\xi \left[\frac{\phi^4}{\xi^2} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] \right] \frac{\partial_t \phi}{\phi}$
 $= \boxed{-\frac{1}{\xi} \partial_\xi \left[\frac{\phi}{\xi^2} \partial_\xi \left[\frac{\phi^6}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] \right]} + V V^* V(\xi\phi) \frac{\partial_t \phi}{\phi} + \frac{V^* V(\xi\phi)}{\xi} \cdot \frac{\phi^2}{\xi} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]$

As we can see in the above, the term $\frac{\partial_t \phi}{\phi}$ and its derivatives naturally appear and there are many ways to write them. The key idea is not to separate them randomly when distributing spatial derivatives, but to find the right form of each term. The boxed terms in (36) have been chosen in such a way that the remaining terms in the right hand sides have the better or the same integrability as the left hand sides.

We analyze the most intriguing full derivative terms $V^*V_tV^*V(\xi\phi)$ and $VV^*V_tV^*(\xi u)$. Other terms can be handled in a rather direct way.

Claim 4.2.

$$\|V^*V_tV^*V(\xi\phi)\|_{L_\xi^2}^2 \leq C_1(\mathcal{E}^{1,4}(\phi, u))$$

where $C_1(\mathcal{E}^{1,4}(\phi, u))$ is a continuous function of $\mathcal{E}^{1,4}(\phi, u)$.

Proof.

$$\begin{aligned} \frac{1}{6}V^*V_tV^*V(\xi\phi) &= \frac{\phi^2}{\xi}\partial_\xi\left[\frac{1}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\frac{\phi^4}{\xi^2}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right]\right]\right] \\ &= \frac{\phi^2}{\xi}\partial_\xi\left[\frac{1}{\xi^2}\frac{\partial_t\phi}{\phi}\partial_\xi\left[\frac{\phi^4}{\xi^2}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right]\right]\right] + \frac{\phi^2}{\xi}\partial_\xi\left[\frac{1}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\frac{\phi^4}{\xi^2}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right]\right] \\ &= \frac{\partial_t\phi}{\phi}\frac{\phi^2}{\xi}\partial_\xi\left[\frac{1}{\xi^2}\partial_\xi\left[\frac{\phi^4}{\xi^2}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right]\right]\right] + 2\frac{\phi^2}{\xi}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\cdot\frac{1}{\xi^2}\partial_\xi\left[\frac{\phi^4}{\xi^2}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right]\right] \\ &\quad + \frac{\phi^2}{\xi}\partial_\xi\left[\frac{1}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right]\cdot\frac{\phi^4}{\xi^2}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right] \\ &\equiv (I) + (II) + (III) \end{aligned}$$

Since $(I) = \frac{\partial_t\phi}{\phi}V^*VV^*V(\xi\phi)$, (I) is controllable:

$$\|(I)\|_{L_\xi^2} \leq \left\|\frac{\partial_t\phi}{\phi}\right\|_{L_\xi^\infty}\|V^*VV^*V(\xi\phi)\|_{L_\xi^2}$$

The second term is written as

$$(II) = -\frac{2}{3}\frac{\phi^2}{\xi}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\cdot\frac{VV^*V(\xi\phi)}{\xi}$$

From (36), we get

$$\frac{\phi^2}{\xi}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right] = \frac{VV^*(\xi u)}{\phi} - \frac{V(\xi\phi)}{\phi}\frac{\partial_t\phi}{\phi} \in L_\xi^\infty.$$

And since $\left\|\frac{VV^*V(\xi\phi)}{\xi}\right\|_{L_\xi^2} \leq \|V^*VV^*V(\xi\phi)\|_{L_\xi^2}$, (II) is also controlled by the energy. Now we turn into (III) . First, we note that by Lemma 3.3 and Lemma 3.6,

$$\frac{1}{\xi}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right] \in L_\xi^2 \quad \text{and} \quad \partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right] \in L_\xi^\infty.$$

Next let us take a look at the other factor and rewrite it by using boxed terms in (36):

$$\begin{aligned} \frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{1}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right] &= \frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{1}{\phi^6}\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right] = \frac{1}{\xi^2}\partial_\xi\left[\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right] - 6\frac{\phi^2}{\xi^2}\partial_\xi\phi\cdot\frac{\phi^3}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right] \\ &= -\frac{\phi}{\xi}V^*VV^*(\xi u) + \frac{V^*V(\xi\phi)}{\xi}\frac{V^*(\xi u)}{\xi} - 2\frac{V(\xi\phi)}{\xi}\frac{VV^*(\xi u)}{\xi} + 2\frac{\xi}{\phi}\left|\frac{V(\xi\phi)}{\xi}\right|^2\frac{V^*(\xi u)}{\xi^2} \end{aligned}$$

Thus (III) can be rewritten as follows:

$$(III) = -\frac{\phi}{\xi} \frac{V^* V V^*(\xi u)}{\xi} \cdot \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] - 2 \frac{V(\xi \phi)}{\xi} \frac{V V^*(\xi u)}{\xi} \cdot \frac{1}{\xi} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] \\ + \frac{V^* V(\xi \phi)}{\xi} \frac{V^*(\xi u)}{\xi} \cdot \frac{1}{\xi} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] + 2 \frac{\xi}{\phi} \left| \frac{V(\xi \phi)}{\xi} \right|^2 \frac{V^*(\xi u)}{\xi^2} \cdot \frac{1}{\xi} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right]$$

Hence (III) can be controlled by $\mathcal{E}^{1,4}(\phi, u)$. \square

Now let us move onto $V V^* V_t V^*(\xi u)$. The treatment of this term contains another flavor.

Claim 4.3.

$$\|V V^* V_t V^*(\xi u)\|_{L_\xi^2}^2 \leq \mathcal{C}_2(\mathcal{E}^{1,4}(\phi, u))$$

where $\mathcal{C}_2(\mathcal{E}^{1,4}(\phi, u))$ is a continuous function of $\mathcal{E}^{1,4}(\phi, u)$.

Proof. We use the continuity equation in (22) first to deal with $V_t V^*(\xi u)$. Since $V^*(\xi u) = \xi \partial_t \phi$ and $V(\xi \partial_t \phi) = \frac{1}{\xi} \partial_\xi [\phi^2 \partial_t \phi] = \frac{1}{\xi} [\phi^2 \partial_\xi \partial_t \phi + 2\phi \partial_\xi \phi \partial_t \phi]$, we can write $V_t V^*(\xi u)$ as the following:

$$\frac{1}{2} V_t V^*(\xi u) = \frac{1}{\xi} \partial_\xi [\phi |\partial_t \phi|^2] = \frac{1}{\xi} [\partial_\xi \phi |\partial_t \phi|^2 + 2\phi \partial_t \partial_\xi \phi \partial_t \phi] = 2 \frac{\partial_t \phi}{\phi} V(\xi \partial_t \phi) - 3 \frac{\partial_\xi \phi |\partial_t \phi|^2}{\xi}$$

Apply V^* :

$$\frac{1}{2} V^* V_t V^*(\xi u) = -\frac{\phi^2}{\xi} \partial_\xi \left[2 \frac{\partial_t \phi}{\phi} \frac{V V^*(\xi u)}{\xi} - 3 \frac{\partial_\xi \phi |\partial_t \phi|^2}{\xi^2} \right] \\ = 2 \frac{\partial_t \phi}{\phi} V^* V V^*(\xi u) - 2 \frac{\phi^2}{\xi} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \cdot \frac{V V^*(\xi u)}{\xi} + 3 \frac{\phi^2}{\xi} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] \frac{|\partial_t \phi|^2}{\phi} \\ = 2 \frac{\partial_t \phi}{\phi} V^* V V^*(\xi u) - 2 \frac{\phi^2}{\xi} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \underbrace{\left\{ \frac{V V^*(\xi u)}{\xi} - 3 \frac{\partial_t \phi \phi^2}{\phi \xi^2} \partial_\xi \phi \right\}}_{(*)} + 3 \frac{\phi^2}{\xi} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] \frac{|\partial_t \phi|^2}{\phi}$$

Note that $(*)$ reduces to:

$$(*) = \frac{1}{\xi^2} \partial_\xi \left[\phi^3 \frac{\partial_t \phi}{\phi} \right] - 3 \frac{\partial_t \phi \phi^2}{\phi \xi^2} \partial_\xi \phi = \frac{\phi^3}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]$$

Apply V :

$$\frac{1}{2} V V^* V_t V^*(\xi u) \\ = 2 \frac{1}{\xi} \partial_\xi \left[\frac{\phi^2}{\xi} \frac{\partial_t \phi}{\phi} V^* V V^*(\xi u) \right] - 2 \frac{1}{\xi} \partial_\xi \left[\frac{1}{\phi^5} \left| \frac{\phi^6}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right|^2 \right] + 3 \frac{1}{\xi} \partial_\xi \left[\frac{\phi^4}{\xi^2} \partial_\xi \left[\frac{\phi^2}{\xi^2} \partial_\xi \phi \right] \frac{|\partial_t \phi|^2}{\phi} \right] \\ = (I) + (II) + (III)$$

We rewrite (I), (II), (III) as follows:

$$\begin{aligned}
(I) &= 2\frac{\partial_t\phi}{\phi}VV^*VV^*(\xi u) + 2\frac{\phi^2}{\xi}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\frac{V^*VV^*(\xi u)}{\xi} \\
(II) &= 10\frac{\partial_\xi\phi}{\xi\phi^6}\left|\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right|^2 - 4\frac{1}{\xi\phi^5}\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right] \cdot \partial_\xi\left[\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right] \\
(III) &= 3\frac{1}{\xi}\partial_\xi\left[\frac{\phi^4}{\xi^2}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right]\right]\left|\frac{\partial_t\phi}{\phi}\right|^2 + 6\frac{1}{\xi}\frac{\phi^4}{\xi^2}\partial_\xi\left[\frac{\phi^2}{\xi^2}\partial_\xi\phi\right] \cdot \frac{\partial_t\phi}{\phi}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]
\end{aligned}$$

It is easy to see that (I) and (III) can be controlled by the energy functional. On the other hand, in order to take care of (II), a special attention is needed. Since $\frac{V^*VV^*(\xi u)}{\xi}$ is bounded by $VV^*VV^*(\xi u)$ in L_ξ^2 , we obtain

$$h \equiv \frac{1}{\xi^3}\partial_\xi\left[\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right] \in L_\xi^2. \quad (37)$$

Thus the second term in (II) is bounded by the energy functional. To prove that the first term is also bounded, we claim

$$\frac{1}{\xi^7}\left|\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right|^2 \in L_\xi^2.$$

By using (37), rewrite $\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]$ as follows:

$$\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right] = \int_0^\xi \zeta^3 h d\zeta$$

Applying Hölder inequality, we observe that

$$\left|\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right|^2 \leq \xi^7 \|h\|_{L_\xi^2}^2.$$

Hence we get

$$\left\|\frac{1}{\xi^7}\left|\frac{\phi^6}{\xi^2}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right|^2\right\|_{L_\xi^2} \leq \|h\|_{L_\xi^2}^2.$$

This finishes the a priori estimates for $k = 1$ as well as the claim. \square

4.2 The case when $\frac{1}{2} < k < 1$ ($\gamma > 3$)

In this subsection, we prove Proposition 4.1 for the case $\frac{1}{2} < k < 1$ and $s = 4$. First, by Lemma 3.6, one finds that $\partial_\xi\phi$, $\frac{\xi}{\phi}$, and $\frac{\partial_t\phi}{\xi}$ are bounded by the energy functional $\mathcal{E}^{k,4}(\phi, u)$. We recall the reformulated Euler equations (10). One can apply V, V^* alternately as in the case $k = 1$, and integrate to get

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^4\int\frac{1}{(2k+1)^2}|(V)^i(\xi^k\phi)|^2+|(V^*)^i(\xi^k u)|^2d\xi\leq\text{mixed nonlinear terms},$$

where the mixed terms have the same form in (35). Thus it suffices to show that those nonlinear terms are bounded by $\mathcal{E}^{k,4}(\phi, u)$. We focus on the intriguing term $VV^*V_tV^*(\xi^k u)$

as well as $V^*V_tV^*(\xi^k u)$, $V_tV^*(\xi^k u)$. As we saw in the case $k = 1$, in order to treat the mixed terms, the careful analysis of $\partial_t\phi$ terms is required. First, take a look at $V^*(\xi^k u)$ and $VV^*(\xi^k u)$.

- $V^*(\xi^k u) = -\frac{\phi^{2k}}{\xi^k} \partial_\xi u = \xi^k \partial_t \phi$
- $VV^*(\xi^k u) = \frac{1}{\xi^k} \partial_\xi [\phi^{2k} \partial_t \phi] = \frac{\phi^{2k+1}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] + (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \phi \frac{\partial_t \phi}{\phi}$
 $= \boxed{\frac{\phi^{2k+1}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]} + V(\xi^k \phi) \frac{\partial_t \phi}{\phi}$

Because $\|\frac{V(\xi^k \phi)}{\xi}\|_{L_\xi^2}$ and $\|\frac{VV^*(\xi^k u)}{\xi}\|_{L_\xi^2}$ are bounded by $\mathcal{E}^{k,4}$ as an application of Lemma 3.3, we deduce that $\|\phi^k \partial_\xi [\frac{\partial_t \phi}{\phi}]\|_{L_\xi^2}$ is also bounded by $\mathcal{E}^{k,4}$. And by Lemma 3.6, we obtain $\phi \partial_\xi [\frac{\partial_t \phi}{\phi}] \in L_\xi^\infty$. Now let us compute $V_tV^*(\xi^k u)$ and write in terms of the energy:

$$\begin{aligned} \frac{1}{2k} V_tV^*(\xi^k u) &= \frac{1}{\xi^k} \partial_\xi [\phi^{2k+1} |\frac{\partial_t \phi}{\phi}|^2] = \frac{\partial_t \phi}{\phi} \left\{ 2 \frac{\phi^{2k+1}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] + (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \phi \frac{\partial_t \phi}{\phi} \right\} \\ &= 2 \frac{\partial_t \phi}{\phi} VV^*(\xi^k u) - |\frac{\partial_t \phi}{\phi}|^2 V(\xi^k \phi) \end{aligned}$$

It is clear that this term is bounded by the energy functional. Next we write out $V^*VV^*(\xi^k u)$.

- $V^*VV^*(\xi^k u) = -\frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k+1}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right]$
 $- (2k+1) \left\{ \frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi} + \frac{\phi^{4k}}{\xi^{3k}} \partial_\xi \phi \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right\}$
 $= -\frac{1}{\xi^k \phi} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] - (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi}$
 $= \boxed{-\frac{1}{\xi^k \phi} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right]} + V^*V(\xi^k \phi) \frac{\partial_t \phi}{\phi}$

(38)

Thus we note that the boxed term $\frac{1}{\xi^k \phi} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right]$ is the right form of the second derivative of $\partial_t \phi$ of which structure we do not want to destroy. Here is $V^*V_tV^*(\xi^k u)$:

$$\begin{aligned} \frac{1}{2k} V^*V_tV^*(\xi^k u) &= -\frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{1}{\xi^k} \frac{\partial_t \phi}{\phi} \left\{ 2 \frac{\phi^{2k+1}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] + (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \phi \frac{\partial_t \phi}{\phi} \right\} \right] \\ &= -2 \frac{\partial_t \phi}{\phi} \frac{1}{\xi^k \phi} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] - 2 \frac{\phi^{4k+1}}{\xi^{3k}} |\partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]|^2 - (2k+1) \left| \frac{\partial_t \phi}{\phi} \right|^2 \cdot \frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \end{aligned}$$

It is easy to see that the first and third terms are bounded by the energy functional. For the second term, writing it as

$$-2 \frac{\phi^{4k+1}}{\xi^{3k}} |\partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]|^2 = -2 \underbrace{\frac{\phi^{3k+1}}{\xi^{3k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]}_{L_\xi^\infty} \cdot \underbrace{\phi^k \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]}_{L_\xi^2},$$

we deduce that it is also controlled by $\mathcal{E}^{k,4}$, and in result, we conclude that $V^*V_tV^*(\xi^k u)$ is bounded by $\mathcal{E}^{k,4}$. Next $VV^*VV^*(\xi^k u)$:

$$\begin{aligned} \bullet VV^*VV^*(\xi^k u) &= -\frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{2k-1}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] \right] - (2k+1) \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{4k}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi} \right] \\ &= \boxed{-\frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{2k-1}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] \right]} + VV^*V(\xi^k \phi) \frac{\partial_t \phi}{\phi} + \frac{\phi^{2k}}{\xi^{2k}} V^*V(\xi^k \phi) \cdot \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \end{aligned}$$

Note that the boxed term is bounded in L_ξ^2 by the energy functional. Now we write $VV^*V_tV^*(\xi^k u)$ in terms of the boxed terms as well as the energy:

$$\begin{aligned} \frac{1}{2k} VV^*V_tV^*(\xi^k u) &= -2 \frac{\partial_t \phi}{\phi} \cdot \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{2k-1}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] \right] \\ &\quad - 6 \frac{\phi^{2k+1}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \cdot \frac{1}{\xi^k \phi^2} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] + (4k+6) \frac{\partial_\xi \phi \phi^{6k}}{\xi^{5k}} \left| \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right|^2 \quad \text{---}(\star) \\ &\quad - (2k+1) \left| \frac{\partial_t \phi}{\phi} \right|^2 \cdot \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{4k}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \right] - (4k+2) \frac{\partial_t \phi \phi^{3k}}{\phi \xi^{3k}} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \cdot \phi^k \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \end{aligned}$$

It is clear that except (\star) , the first factor of each term in the right hand side is bounded in L_ξ^∞ and the second factor is bounded in L_ξ^2 by $\mathcal{E}^{k,4}$. In order to show that (\star) is bounded in L_ξ^2 , first note that by Lemma 3.3, $\| \frac{V^*VV^*(\xi^k u)}{\xi} \|_{L_\xi^2}$ and $\| \frac{V^*V(\xi^k \phi)}{\xi} \|_{L_\xi^2}$ are bounded by $\mathcal{E}^{k,4}(\phi, u)$ due to the fact $k > \frac{1}{2}$, from the relation (38), thus we get

$$h \equiv \frac{1}{\xi^{k+2}} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] \in L_\xi^2,$$

and in turn we obtain

$$\begin{aligned} \frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] &= \int_0^\xi |\xi'|^{k+2} h d\xi' \leq \xi^{k+\frac{5}{2}} \|h\|_{L_\xi^2} \\ \implies \|\xi^k \left| \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right|^2\|_{L_\xi^2} &\leq \|h\|_{L_\xi^2}^4 \int_0^1 \xi^{2k+(4k+10)-(8k+8)} d\xi \end{aligned}$$

Note that the last integral is bounded. Therefore we conclude that $\|VV^*V_tV^*(\xi^k u)\|_{L_\xi^2}$ is bounded by $\mathcal{E}^{k,4}$. Similarly, we deduce the same conclusion for other mixed terms and it finishes the proof of Proposition 4.1 for $\frac{1}{2} < k < 1$.

4.3 The case when $[k] \geq 2$ ($1 < \gamma < 3$)

We now turn into the general k . The spirit is the same as the case when $k = 1$: we need to carry out L_ξ^∞ estimates and nonlinear estimates. For large k , however, the number of mixed nonlinear terms increases accordingly and it is not an easy task to work term by term. We will present a systematic way of treating those terms involving derivatives of $\partial_t \phi$.

The following lemma is a direct result from Lemma 3.6 and 3.7, and it is useful to justify the assumption (21) in $\mathcal{E}^{k, [k]+3}(\phi, u)$.

Lemma 4.4. (1) We obtain the following L_ξ^∞ estimates:

$$\left\| \frac{V(\xi^k \phi)}{\xi^k} \right\|_{L_\xi^\infty}, \left\| \frac{V^* V(\xi^k \phi)}{\xi^k} \right\|_{L_\xi^\infty}, \left\| \frac{V^*(\xi^k u)}{\xi^k} \right\|_{L_\xi^\infty}, \left\| \frac{V V^*(\xi^k u)}{\xi^k} \right\|_{L_\xi^\infty} \leq \mathcal{C}_3(\mathcal{E}^{k, [k]+3}(\phi, u))$$

where $\mathcal{C}_3(\mathcal{E}^{k, [k]+3}(\phi, u))$ is a continuous function of $\|\frac{\xi}{\phi}\|_{L_\xi^\infty}$ and $\mathcal{E}^{k, [k]+3}(\phi, u)$.

(2) For $3 \leq i \leq [k] + 1$

$$\left\| \frac{(V)^i(\xi^k \phi)}{\xi^{[k]+2-i}} \right\|_{L_\xi^\infty}, \left\| \frac{(V^*)^i(\xi^k u)}{\xi^{[k]+2-i}} \right\|_{L_\xi^\infty} \leq \mathcal{C}_4(\mathcal{E}^{k, [k]+3}(\phi, u))$$

where $\mathcal{C}_4(\mathcal{E}^{k, [k]+3}(\phi, u))$ is a continuous function of $\|\frac{\xi}{\phi}\|_{L_\xi^\infty}$ and $\mathcal{E}^{k, [k]+3}(\phi, u)$.

We start with L_ξ^∞ estimate of ϕ . We list out V , V^*V , VV^*V of $\xi^k \phi$ for references.

- $V(\xi^k \phi) = \frac{1}{\xi^k} \partial_\xi [\phi^{2k+1}] = (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \phi$
- $V^*V(\xi^k \phi) = -(2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right]$
- $VV^*V(\xi^k \phi) = -(2k+1) \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{4k}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \right]$

One advantage of V and V^* formulation is that L_ξ^∞ control of ϕ is cheap to get:

$$\int \xi^k \phi \cdot V(\xi^k \phi) d\xi = \int \phi \partial_\xi [\phi^{2k+1}] d\xi = \frac{2k+1}{2k+2} \phi^{2k+2}$$

Applying Lemma 4.4, we obtain that $\partial_\xi \phi$ is bounded and continuous if the assumption (21) holds and $\mathcal{E}^{k, [k]+3}(\phi, u)$ is bounded.

Next we turn to u variable.

- $V^*(\xi^k u) = -\frac{\phi^{2k}}{\xi^k} \partial_\xi u$
 - $VV^*(\xi^k u) = -\frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{4k}}{\xi^{2k}} \partial_\xi u \right]$
 - $V^*VV^*(\xi^k u) = \frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{1}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{4k}}{\xi^{2k}} \partial_\xi u \right] \right]$
- (39)

Note that $\frac{\partial_t \phi}{\xi}$ can be estimated in terms of $\partial_\xi u$ via the equation (5):

$$\partial_t \phi = -\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi u = \frac{V^*(\xi^k u)}{\xi^k}$$

By Lemma 4.4, we deduce that $\frac{VV^*(\xi^k u)}{\xi^k} = \frac{1}{\xi^{2k}} \partial_\xi [\phi^{2k} \partial_t \phi]$ is bounded and continuous if $\mathcal{E}^{k, [k]+3}(\phi, u)$ is bounded. Letting h be $\frac{1}{\xi^{2k}} \partial_\xi [\phi^{2k} \partial_t \phi]$, we can write $\phi^{2k} \partial_t \phi = \int_0^\xi \xi'^{2k} h d\xi'$, and therefore we conclude $\frac{\partial_t \phi}{\xi}$ is also bounded. By the same continuity argument as in $k=1$ case, we can verify the same boundary behavior for a short time and the assumption (21).

We now perform the energy estimates. From (10), we have the conservation of the zeroth

order energy:

$$\frac{1}{2} \frac{d}{dt} \int \frac{1}{2k+1} |\xi^k \phi|^2 + |\xi^k u|^2 d\xi = 0$$

Apply V and V^* to (10) and use $V_t(\xi^k \phi) = \frac{2k}{2k+1} \partial_t V(\xi^k \phi)$ to get

$$\begin{aligned} \partial_t V(\xi^k \phi) - (2k+1) V V^*(\xi^k u) &= 0 \\ \partial_t V^*(\xi^k u) + \frac{1}{2k+1} V^* V(\xi^k \phi) &= V_t^*(\xi^k u) \end{aligned} \quad (40)$$

Apply V^* and V to (40) and use $V V_t^* = V_t V^*$ to get

$$\begin{aligned} \partial_t V^* V(\xi^k \phi) - (2k+1) V^* V V^*(\xi^k u) &= V_t^* V(\xi^k \phi) \\ \partial_t V V^*(\xi^k u) + \frac{1}{2k+1} V V^* V(\xi^k \phi) &= 2 V_t V^*(\xi^k u) \end{aligned} \quad (41)$$

By keeping taking V and V^* alternately, one obtains higher order equations for $(V)^i(\xi^k \phi)$ and $(V^*)^i(\xi^k u)$ for any i . Indeed, the mixed terms in the right hand sides can be written in a systematic way. For $i = 2j + 1$ where $j \geq 1$:

$$\begin{aligned} \partial_t (V)^{2j+1}(\xi^k \phi) - (2k+1) (V^*)^{2j+2}(\xi^k u) &= 2 \sum_{l=0}^{j-1} (V^*)^{2j-2l-2} V_t V^*(V)^{2l+1}(\xi^k \phi) \\ \partial_t (V^*)^{2j+1}(\xi^k u) + \frac{1}{2k+1} (V)^{2j+2}(\xi^k \phi) &= 2 \sum_{l=0}^{j-1} (V^*)^{2j-2l-1} V_t V^*(V^*)^{2l}(\xi^k u) \\ &\quad + V_t^*(V^*)^{2j}(\xi^k u) \end{aligned} \quad (42)$$

For $i = 2j$ where $j \geq 2$:

$$\begin{aligned} \partial_t (V)^{2j}(\xi^k \phi) - (2k+1) (V^*)^{2j+1}(\xi^k u) &= 2 \sum_{l=0}^{j-2} (V^*)^{2j-2l-3} V_t V^*(V)^{2l+1}(\xi^k \phi) \\ &\quad + V_t^*(V)^{2j-1}(\xi^k \phi) \\ \partial_t (V^*)^{2j}(\xi^k u) + \frac{1}{2k+1} (V)^{2j+1}(\xi^k \phi) &= 2 \sum_{l=0}^{j-1} (V^*)^{2j-2l-2} V_t V^*(V^*)^{2l}(\xi^k u) \end{aligned} \quad (43)$$

Our main goal is to estimate the mixed terms in the right hand sides of (42) and (43). Note that the most intriguing cases seem to be when $l = 0$, where the most spatial derivatives of $\frac{\partial_t \phi}{\phi}$ are present. Before getting into the estimates, let us get the better understanding of the effect of the operator V_t . First, recall $(V^*)^i(\xi^k u)$'s. They give rise to $\partial_t \phi$ terms via the equations (5) and furthermore they predict the right form of spatial derivatives of $\partial_t \phi$ terms.

We borrow the computations from the previous section for $i \leq 4$.

- $V^*(\xi^k u) = -\frac{\phi^{2k}}{\xi^k} \partial_\xi u = \xi^k \partial_t \phi$
- $VV^*(\xi^k u) = \frac{1}{\xi^k} \partial_\xi [\phi^{2k} \partial_t \phi] = \frac{\phi^{2k+1}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] + (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \phi \frac{\partial_t \phi}{\phi}$

$$= \boxed{\frac{\phi^{2k+1}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right]} + V(\xi^k \phi) \frac{\partial_t \phi}{\phi} \quad (44)$$

Now let us compute $V_t V^*(\xi^k u)$ and write in terms of the energy:

$$\begin{aligned} \frac{1}{2k} V_t V^*(\xi^k u) &= \frac{1}{\xi^k} \partial_\xi [\phi^{2k+1} \left| \frac{\partial_t \phi}{\phi} \right|^2] = \frac{\partial_t \phi}{\phi} \left\{ 2 \frac{\phi^{2k+1}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] + (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \phi \frac{\partial_t \phi}{\phi} \right\} \\ &= 2 \frac{\partial_t \phi}{\phi} VV^*(\xi^k u) - \left| \frac{\partial_t \phi}{\phi} \right|^2 V(\xi^k \phi) \end{aligned} \quad (45)$$

Next we write out $V^* VV^*(\xi^k u)$.

- $V^* VV^*(\xi^k u) = -\frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k+1}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] - (2k+1) \left\{ \frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi} + \frac{\phi^{4k}}{\xi^{3k}} \partial_\xi \phi \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right\}$

$$= -\frac{1}{\xi^k \phi} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] - (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi}$$

$$= \boxed{-\frac{1}{\xi^k \phi} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right]} + V^* V(\xi^k \phi) \frac{\partial_t \phi}{\phi}$$

Thus we note that $\frac{1}{\xi^k \phi} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right]$ is the right form of the second derivative of $\partial_t \phi$ that we do not want to destroy. Here is $V^* V_t V^*(\xi^k u)$.

$$\begin{aligned} \frac{1}{2k} V^* V_t V^*(\xi^k u) &= -\frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{1}{\xi^k} \frac{\partial_t \phi}{\phi} \left\{ 2 \frac{\phi^{2k+1}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] + (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \phi \frac{\partial_t \phi}{\phi} \right\} \right] \\ &= -2 \frac{\partial_t \phi}{\phi} \frac{1}{\xi^k \phi} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] - 2 \frac{\phi^{4k+1}}{\xi^{3k}} \left| \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right|^2 - (2k+1) \frac{\phi^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi} \end{aligned} \quad (46)$$

Next $VV^* VV^*(\xi^k u)$:

- $VV^* VV^*(\xi^k u) = -\frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{2k-1}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] \right] - (2k+1) \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{4k}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{2k}}{\xi^{2k}} \partial_\xi \phi \right] \frac{\partial_t \phi}{\phi} \right]$

$$= \boxed{-\frac{1}{\xi^k} \partial_\xi \left[\frac{\phi^{2k-1}}{\xi^{2k}} \partial_\xi \left[\frac{\phi^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \right] \right]} + VV^* V(\xi^k \phi) \frac{\partial_t \phi}{\phi} + \frac{\phi^{2k}}{\xi^{2k}} V^* V(\xi^k \phi) \cdot \partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] \quad (47)$$

In turn, $VV^*V_tV^*(\xi^k u)$:

$$\begin{aligned} \frac{1}{2k}VV^*V_tV^*(\xi^k u) &= -2\frac{\partial_t\phi}{\phi}\frac{1}{\xi^k}\partial_\xi\left[\frac{\phi^{2k-1}}{\xi^{2k}}\partial_\xi\left[\frac{\phi^{4k+2}}{\xi^{2k}}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right]\right] \\ &\quad - 6\frac{\phi^{2k+1}}{\xi^{2k}}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right] \cdot \frac{1}{\xi^k\phi^2}\partial_\xi\left[\frac{\phi^{4k+2}}{\xi^{2k}}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right] + (4k+6)\partial_\xi\phi\frac{\phi^{6k}}{\xi^{5k}}|\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]|^2 \\ &\quad - (2k+1)\left|\frac{\partial_t\phi}{\phi}\right|^2 \cdot \frac{1}{\xi^k}\partial_\xi\left[\frac{\phi^{4k}}{\xi^{2k}}\partial_\xi\left[\frac{\phi^{2k}}{\xi^{2k}}\partial_\xi\phi\right]\right] - (4k+2)\frac{\partial_t\phi}{\phi}\frac{\phi^{3k}}{\xi^{3k}}\partial_\xi\left[\frac{\phi^{2k}}{\xi^{2k}}\partial_\xi\phi\right] \cdot \phi^k\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right] \end{aligned}$$

As in the case $\frac{1}{2} < k \leq 1$, we chose the above specific expansion of $V^*VV^*(\xi^k u)$ in order to find the right expression of the second derivative of $\partial_t\phi$ and then estimate the mixed term $V^*V_tV^*(\xi^k u)$. This is because the nonlinearity of (5) is prevalent at this level in V, V^* formulation and an arbitrary expansion may destroy the structure of (5). For higher order terms, we employ a rather crude expansion in a systematic way. In the below, we present a representation for $(V^*)^i(\xi^k u)$, get the information of spatial derivatives of $\partial_t\phi$, which we denote by T_i 's, and derive the estimates of $(V^*)^{i-2}V_tV^*(\xi^k u)$.

Representation of $(V^*)^i(\xi^k u)$ and T_i for $i \geq 2$: First, we define T_i for $2 \leq i \leq [k] + 3$:

$$T_2 \equiv \frac{\phi^{2k+1}}{\xi^k}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]; \quad T_3 \equiv -\frac{1}{\xi^k\phi}\partial_\xi\left[\frac{\phi^{4k+2}}{\xi^{2k}}\partial_\xi\left[\frac{\partial_t\phi}{\phi}\right]\right]; \quad T_i \equiv (V)^{i-3}T_3 \text{ for } i \geq 4 \quad (48)$$

We chose T_2 and T_3 so that

$$(V^*)^2(\xi^k u) = T_2 + V(\xi^k\phi)\frac{\partial_t\phi}{\phi}, \quad (V^*)^3(\xi^k u) = T_3 + (V)^2(\xi^k\phi)\frac{\partial_t\phi}{\phi}.$$

In other words, they are boxed terms in the above as well as in the previous sections. Note that by Lemma 4.4,

$$\frac{T_2}{\xi^k} \in L_\xi^\infty, \quad \frac{T_3}{\xi^{[k]-1}} \in L_\xi^\infty. \quad (49)$$

The validity of T_i will follow from the construction of T_i in the below. Now we claim for any $i \geq 4$, $(V^*)^i(\xi^k u)$ has the following representation which extends the case $i = 2, 3$:

$$(V^*)^i(\xi^k u) = T_i + (V)^{i-1}(\xi^k\phi)\frac{\partial_t\phi}{\phi} + \sum_{j=2}^{i-2} \Phi_{i-j}T_j \quad (50)$$

where

$$\Phi_{i-j} \equiv \sum_{r=0}^{i-j-2} \left\{ C_r \frac{1}{\xi^r} \sum_{\substack{l_1+\dots+l_p=i-j-r \\ l_1, \dots, l_p \geq 1}} C_{l_1 \dots l_p} \prod_{q=1}^p \frac{(V)^{l_q}(\xi^k\phi)}{\xi^k\phi} \right\} \quad (51)$$

for some functions $C_r, C_{l_1 \dots l_p}$ which may only depend on $\frac{\phi}{\xi}$ and $\frac{\xi}{\phi}$ and therefore $C_r, C_{l_1 \dots l_p}$ are bounded by $\|\frac{\phi}{\xi}\|_{L_\xi^\infty}$ and $\|\frac{\xi}{\phi}\|_{L_\xi^\infty}$. Furthermore, T_i and the last term in (50) have the following property: for each $4 \leq i \leq [k] + 3$,

$$\left\| \frac{1}{\xi^{[k]+3-i}} \sum_{j=2}^{i-2} \Phi_{i-j}T_j \right\|_{L_\xi^2} \text{ and } \left\| \frac{T_i}{\xi^{[k]+3-i}} \right\|_{L_\xi^2} \text{ are bounded by } \mathcal{E}^{k, [k]+3}(\phi, u), \quad (52)$$

and for $4 \leq i \leq [k] + 1$,

$$\left\| \frac{1}{\xi^{[k]+2-i}} \sum_{j=2}^{i-2} \Phi_{i-j} T_j \right\|_{L_\xi^\infty} \text{ and } \left\| \frac{T_i}{\xi^{[k]+2-i}} \right\|_{L_\xi^\infty} \text{ are bounded by } \mathcal{E}^{k, [k]+3}(\phi, u). \quad (53)$$

In particular, each T_i is well-defined. We are now ready to prove the representation formula (50). From the previous computation (47),

$$(V^*)^4(\xi^k u) = T_4 + (V)^3(\xi^k \phi) \frac{\partial_t \phi}{\phi} + \frac{(V)^2(\xi^k \phi)}{\xi^k \phi} T_2. \quad (54)$$

Setting $\Phi_{4-2} = \frac{(V)^2(\xi^k \phi)}{\xi^k \phi}$, the formula (50) holds for $i = 4$. Moreover, by Lemma 4.4, the properties (52) and (53) are satisfied for $i = 4$. For $i \geq 5$, based on the induction on i , we will show that V or V^* of each term in (50) can be decomposed in the same fashion. With the aid of Lemma 3.8, applying V or V^* of the second term $(V)^{i-1}(\xi^k \phi) \frac{\partial_t \phi}{\phi}$ in the right hand side of (50), we obtain

$$\begin{aligned} \bullet V((V)^{2j}(\xi^k \phi) \frac{\partial_t \phi}{\phi}) &= (V)^{2j+1}(\xi^k \phi) \frac{\partial_t \phi}{\phi} + \frac{(V)^{2j}(\xi^k \phi)}{\xi^k \phi} T_2 \\ \bullet V^*((V)^{2j+1}(\xi^k \phi) \frac{\partial_t \phi}{\phi}) &= (V)^{2j+2}(\xi^k \phi) \frac{\partial_t \phi}{\phi} - \frac{(V)^{2j+1}(\xi^k \phi)}{\xi^k \phi} T_2 \end{aligned}$$

Note that the right hand sides are of the right form. In the same spirit, we can apply V or V^* to the last term in (50). Here is the estimate of the simplest case when $j = 2$, $r = 0$, $l_1 = i - 2$. Consider $i = 2j + 2$ or $i = 2j + 3$.

$$\begin{aligned} \bullet V^*\left(\frac{(V)^{2j}(\xi^k \phi)}{\xi^k \phi} T_2\right) &= 2 \frac{V(\xi^k \phi)}{\xi^k \phi} \frac{V^{2j}(\xi^k \phi)}{\xi^k \phi} T_2 - \frac{V^{2j+1}(\xi^k \phi)}{\xi^k \phi} T_2 + \frac{V^{2j}(\xi^k \phi)}{\xi^k \phi} T_3 \\ \bullet V\left(\frac{(V)^{2j+1}(\xi^k \phi)}{\xi^k \phi} T_2\right) &= -\frac{2}{2k+1} \frac{V(\xi^k \phi)}{\xi^k \phi} \frac{V^{2j+1}(\xi^k \phi)}{\xi^k \phi} T_2 - \frac{V^{2j+2}(\xi^k \phi)}{\xi^k \phi} T_2 + \frac{V^{2j+1}(\xi^k \phi)}{\xi^k \phi} T_3 \end{aligned}$$

Each term in the right hand sides has a desirable form. From Lemma 3.8, we deduce that when V or V^* act on a function h , depending on that function, they can yield $\frac{h}{\xi}$ or $\frac{V(\xi^k \phi)}{\xi^k} \frac{h}{\phi}$ or Vh or V^*h . Therefore, V or V^* of other cases of $\Phi_{i-j} T_j$ falls into the right form of the case when $i + 1$. Next we verify (52) and (53). Let $s \geq 4$ be given. Assume that they hold for $i \leq s$ and we first claim that

$$\left\| \frac{1}{\xi^{[k]+3-(s+1)}} \sum_{j=2}^{(s+1)-2} \Phi_{(s+1)-j} T_j \right\|_{L_\xi^2} \text{ is bounded by } \mathcal{E}^{k, [k]+3}(\phi, u).$$

This can be justified by counting derivatives and $\frac{1}{\xi}$ and distributing $\frac{1}{\xi}$ factors in a right way. The highest derivative term with appropriate factor of $\frac{1}{\xi}$ will be taken as the main L_ξ^2 term and others will be bounded by taking the sup. Let $z \equiv \max\{j, l_1, \dots, l_p\}$ for each term. If $z = j$, we consider $\frac{T_j}{\xi^{[k]+3-j}}$ whose L_ξ^2 norm is bounded by $\mathcal{E}^{k, [k]+3}(\phi, u)$ from the induction

hypothesis, since $j \leq s - 1$, as an L_ξ^2 term. Now it remains to show that the rest of factors

$$\frac{\xi^{\lceil k \rceil + 3 - j}}{\xi^{\lceil k \rceil + 3 - (s+1)}} \sum_{r=0}^{(s+1)-j-2} \frac{1}{\xi^r} \sum_{\substack{l_1 + \dots + l_p = (s+1) - j - r \\ l_1, \dots, l_p \geq 1}} \prod_{q=1}^p \frac{(V)^{l_q}(\xi^k \phi)}{\xi^k \phi}$$

are bounded. It is enough to look at the following: for each $r \leq s - j - 1$

$$\frac{\xi^{s+1-j}}{\xi^r} \sum_{\substack{l_1 + \dots + l_p = s+1-j-r \\ l_1, \dots, l_p \geq 1}} \prod_{q=1}^p \frac{(V)^{l_q}(\xi^k \phi)}{\xi^k \phi}$$

which is clearly bounded by $\mathcal{E}^{k, \lceil k \rceil + 3}(\phi, u)$ because of Lemma 4.4 and since $k \leq \lceil k \rceil$. When z is one of l_q 's, one can derive the same conclusion. Thus from (50), we deduce that $\|\frac{T_{s+1}}{\xi^{\lceil k \rceil + 3 - (s+1)}}\|_{L_\xi^2}$ is also bounded and it finishes the verification of (52). For the L_ξ^∞ boundedness for $s + 1 \leq \lceil k \rceil + 1$

$$\|\frac{1}{\xi^{\lceil k \rceil + 2 - (s+1)}} \sum_{j=2}^{(s+1)-2} \Phi_{(s+1)-j} T_j\|_{L_\xi^\infty}$$

we employ the same counting argument except that we have to use the fact $\frac{T_2}{\xi^k}, \frac{T_j}{\xi^{\lceil k \rceil + 2 - j}} \in L_\xi^\infty$ for $3 \leq j \leq s$. This finishes the induction argument.

These T_i 's and their properties will be useful to estimate the nonlinear terms involving $V_t V^*$. Based on the representation formula (50) and T_i 's, we further study the representation of more general mixed terms $(V^*)^i V_t f$ for $i \leq \lceil k \rceil + 1$. First $V_t f$ can be written as

$$\frac{1}{2k} V_t f = \frac{1}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi}{\phi} \frac{\phi^{2k}}{\xi^k} f \right] = \frac{\partial_t \phi}{\phi} V f + \frac{f}{\xi^k \phi} T_2.$$

Note that the right hand side has the same structure as the last two terms of the right hand side of (54), in letting f be $(V)^2(\xi^k \phi)$. Thus we can apply the same technique to obtain the following: for each $i \leq \lceil k \rceil + 1$

$$\frac{1}{2k} (V^*)^i V_t f = \frac{\partial_t \phi}{\phi} (V)^{i+1} f + \sum_{m=0}^i \left\{ \sum_{j=2}^{m+2} \Phi_{(m+2)-j} T_j \right\} (V)^{i-m} f. \quad (55)$$

Proposition 4.5. *(Nonlinear estimates) The right hand sides in (42) and (43) are bounded in L_ξ^2 by a continuous function of $\mathcal{E}^{k, \lceil k \rceil + 3}(\phi, u)$ and C .*

Proof. We only treat the most intriguing term $(V^*)^i V_t V^*(\xi^k u)$ for $i \leq \lceil k \rceil + 1$. The idea is to estimate it in terms of T_j 's and to use their properties. First, note that $V^* V_t V^*(\xi^k u)$ can be written in terms of T_2, T_3 as in

$$\frac{1}{2k} V^* V_t V^*(\xi^k u) = 2 \frac{\partial_t \phi}{\phi} T_3 - 2 \frac{1}{\xi^k \phi} |T_2|^2 + V^* V(\xi^k \phi) \left| \frac{\partial_t \phi}{\phi} \right|^2 \quad (56)$$

We would like to compute $(V)^i V^* V_t V^*(\xi^k u)$ for $1 \leq i \leq \lceil k \rceil$. Consider $i = 1$. We apply V to

each term in (56). Since

$$\partial_\xi \left[\frac{\partial_t \phi}{\phi} \right] = \frac{\xi^k}{\phi^{2k+1}} T_2,$$

we have

$$\begin{aligned} V \left(\frac{\partial_t \phi}{\phi} T_3 \right) &= \frac{\partial_t \phi}{\phi} T_4 + \frac{\xi^k}{\phi^{2k+1}} T_2 \cdot T_3 \\ V \left(\frac{1}{\xi^k \phi} |T_2|^2 \right) &= \frac{2}{\xi^k \phi} T_2 \cdot T_3 - \frac{4k+3}{2k+1} \frac{1}{\xi^k \phi} \frac{V(\xi^k \phi)}{\xi^k \phi} |T_2|^2 \\ V(V^* V(\xi^k \phi) \left| \frac{\partial_t \phi}{\phi} \right|^2) &= V V^* V(\xi^k \phi) \left| \frac{\partial_t \phi}{\phi} \right|^2 + 2 \frac{\xi^k}{\phi^{2k+1}} T_2 \cdot V^* V(\xi^k \phi) \frac{\partial_t \phi}{\phi} \end{aligned}$$

Thus we deduce that $V V^* V_t V^*(\xi^k u)$ is bounded in L_ξ^2 by the energy functional. In the same vein, by keeping applying V^* and V , for any $i \geq 2$, we can write $(V)^i V^* V_t V^*(\xi^k u)$ as follows:

$$\begin{aligned} \frac{1}{2k} (V)^i V^* V_t V^*(\xi^k u) &= 2 \frac{\partial_t \phi}{\phi} T_{i+3} + (V)^{i+2} (\xi^k \phi) \left| \frac{\partial_t \phi}{\phi} \right|^2 + \frac{\partial_t \phi}{\phi} \cdot \sum_{j=2}^{i+1} \Phi_{(i+3)-j} T_j \\ &\quad + \sum_{j=2}^{i+2} C_j \frac{1}{\xi^k \phi} T_{i+4-j} T_j + \sum_{s=4}^{i+2} \left\{ \Phi_{(i+4)-s} \left(\sum_{j=2}^{s-2} C_{sj} \frac{1}{\xi^k \phi} T_{s-j} T_j \right) \right\} \end{aligned} \quad (57)$$

where Φ is given as in (51) with possibly different coefficient functions and C_j and C_{sj} are some functions bounded by $\|\frac{\phi}{\xi}\|_{L_\xi^\infty}$ and $\|\frac{\xi}{\phi}\|_{L_\xi^\infty}$. This formula (57) can be also obtained by plugging (50) into (55).

Now we claim that for each $1 \leq i \leq [k]$, $\|(V)^i V^* V_t V^*(\xi^k u)\|_{L_\xi^2}$ is bounded by $\mathcal{E}^{k, [k]+3}(\phi, u)$. The first three terms in the right hand side of (57) are bounded since they are of the form as in (50) multiplied with $\frac{\partial_t \phi}{\phi}$. The rest terms consist of quadratic or higher of energy terms or T_j 's. For each term, the highest derivative with appropriate factor of $\frac{1}{\xi}$ is considered the main L_ξ^2 term and other factors are bounded by taking the sup. This can be done by employing the counting and distributing $\frac{1}{\xi}$ argument as well as the estimates of T_j 's (52) and (53) as before. \square

5 Approximate Scheme

In this section, we implement the linear approximate scheme and prove that the linear system is well-posed in some energy space.

Let the initial data $\phi_0(\xi)$ and $u_0(\xi)$ of the Euler equations (5) be given such that $\frac{1}{C_0} \leq \frac{\phi_0}{\xi} \leq C_0$ for a constant $C_0 > 1$, and $\mathcal{E}^{k, [k]+3}(\phi_0, u_0) \leq A$ for a constant $A > 0$. Here $\mathcal{E}^{k, [k]+3}(\phi_0, u_0)$ is the energy functional (24) induced by ϕ_0 . Note that from the energy bound, we obtain $\frac{\partial_\xi u_0}{\xi} \in L_\xi^\infty$. We will construct approximate solutions $\phi_n(t, \xi)$ and $u_n(t, \xi)$ for each n by induction satisfying the following properties:

$$\phi_n|_{t=0} = \phi_0, \quad u_n|_{t=0} = u_0; \quad \phi_n|_{\xi=0} = u_n|_{\xi=1} = 0; \quad \frac{1}{C_n} \leq \frac{\phi_n}{\xi} \leq C_n, \quad \text{for } C_n > 1 \quad (58)$$

Note that ϕ_0, u_0 automatically satisfy (58). Define the operators V_n and V_n^* as follows:

$$V_n(f) \equiv \frac{1}{\xi^k} \partial_\xi \left(\frac{\phi_n^{2k}}{\xi^k} f \right), \quad V_n^*(g) \equiv -\frac{\phi_n^{2k}}{\xi^k} \partial_\xi \left(\frac{1}{\xi^k} g \right) \quad (59)$$

In addition, we define commutator operators $(V_n)_t$ and $(V_n^*)_t$:

$$(V_n)_t(f) \equiv 2k \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi_n^{2k-1} \partial_t \phi_n}{\xi^k} f \right], \quad (V_n^*)_t(g) \equiv -2k \frac{\phi_n^{2k-1} \partial_t \phi_n}{\xi^k} \partial_\xi \left[\frac{g}{\xi^k} \right]$$

We define $\partial_t \phi_0$ through the equation by

$$\partial_t \phi_0 \equiv -\frac{\phi_0^{2k}}{\xi^{2k}} \partial_\xi u_0.$$

For the linear iteration scheme, we approximate $(V)^3(\xi^k \phi) \equiv G$ and $(V^*)^3(\xi^k u) \equiv F$ and the equations (42) when $j = 1$, instead of ϕ and u themselves. Let

$$D_0 \equiv V_0(\xi^k \phi_0), \quad H_0 \equiv V_0^*(\xi^k u_0), \quad G_0 \equiv V_0 V_0^* V_0(\xi^k \phi_0), \quad F_0 \equiv V_0^* V_0 V_0^*(\xi^k u_0).$$

For each $n \geq 0$, consider the following approximate equations

$$\begin{aligned} \partial_t G_{n+1} - (2k+1) V_n F_{n+1} &= J_n^1 \\ \partial_t F_{n+1} + \frac{1}{2k+1} V_n^* G_{n+1} &= J_n^2 \end{aligned} \quad (60)$$

where J_n^1 and J_n^2 are given as follows:

$$\begin{aligned} J_n^1 &\equiv 4k \frac{\partial_t \phi_n}{\phi_n} G_n + 4k \frac{\phi_n^{2k}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \frac{V_n^* D_n}{\xi^k} \\ J_n^2 &\equiv 2k \frac{\partial_t \phi_n}{\phi_n} F_n + 4k \left| \frac{\partial_t \phi_n}{\phi_n} \right|^2 V_n^* D_n - 8k \frac{\phi_n^{4k+1}}{\xi^{3k}} \left| \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \right|^2 - 8k \frac{\partial_t \phi_n}{\phi_n} \frac{1}{\xi^k \phi_n} \partial_\xi \left[\frac{\phi_n^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \right] \end{aligned}$$

The initial and boundary conditions are inherited from the original system:

$$G_{n+1}|_{t=0} = V_0 V_0^* V_0(\xi^k \phi_0), \quad F_{n+1}|_{t=0} = V_0^* V_0 V_0^*(\xi^k u_0), \quad G_{n+1}|_{\xi=1} = 0.$$

Note that $F_{n+1}|_{\xi=0}$ is built in the equations (60). In turn, from these F_{n+1}, G_{n+1} , we define $D_{n+1}, H_{n+1}, \phi_{n+1}, u_{n+1}$:

$$\begin{aligned} D_{n+1} &\equiv -\xi^k \int_1^\xi \frac{\xi'^{2k}}{\phi_n^{4k}} \int_0^{\xi'} \xi_1^k G_{n+1} d\xi_1 d\xi', \quad H_{n+1} \equiv -\frac{\xi^k}{\phi_n^{2k}} \int_0^\xi \xi'^k \int_1^{\xi'} \frac{\xi_1^k}{\phi_n^{2k}} F_{n+1} d\xi_1 d\xi' \\ \phi_{n+1} &\equiv \left\{ \int_0^\xi \xi'^k D_{n+1}(t, \xi') d\xi' \right\}^{\frac{1}{2k+1}}, \quad \partial_\xi u_{n+1} \equiv -\frac{\xi^k H_{n+1}}{\phi_{n+1}^{2k}}, \quad u_{n+1} \equiv \int_1^\xi \partial_\xi u_{n+1} d\xi' \end{aligned}$$

Note that we have used the boundary condition at $\xi = 1$ in order to invert V_n^* . Also note that from the above definitions the following identities hold

$$V_n V_n^* D_{n+1} = G_{n+1}, \quad V_n^* V_n H_{n+1} = F_{n+1}, \quad D_{n+1} = V_{n+1}(\xi^k \phi_{n+1}), \quad H_{n+1} = V_{n+1}^*(\xi^k u_{n+1}).$$

In view of Proposition 5.1, it is easy to deduce that $(V_n^*)^i D_{n+1}$ and $(V_n)^i H_{n+1}$ for $0 \leq i \leq$

$[k] + 2$ are well-defined, namely bounded in L_ξ^2 . Also V_{n+1} and V_{n+1}^* are defined as in (59) with ϕ_{n+1} . The right hand sides J_n^1, J_n^2 of (60) are approximations of $2(V_n)_t V_n^* V_n(\xi^k \phi_n)$ and $2V_n^*(V_n)_t V_n^*(\xi^k u_n) + (V_n^*)_t V_n V_n^*(\xi^k u_n)$ in the following manner:

$$\begin{aligned}
& \frac{1}{2k}(V_n)_t V_n^* V_n(\xi^k \phi_n) = -(2k+1) \frac{1}{\xi^k} \partial_\xi \left[\frac{\phi_n^{2k-1} \partial_t \phi_n}{\xi^k} \frac{\phi_n^{2k}}{\xi^k} \partial_\xi \left[\frac{\phi_n^{2k}}{\xi^{2k}} \partial_\xi \phi_n \right] \right] \\
& = -(2k+1) \frac{1}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \cdot \frac{\phi_n^{4k}}{\xi^{2k}} \partial_\xi \left[\frac{\phi_n^{2k}}{\xi^{2k}} \partial_\xi \phi_n \right] \right] = \frac{\partial_t \phi_n}{\phi_n} V_n V_n^* V_n(\xi^k \phi_n) + \frac{\phi_n^{2k}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \frac{V_n^* V_n(\xi^k \phi_n)}{\xi^k} \\
& \sim \frac{\partial_t \phi_n}{\phi_n} G_n + \frac{\phi_n^{2k}}{\xi^k} \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \frac{V_n^* D_n}{\xi^k} \\
& \frac{1}{2k} V_n^*(V_n)_t V_n^*(\xi^k u_n) \sim -2 \frac{\partial_t \phi_n}{\phi_n} \frac{1}{\xi^k \phi_n} \partial_\xi \left[\frac{\phi_n^{4k+2}}{\xi^{2k}} \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \right] - 2 \frac{\phi_n^{4k+1}}{\xi^{3k}} \left| \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \right|^2 + \left| \frac{\partial_t \phi_n}{\phi_n} \right|^2 V_n^* D_n \\
& \frac{1}{2k} (V_n^*)_t V_n V_n^*(\xi^k u_n) = \frac{\partial_t \phi_n}{\phi_n} V_n^* V_n V_n^*(\xi^k u_n) \sim \frac{\partial_t \phi_n}{\phi_n} F_n
\end{aligned}$$

In particular, note that the approximation of $V_n^*(V_n)_t V_n^*(\xi^k u_n)$, which has the strongest nonlinearity in view of the a priori estimates, is based on the expression (46). Also note that the equation (60) converges to (42) for $j = 1$ in the formal limit.

We define the approximate energy functional $\tilde{\mathcal{E}}_{n+1}^k$ at the n -th step:

$$\begin{aligned}
\tilde{\mathcal{E}}_{n+1}^k(t) & \equiv \sum_{i=0}^{[k]} \int \frac{1}{(2k+1)^2} |(V_n^*)^i G_{n+1}|^2 + |(V_n)^i F_{n+1}|^2 d\xi \\
& \equiv \left\| \frac{1}{2k+1} G_{n+1} \right\|_{Y_n^{k, [k]}}^2 + \|F_{n+1}\|_{X_n^{k, [k]}}^2
\end{aligned} \tag{61}$$

where $X_n^{k,s}$ and $Y_n^{k,s}$ denote $X^{k,s}$ and $Y^{k,s}$ induced by V_n and V_n^* in (9).

We now state and prove that the approximate system (60) are well-posed in the energy space generated by X_n, Y_n under the following induction hypotheses:

- (HP1) $\tilde{\mathcal{E}}_n^k < \infty$ and ϕ_n, u_n satisfy (58);
- (HP2) when $k \leq 1$, $\left\| \frac{\partial_t \phi_n}{\phi_n} \right\|_{L_\xi^\infty}$ and $\left\| \xi \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \right\|_{L_\xi^\infty}$ are bounded by $\tilde{\mathcal{E}}_n^k, \tilde{\mathcal{E}}_{n-1}^k$;
- when $k > 1$, $\left\| \frac{\partial_t \phi_n}{\phi_n} \right\|_{L_\xi^\infty}, \left\| \frac{T_{2,n}}{\xi^k} \right\|_{L_\xi^\infty}, \left\| \frac{T_{i,n}}{\xi^{[k]+2-i}} \right\|_{L_\xi^\infty}, \left\| \frac{(V_n)^i(\xi^k \phi_n)}{\xi^{[k]+2-i}} \right\|_{L_\xi^\infty}$
for $3 \leq i \leq [k] + 1$ are bounded by $\tilde{\mathcal{E}}_n^k, \tilde{\mathcal{E}}_{n-1}^k$;
- (HP3) J_n^1 and J_n^2 in (60) are bounded in $Y_n^{k, [k]}, X_n^{k, [k]}$ respectively.

Here $T_{i,n}$ is the T_i defined in (48) where ϕ is taken as ϕ_n .

Proposition 5.1. *(Well-posedness of approximate system and regularity) Under the hypotheses (HP1), (HP2), and (HP3), the linear system (60) admits a unique solution (G_{n+1}, H_{n+1}) in $Y_n^{k, [k]}, X_n^{k, [k]}$ space. Furthermore, we obtain the following energy bounds.*

$$\tilde{\mathcal{E}}_{n+1}^k(t) \leq \tilde{\mathcal{E}}_{n+1}^k(0) + \int_0^t C_5(\tilde{\mathcal{E}}_{n-1}^k, \tilde{\mathcal{E}}_n^k, \tilde{\mathcal{E}}_{n+1}^k)(\tilde{\mathcal{E}}_{n+1}^k)^{\frac{1}{2}} d\tau$$

where $\mathcal{C}_5(\tilde{\mathcal{E}}_{n-1}^k, \tilde{\mathcal{E}}_n^k, \tilde{\mathcal{E}}_{n+1}^k)$ is a continuous function of $\tilde{\mathcal{E}}_{n-1}^k$, $\tilde{\mathcal{E}}_n^k$, $\tilde{\mathcal{E}}_{n+1}^k$ and C_0 .

Proposition 5.1 directly follows from Proposition 7.1. In the next subsection, we verify the induction hypotheses.

5.1 Induction procedure

In order to finish the induction procedure of approximate schemes, it now remains to verify the induction hypotheses (HP1), (HP2), and (HP3) for $n+1$ as described in Proposition 5.1. The spirit is the same as in the a priori estimates. However, since n and $n+1$ are mingled in the energy functional (61), the verification of the induction hypotheses needs an attention. We only treat the case when $k=1$. Other cases can be estimated similarly. First, it is easy to see that ϕ_{n+1} and u_{n+1} constructed in the above satisfy the initial boundary conditions in (58) for $n+1$. The boundedness of $\frac{\phi_{n+1}}{\xi}$ will follow from the continuity argument by using the estimate of $\frac{\partial_t \phi_{n+1}}{\xi}$. Write the equation for $\partial_t(\phi_{n+1}^3)$ from the definition of ϕ_{n+1} 's:

$$\begin{aligned} 3\phi_{n+1}^2 \partial_t \phi_{n+1} &= 4 \int_0^\xi (\xi')^2 \int_1^{\xi'} \frac{\xi_2^2 \partial_t \phi_n}{\phi_n^5} \int_0^{\xi_2} \xi_1 G_{n+1} d\xi_1 d\xi_2 d\xi' \\ &\quad - \int_0^\xi (\xi')^2 \int_1^{\xi'} \frac{\xi_2^2}{\phi_n^4} \int_0^{\xi_2} \xi_1 \partial_t G_{n+1} d\xi_1 d\xi_2 d\xi' \\ &= 4 \int_0^\xi (\xi')^2 \int_1^{\xi'} \frac{\xi_2^2 \partial_t \phi_n}{\phi_n^5} \int_0^{\xi_2} \xi_1 G_{n+1} d\xi_1 d\xi_2 d\xi' \\ &\quad - 3 \int_0^\xi (\xi')^2 \int_1^{\xi'} \frac{\xi_2}{\phi_n^2} F_{n+1} d\xi_2 d\xi' - \int_0^\xi (\xi')^2 \int_1^{\xi'} \frac{\xi_2^2}{\phi_n^4} \int_0^{\xi_2} \xi_1 J_n^1 d\xi_1 d\xi_2 d\xi' \end{aligned} \tag{62}$$

The first term can be controlled as follows: since

$$\int_0^{\xi_2} \xi_1 G_{n+1} d\xi_1 \leq \xi_2^{\frac{5}{2}} \left\| \frac{G_{n+1}}{\xi} \right\|_{L_\xi^2}, \tag{63}$$

it follows that

$$\int_0^\xi (\xi')^2 \int_1^{\xi'} \frac{\xi_2^2 \partial_t \phi_n}{\phi_n^5} \int_0^{\xi_2} \xi_1 G_{n+1} d\xi_1 d\xi_2 d\xi' \leq \xi^{\frac{7}{2}} \left\| \frac{\xi}{\phi_n} \right\|_{L_\xi^\infty}^4 \left\| \frac{\partial_t \phi_n}{\phi_n} \right\|_{L_\xi^\infty} \left\| \frac{G_{n+1}}{\xi} \right\|_{L_\xi^2}.$$

Thus $\frac{1}{\xi^3} \int_0^\xi (\xi')^2 \int_1^{\xi'} \frac{\xi_2^2 \partial_t \phi_n}{\phi_n^5} \int_0^{\xi_2} \xi_1 G_{n+1} d\xi_1 d\xi_2 d\xi'$ is bounded by $\tilde{\mathcal{E}}_{n+1}$ and the previous energies. For the second term, note that

$$\int_1^{\xi'} \frac{\xi_2}{\phi_n^2} F_{n+1} d\xi_2 \leq \left\| \frac{\xi}{\phi_n} \right\|_{L_\xi^\infty}^2 \left\| \frac{F_{n+1}}{\xi} \right\|_{L_\xi^2}.$$

Hence, we obtain

$$\frac{1}{\xi^3} \int_0^\xi (\xi')^2 \int_1^{\xi'} \frac{\xi_2}{\phi_n^2} F_{n+1} d\xi_2 d\xi' \leq \left\| \frac{\xi}{\phi_n} \right\|_{L_\xi^\infty}^2 \left\| \frac{F_{n+1}}{\xi} \right\|_{L_\xi^2}.$$

Finally, since

$$\begin{aligned} \int_0^{\xi_2} \xi_1 J_n^1 d\xi_1 &\leq \xi_2^{\frac{5}{2}} \left\| \frac{\partial_t \phi_n}{\phi_n} \right\|_{L_\xi^\infty} \left\| \frac{G_n}{\xi} \right\|_{L_\xi^2} \\ &+ \xi_2^{\frac{5}{2}} \left\| \frac{\phi_n}{\xi} \right\|_{L_\xi^\infty}^2 \left\| \frac{\phi_n}{\phi_{n-1}} \right\|_{L_\xi^\infty}^2 \left\| \xi \partial_\xi \left[\frac{\partial_t \phi_n}{\phi_n} \right] \right\|_{L_\xi^\infty} \left\| \frac{V_{n-1}^* D_n}{\xi} \right\|_{L_\xi^2} \end{aligned} \quad (64)$$

the last term is also bounded and therefore we conclude that $\partial_t(\frac{\phi_{n+1}^3}{\xi^3})$ is bounded by $\tilde{\mathcal{E}}_{n+1}$ and the previous energies. Note that the nontrivial contribution near the boundary $\xi = 0$ is coming from the second term F^{n+1} . Since

$$\frac{\phi_{n+1}^3}{\xi^3}(t) = \frac{\phi_{n+1}^3}{\xi^3}(0) + \int_0^t \partial_t \left(\frac{\phi_{n+1}^3}{\xi^3} \right) d\tau = \frac{\phi_0^3}{\xi^3}(0) + \int_0^t \partial_t \left(\frac{\phi_{n+1}^3}{\xi^3} \right) d\tau,$$

we get

$$\frac{1}{C_0^3} - T \left\| \partial_t \left(\frac{\phi_{n+1}^3}{\xi^3} \right) \right\|_{L_\xi^\infty} \leq \frac{\phi_{n+1}^3}{\xi^3} \leq C_0^3 + T \left\| \partial_t \left(\frac{\phi_{n+1}^3}{\xi^3} \right) \right\|_{L_\xi^\infty}$$

and in result, we also obtain C_{n+1} in (58) depending on T and energy bounds.

We move onto (HP2). Since $\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} = \frac{\partial_t \phi_{n+1}}{\xi} \frac{\xi}{\phi_{n+1}}$, we immediately deduce that $\left\| \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right\|_{L_\xi^\infty}$ is bounded. Take ∂_ξ of (62):

$$\begin{aligned} 3\phi_{n+1}^3 \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right] + 9\phi_{n+1} \partial_\xi \phi_{n+1} \partial_t \phi_{n+1} &= 4\xi^2 \int_1^\xi \frac{\xi_2^2 \partial_t \phi_n}{\phi_n^5} \int_0^{\xi_2} \xi_1 G_{n+1} d\xi_1 d\xi_2 \\ &- 3\xi^2 \int_1^\xi \frac{\xi_2}{\phi_n^2} F_{n+1} d\xi_2 - \xi^2 \int_1^\xi \frac{\xi_2^2}{\phi_n^4} \int_0^{\xi_2} \xi_1 J_n^1 d\xi_1 d\xi_2 \end{aligned} \quad (65)$$

Therefore the boundedness of $\left\| \xi \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right] \right\|_{L_\xi^\infty}$ follows by the same reasoning.

For (HP3), we now show that J_{n+1}^1 and J_{n+1}^2 of (60) at the next step are bounded in $Y_{n+1}^{1,1}$, $X_{n+1}^{1,1}$ accordingly.

Claim 5.2. $V_{n+1}^* J_{n+1}^1$ and $V_{n+1} J_{n+1}^2$ are bounded in L_ξ^2 .

Proof. The sprit of the proof is same as in the nonlinear estimates of the a priori estimates. We present the detailed computation for J_{n+1}^2 , which is more complicated than J_{n+1}^1 . We start with $V_{n+1}(\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} F_{n+1})$.

$$\begin{aligned} V_{n+1} \left(\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} F_{n+1} \right) &= \frac{1}{\xi} \partial_\xi \left[\frac{\phi_{n+1}^2}{\xi} \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} F_{n+1} \right] = \frac{1}{\xi} \partial_\xi \left[\frac{\phi_{n+1}^2}{\phi_n^2} \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \frac{\phi_n^2}{\xi} F_{n+1} \right] \\ &= \underbrace{\frac{\phi_{n+1}^2}{\phi_n^2} \frac{\partial_t \phi_{n+1}}{\phi_{n+1}}}_{L_\xi^\infty} \underbrace{V_n F_{n+1}}_{L_\xi^2} + \underbrace{\frac{\phi_{n+1}^2}{\xi^2} \xi \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right]}_{L_\xi^\infty} \underbrace{\frac{F_{n+1}}{\xi}}_{L_\xi^2} \\ &+ 2 \underbrace{\left(\frac{\phi_{n+1}}{\xi} \partial_\xi \phi_{n+1} - \frac{\phi_{n+1}^2}{\xi \phi_n} \partial_\xi \phi_n \right)}_{L_\xi^\infty} \underbrace{\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \frac{F_{n+1}}{\xi}}_{L_\xi^2} \end{aligned}$$

Thus $\|V_{n+1}(\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} F_{n+1})\|_{L_\xi^2}$ is bounded by $\tilde{\mathcal{E}}_{n+1}$. Next we compute $V_{n+1}(|\frac{\partial_t \phi_{n+1}}{\phi_{n+1}}|^2 V_{n+1}^* D_{n+1})$.

$$\begin{aligned} V_{n+1}(|\frac{\partial_t \phi_{n+1}}{\phi_{n+1}}|^2 V_{n+1}^* D_{n+1}) &= -\frac{1}{\xi} \partial_\xi \left[\frac{\phi_{n+1}^4}{\xi^2} \partial_\xi \left[\frac{D_{n+1}}{\xi} \right] \right] \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} |^2 \\ &= \underbrace{\frac{\phi_{n+1}^4}{\phi_n^4} \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} |^2}_{L_\xi^\infty} \underbrace{G_{n+1}}_{L_\xi^2} + \underbrace{\frac{\phi_{n+1}^4}{\xi^2 \phi_n^2} \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \xi \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right]}_{L_\xi^\infty} \underbrace{\frac{V_n^* D_{n+1}}{\xi}}_{L_\xi^2} \\ &\quad + 4 \underbrace{\left(\frac{\phi_{n+1}^3}{\xi \phi_n^2} \partial_\xi \phi_{n+1} - \frac{\phi_{n+1}^4}{\xi \phi_n^3} \partial_\xi \phi_n \right)}_{L_\xi^\infty} \underbrace{\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} |^2 \frac{V_n^* D_{n+1}}{\xi}}_{L_\xi^2} \end{aligned}$$

Thus $V_{n+1}(|\frac{\partial_t \phi_{n+1}}{\phi_{n+1}}|^2 V_{n+1}^* D_{n+1})$ is bounded in L_ξ^2 . In order to take care of the rest of J_{n+1}^2 , first we claim that

$$\left\| \frac{1}{\xi^3} \partial_\xi \left[\frac{\phi_{n+1}^6}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right] \right] \right\|_{L_\xi^2} \text{ is bounded by } \tilde{\mathcal{E}}_{n+1}, \tilde{\mathcal{E}}_n. \quad (66)$$

In order to do so, multiply (65) by $\frac{\phi_{n+1}^3}{3\xi^2}$ and take ∂_ξ to get

$$\begin{aligned} &\partial_\xi \left[\frac{\phi_{n+1}^6}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right] \right] + 3 \partial_\xi \left[\phi_{n+1}^3 \right] \frac{\phi_{n+1}^2}{\xi^2} \partial_\xi \phi_{n+1} \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \\ &\quad + 3 \phi_{n+1}^3 \partial_\xi \left[\frac{\phi_{n+1}^2}{\xi^2} \partial_\xi \phi_{n+1} \right] \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} + 3 \phi_{n+1}^3 \frac{\phi_{n+1}^2}{\xi^2} \partial_\xi \phi_{n+1} \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right]_{(\star)} \\ &= \phi_{n+1}^2 \partial_\xi \phi_{n+1} \left\{ 4 \int_1^\xi A d\xi' - 3 \int_1^\xi B d\xi' - \int_1^\xi C d\xi' \right\} + \frac{\phi_{n+1}^3}{3} \{4A - 3B - C\} \end{aligned}$$

where A, B, C are defined so that (65) can be written as

$$3 \phi_{n+1}^3 \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right] + 9 \phi_{n+1} \partial_\xi \phi_{n+1} \partial_t \phi_{n+1} = \xi^2 \left\{ 4 \int_1^\xi A d\xi' - 3 \int_1^\xi B d\xi' - \int_1^\xi C d\xi' \right\}.$$

The term (\star) becomes

$$(\star) = \frac{\phi_{n+1}^2}{\xi^2} \partial_\xi \phi_{n+1} \left\{ -9 \phi_{n+1} \partial_\xi \phi_{n+1} \partial_t \phi_{n+1} + \xi^2 \left(4 \int_1^\xi A d\xi' - 3 \int_1^\xi B d\xi' - \int_1^\xi C d\xi' \right) \right\}$$

Hence by canceling terms, we obtain

$$\begin{aligned} \partial_\xi \left[\frac{\phi_{n+1}^6}{\xi^2} \partial_\xi \left[\frac{\partial_t \phi_{n+1}}{\phi_{n+1}} \right] \right] &= -3 \phi_{n+1}^3 \partial_\xi \left[\frac{\phi_{n+1}^2}{\xi^2} \partial_\xi \phi_{n+1} \right] \frac{\partial_t \phi_{n+1}}{\phi_{n+1}} + \frac{\phi_{n+1}^3}{3} \{4A - 3B - C\} \\ &= \phi_{n+1}^3 \frac{\xi^2}{\phi_n^2} \frac{V_n^* D_{n+1}}{\xi} + \frac{\phi_{n+1}^3}{3} \{4A - 3B - C\} \end{aligned} \quad (67)$$

Indeed, A, B, C were treated during the estimates of $\|\frac{\partial_t \phi_{n+1}}{\xi}\|_{L_\xi^\infty}$. From the same analysis, we deduce that L_ξ^2 norms of A, B, C are bounded by $\tilde{\mathcal{E}}_{n+1}, \tilde{\mathcal{E}}_n$. Thus the claim (66) follows.

Now we consider $V_{n+1}(\frac{\phi_{n+1}^5}{\xi^3}|\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]|^2)$.

$$\begin{aligned} V_{n+1}(\frac{\phi_{n+1}^5}{\xi^3}|\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]|^2) &= \frac{1}{\xi}\partial_\xi[\frac{1}{\phi_{n+1}^5}|\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]|^2] \\ &= 2\phi_{n+1}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}] \cdot \frac{1}{\xi^3}\partial_\xi[\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]] - 5\frac{\partial_\xi\phi_{n+1}}{\xi\phi_{n+1}^6}|\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]|^2 \end{aligned}$$

By (66), the first term in the right hand side is controllable, and again due to (66), since $|\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]|^2 \leq \xi^7\|\frac{1}{\xi^3}\partial_\xi[\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]]\|_{L_\xi^2}^2$, we obtain

$$\|\frac{1}{\xi^7}|\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]|^2\|_{L_\xi^2} \leq \|\frac{1}{\xi^3}\partial_\xi[\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]]\|_{L_\xi^2}^2$$

and this completes the estimate. For the last term in J_{n+1}^2 , note that

$$\begin{aligned} V_{n+1}(\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}\frac{1}{\xi\phi_{n+1}}\partial_\xi[\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]]) \\ = \frac{\partial_t\phi_{n+1}}{\phi_{n+1}}\frac{1}{\xi}\partial_\xi[\frac{\phi_{n+1}}{\xi^2}\partial_\xi[\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]]] + \phi_{n+1}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]\frac{1}{\xi^3}\partial_\xi[\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]] \end{aligned}$$

Thus it remains to show that $\frac{1}{\xi}\partial_\xi[\frac{\phi_{n+1}}{\xi^2}\partial_\xi[\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]]]$ is bounded by $\tilde{\mathcal{E}}_{n+1}$, $\tilde{\mathcal{E}}_n$. From (67), one can write it as

$$\begin{aligned} \frac{1}{\xi}\partial_\xi[\frac{\phi_{n+1}}{\xi^2}\partial_\xi[\frac{\phi_{n+1}^6}{\xi^2}\partial_\xi[\frac{\partial_t\phi_{n+1}}{\phi_{n+1}}]]] &= \frac{1}{\xi}\partial_\xi[\frac{\phi_{n+1}^4}{\phi_n^4} \cdot \frac{\phi_n^2}{\xi}V_n^*D_{n+1}] + \frac{1}{3\xi}\partial_\xi[\frac{\phi_{n+1}^4}{\xi^2}(4A - 3B - C)] \\ &\equiv (I) + (II) \end{aligned}$$

(I) can be decomposed as follows:

$$(I) = \frac{\phi_{n+1}^4}{\phi_n^4}G_{n+1} + \xi\partial_\xi[\frac{\phi_{n+1}^4}{\phi_n^4}]\frac{\phi_n^2}{\xi^2}\frac{V_n^*D_{n+1}}{\xi}$$

hence it is bounded. For (II), we put A, B, C back into the expression:

$$\begin{aligned} 3(II) &= \frac{1}{\xi}\partial_\xi[\frac{\phi_{n+1}^4}{\phi_n^4}\{4\frac{\partial_t\phi_n}{\phi_n}\int_0^\xi \xi_1 G_{n+1}d\xi_1 - 3\frac{\phi_n^2}{\xi}F_{n+1} - \int_0^\xi \xi_1 J_n^1 d\xi_1\}] \\ &= \frac{\phi_{n+1}^4}{\phi_n^4}\{4\frac{\partial_t\phi_n}{\phi_n}G_{n+1} + \xi\partial_\xi[\frac{\partial_t\phi_n}{\phi_n}]\cdot\frac{1}{\xi^2}\int_0^\xi \xi_1 G_{n+1}d\xi_1 - 3V_n F_{n+1} - J_n^1\} \\ &\quad + \xi\partial_\xi[\frac{\phi_{n+1}^4}{\phi_n^4}]\{4\frac{\partial_t\phi_n}{\phi_n}\frac{1}{\xi^2}\int_0^\xi \xi_1 G_{n+1}d\xi_1 - 3\frac{\phi_n^2}{\xi^2}\frac{F_{n+1}}{\xi} - \frac{1}{\xi^2}\int_0^\xi \xi_1 J_n^1 d\xi_1\} \end{aligned}$$

Therefore, by the estimates (63) and (64) of $\int_0^\xi \xi_1 G_{n+1}d\xi_1$ and $\int_0^\xi \xi_1 J_n^1 d\xi_1$, we conclude that (II) is also bounded by $\tilde{\mathcal{E}}_{n+1}$, $\tilde{\mathcal{E}}_n$. This finishes the proof of Claim. \square

6 Proof of Theorem 2.1

In order to prove Theorem 2.1, it now remains to show that ϕ_n, u_n converge, the limit functions solve Euler equations (5), and they are unique.

6.1 Existence

First, by applying Gronwall inequality to the energy inequality in Proposition 5.1, we can deduce the following claim.

Claim 6.1. *Suppose that the initial data $\phi_0(\xi)$ and $u_0(\xi)$ of the Euler equations (5) be given such that $\frac{1}{C_0} \leq \frac{\phi_0}{\xi} \leq C_0$ for a constant $C_0 > 1$, and $\mathcal{E}^{k, [k]+3}(\phi_0, u_0) \leq A$ for a constant $A > 0$. Then there exists $T > 0$ such that if for $m \leq n$, $\tilde{\mathcal{E}}_m^k \leq \frac{3}{2}A$ for $t \leq T$, then $\tilde{\mathcal{E}}_{n+1}^k \leq \frac{3}{2}A$ for $t \leq T$ and in addition, for all n , $\frac{1}{2C_0} \leq \frac{\phi_n}{\xi} \leq 2C_0$ for $t \leq T$.*

Thus we get the uniform bound of $\tilde{\mathcal{E}}_n^k$'s as well as the uniform bound of the upper and lower bounds of $\frac{\phi_n}{\xi}$. Since the approximate energy functionals (61) depend on not only the approximate functions G_{n+1}, F_{n+1} but also ϕ_n , the corresponding Banach space changes at every step. In order to take the limit, it is desirable to have the fixed space where G_{n+1}, F_{n+1} live. The plan is as follows: by making use of Proposition 3.10, we prove the equivalence between our energy functionals and the energy functional induced by (18) so that approximate functions have the uniform energy bounds in the Banach space induced by (18) and thus we can apply the fixed point theorem. We present the detailed analysis for $k = 1$.

Recall the homogeneous operators \bar{V} and \bar{V}^* :

$$\bar{V}(f) \equiv \frac{1}{\xi} \partial_\xi [\xi f], \quad \bar{V}^*(g) \equiv -\xi \partial_\xi \left[\frac{g}{\xi} \right]$$

and we define the corresponding energy functional $\bar{\mathcal{E}}_{n+1}$:

$$\bar{\mathcal{E}}_{n+1}(t) \equiv \sum_{i=0}^1 \int \frac{1}{9} |(\bar{V}^*)^i G_{n+1}|^2 + |(\bar{V})^i F_{n+1}|^2 d\xi \quad (68)$$

We claim that $\tilde{\mathcal{E}}_{n+1}$ and $\bar{\mathcal{E}}_{n+1}$ are equivalent in some sense. We compute $\tilde{\mathcal{E}}_{n+1}$. V_n and V_n^* are written in terms of \bar{V} and \bar{V}^* .

$$\begin{aligned} \bullet V_n^* G_{n+1} &= \frac{\phi_n^2}{\xi^2} \bar{V}^* G_{n+1} \\ \bullet V_n F_{n+1} &= \frac{\phi_n^2}{\xi^2} \bar{V} F_{n+1} + 2 \left(\frac{\phi_n}{\xi} \partial_\xi \phi_n - \frac{\phi_n^2}{\xi^2} \right) \frac{F_{n+1}}{\xi} \end{aligned}$$

Note that from the definition of D_n

$$\left| \frac{\phi_n^2}{\xi^2} \partial_\xi \phi_n \right| = \left| \frac{1}{3} \frac{D_n}{\xi} \right| \leq \left\| \frac{\phi_{n-1}^2}{\xi^2} \partial_\xi \phi_{n-1} \right\|_{L_\xi^\infty} + \frac{2}{3^{\frac{3}{2}}} \xi^{\frac{1}{2}} C_{n-1}^4 \|G_n\|_{L_\xi^2} \leq I + C_{n-1}^4 \|G_n\|_{L_\xi^2}$$

where $I = \left\| \frac{\phi_0^2}{\xi^2} \partial_\xi \phi_0 \right\|_{L_\xi^\infty}$ and C_{n-1} is the bound of $\frac{\phi_{n-1}}{\xi}$ as in (58). Therefore, we deduce that

$$\tilde{\mathcal{E}}_{n+1} \leq (1 + M_n) \bar{\mathcal{E}}_{n+1}$$

where $M_n \equiv 5C_n^2\{C_n^2 + I^2 + C_{n-1}^8\bar{\mathcal{E}}_n\}$. To show the the converse, namely $\bar{\mathcal{E}}_{n+1}$ is bounded by $\tilde{\mathcal{E}}_{n+1}$, we rewrite \bar{V} and \bar{V}^* in terms of V^n and V_n^* .

$$\begin{aligned} \bullet \bar{V}^* G_{n+1} &= \frac{\xi^2}{\phi_n^2} V_n^* G_{n+1} \\ \bullet \bar{V} F_{n+1} &= \frac{\xi^2}{\phi_n^2} V_n F_{n+1} + 2\left(1 - \frac{\xi}{\phi_n} \partial_\xi \phi_n\right) \frac{F_{n+1}}{\xi} \end{aligned}$$

Thus we reach the same conclusion:

$$\bar{\mathcal{E}}_{n+1} \leq (1 + M_n) \tilde{\mathcal{E}}_{n+1}$$

Note that M_n 's have the uniform bound over $t \leq T$ by Claim 6.1. Therefore, there exists a sequence n_l so that $G_{n_l}, F_{n_l}, \phi_{n_l}, u_{n_l}$ converge strongly to some G, F, ϕ, u . Due to the uniform energy bound, we also conclude that G, F solve the equations (42) for $j = 1$ and moreover ϕ, u solve Euler equations (5) with the desired properties.

Next, we turn to the general case. Back to the approximate system (60), we define the corresponding homogeneous energy functional:

$$\bar{\mathcal{E}}_{n+1}^k(t) \equiv \left\| \frac{1}{2k+1} G_{n+1} \right\|_{Y^{k, [k]}}^2 + \|F_{n+1}\|_{X^{k, [k]}}^2 \quad (69)$$

From Proposition 3.10, one can derive the equivalence of the associated energy functional (69) and the original approximate functional (61): There exists $M_n > 0$ only depending on initial data, C_n, C_{n-1} , and $\bar{\mathcal{E}}_n$ such that

$$\frac{1}{1 + M_n} \bar{\mathcal{E}}_{n+1}^k \leq \tilde{\mathcal{E}}_{n+1}^k \leq (1 + M_n) \bar{\mathcal{E}}_{n+1}^k.$$

By the same reasoning as the case when $k = 1$, the existence of G, F, ϕ, u follows.

6.2 Uniqueness

In order to prove Theorem 2.1, it only remains to prove the uniqueness. Let (ϕ_1, u_1) and (ϕ_2, u_2) be two regular solutions to (5) with the same initial boundary conditions with $\mathcal{E}^{k, [k]+3}(\phi_1, u_1), \mathcal{E}^{k, [k]+3}(\phi_2, u_2) \leq 2A$. Define $\mathcal{D}(t)$ by

$$\begin{aligned} \mathcal{D}(t) &\equiv \int \frac{1}{2k+1} |\xi^k(\phi_1 - \phi_2)|^2 + |\xi^k(u_1 - u_2)|^2 d\xi \\ &+ \sum_{i=1}^2 \int \frac{1}{(2k+1)^2} |(V_1)^i(\xi^k \phi_1) - (V_2)^i(\xi^k \phi_2)|^2 + |(V_1^*)^i(\xi^k u_1) - (V_2^*)^i(\xi^k u_2)|^2 d\xi, \end{aligned}$$

where V_j, V_j^* are the V, V^* induced by ϕ_j . We will prove that $\frac{d}{dt} \mathcal{D} \leq \mathcal{C} \mathcal{D}$, which immediately gives the uniqueness. Let us consider $i = 2$ case only in $\mathcal{D}(t)$. Recall the system (41). By subtracting two systems from each other, we obtain the equations for $(V_1^* V_1(\xi^k \phi_1) -$

$V_2^*V_2(\xi^k\phi_2), V_1V_1^*(\xi^ku_1) - V_2V_2^*(\xi^ku_2)$):

$$\begin{aligned}
& \partial_t\{V_1^*V_1(\xi^k\phi_1) - V_2^*V_2(\xi^k\phi_2)\} - (2k+1)V_1^*\{V_1V_1^*(\xi^ku_1) - V_2V_2^*(\xi^ku_2)\} \\
&= (2k+1)\{V_1^*V_2V_2^*(\xi^ku_2) - V_2^*V_2V_2^*(\xi^ku_2)\} + \{(V_1^*)_tV_1(\xi^k\phi_1) - (V_2^*)_tV_2(\xi^k\phi_2)\} \\
& \partial_t\{V_1V_1^*(\xi^ku_1) - V_2V_2^*(\xi^ku_2)\} + \frac{1}{2k+1}V_1\{V_1^*V_1(\xi^k\phi_1) - V_2^*V_2(\xi^k\phi_2)\} \\
&= \frac{1}{2k+1}\underbrace{\{V_2V_2^*V_2(\xi^k\phi_2) - V_1V_2^*V_2(\xi^k\phi_2)\}}_{(I)} + 2\underbrace{\{(V_1)_tV_1^*(\xi^ku_1) - (V_2)_tV_2^*(\xi^ku_2)\}}_{(II)}
\end{aligned}$$

We have to show that the L_ξ^2 norm of the right hand sides is bounded by $\mathcal{D}^{\frac{1}{2}}$. We estimate (I) and (II). Other two terms can be treated in the same way. First, we note that

$$\phi_1^{2k+1} - \phi_2^{2k+1} = \int_0^\xi (\xi')^k \{V_1(\xi^k\phi_1) - V_2(\xi^k\phi_2)\} d\xi'.$$

If simply applying Hölder inequality, one gets

$$\left\| \frac{\phi_1^{2k+1}}{\xi^{k+\frac{1}{2}}} - \frac{\phi_2^{2k+1}}{\xi^{k+\frac{1}{2}}} \right\|_{L_\xi^\infty} \leq \|V_1(\xi^k\phi_1) - V_2(\xi^k\phi_2)\|_{L_\xi^2}. \quad (70)$$

Apply Hölder inequality once more:

$$\begin{aligned}
\int_0^\xi (\xi')^{k+\frac{1}{2}} \frac{V_1(\xi^k\phi_1) - V_2(\xi^k\phi_2)}{\sqrt{\xi'}} d\xi' &\leq \xi^{k+1} \{ \|V_1(\xi^k\phi_1) - V_2(\xi^k\phi_2)\|_{L_\xi^2} \\
& \quad + \underbrace{\|\sqrt{\xi}V_1^*(V_1(\xi^k\phi_1) - V_2(\xi^k\phi_2))\|_{L_\xi^2}}_{(**)} \}
\end{aligned}$$

Note that it is not trivial to get the boundedness of (**) in terms of $\mathcal{D}^{\frac{1}{2}}$, since it is not of the right form yet. Here is the estimate of (**):

$$(**) \leq \|\sqrt{\xi}\{V_1^*V_1(\xi^k\phi_1) - V_2^*V_2(\xi^k\phi_2)\}\|_{L_\xi^2} + \|\sqrt{\xi}\{V_2^*V_2(\xi^k\phi_2) - V_1^*V_2(\xi^k\phi_2)\}\|_{L_\xi^2}$$

The second term can be written as

$$\left(\frac{\phi_1^{2k}}{\xi^{k-\frac{1}{2}}} - \frac{\phi_2^{2k}}{\xi^{k-\frac{1}{2}}} \right) \partial_\xi \left[\frac{1}{\xi^k} V_2(\xi^k\phi_2) \right] = - \left(\frac{\phi_1^{2k}}{\xi^{k-\frac{1}{2}}} - \frac{\phi_2^{2k}}{\xi^{k-\frac{1}{2}}} \right) \frac{\xi^{2k} V_2^*V_2(\xi^k\phi_2)}{\phi_2^{2k} \xi^k}.$$

Hence, by using (70), we deduce that

$$\left\| \frac{\phi_1^{2k+1}}{\xi^{k+1}} - \frac{\phi_2^{2k+1}}{\xi^{k+1}} \right\|_{L_\xi^\infty} \sim \left\| \frac{\phi_1^{2k}}{\xi^k} - \frac{\phi_2^{2k}}{\xi^k} \right\|_{L_\xi^\infty} \leq C\mathcal{D}^{\frac{1}{2}}. \quad (71)$$

Note that one cannot hope to get the bound of $\frac{\phi_1^{2k+1}}{\xi^{2k+1}} - \frac{\phi_2^{2k+1}}{\xi^{2k+1}}$ with the \mathcal{D} -regularity. Of course, it is bounded by A , but for the purpose of uniqueness, A -bound is not useful for the difference terms. The idea is to rearrange (I) and (II) so that $\mathcal{D}^{\frac{1}{2}}$ can be extracted from each term. We write the L_ξ^∞ factors first and then L_ξ^2 factor at the last. $\mathcal{D}^{\frac{1}{2}}$ can also come from L_ξ^∞

factor thanks to the above estimate (71). We start with (I):

$$(I) = (1 - \frac{\phi_1^{2k}}{\phi_2^{2k}})V_2V_2^*V_2(\xi^k\phi_2) - \frac{\phi_2^{2k}}{\xi^{2k}}\partial_\xi[\frac{\phi_1^{2k}}{\phi_2^{2k}}]V_2^*V_2(\xi^k\phi_2) \quad (\star)$$

The first term is written as $(\frac{\phi_1^{2k}}{\xi^k} - \frac{\phi_2^{2k}}{\xi^k})\frac{\xi^{2k}}{\phi_2^{2k}}\frac{V_2V_2^*V_2(\xi^k\phi_2)}{\xi^k}$ and thus its L_ξ^2 norm is bounded by A and $\mathcal{D}^{\frac{1}{2}}$. The second term can be treated as follows:

$$\begin{aligned} (\star) &= 2k\frac{\xi}{\phi_1}\left[\frac{\phi_1^{2k}}{\xi^k}\partial_\xi\phi_1 - \frac{\phi_1^{2k}}{\xi^k}\frac{\phi_1}{\phi_2}\partial_\xi\phi_2\right]\frac{V_2^*V_2(\xi^k\phi_2)}{\xi^{k+1}} \\ &= \frac{2k}{2k+1}\frac{\xi}{\phi_1}\frac{V_2^*V_2(\xi^k\phi_2)}{\xi^k}\frac{V_1(\xi^k\phi_1) - V_2(\xi^k\phi_2)}{\xi} + 2k\frac{\xi}{\phi_1}\partial_\xi\phi_2\left(\frac{\phi_2^{2k}}{\xi^k} - \frac{\phi_1^{2k}}{\xi^k}\frac{\phi_1}{\phi_2}\right)\frac{V_2^*V_2(\xi^k\phi_2)}{\xi^{k+1}} \end{aligned}$$

Thus (\star) is controlled by A and $\mathcal{D}^{\frac{1}{2}}$. Next we rearrange (II):

$$\begin{aligned} \frac{1}{2k}(II) &= \frac{\partial_t\phi_1}{\phi_1}\{V_1V_1^*(\xi^ku_1) - V_2V_2^*(\xi^ku_2)\} + \frac{V_2V_2^*(\xi^ku_2)}{\xi^k}\left\{\frac{\xi^k\partial_t\phi_1}{\phi_1} - \frac{\xi^k\partial_t\phi_2}{\phi_2}\right\} \quad (\star\star) \\ &+ \frac{\phi_1^{2k}}{\xi^{2k}}\frac{V_1^*(\xi^ku_1)}{\xi^{k+1}}\{\xi^{k+1}\partial_\xi[\frac{\partial_t\phi_1}{\phi_1}] - \xi^{k+1}\partial_\xi[\frac{\partial_t\phi_2}{\phi_2}]\} + \xi\partial_\xi[\frac{\partial_t\phi_2}{\phi_2}]\frac{\phi_1^{2k}}{\xi^{2k}}\left\{\frac{V_1^*(\xi^ku_1)}{\xi} - \frac{V_2^*(\xi^ku_2)}{\xi}\right\} \\ &+ \xi\partial_\xi[\frac{\partial_t\phi_2}{\phi_2}]\left(\frac{\phi_1^{2k}}{\xi^k} - \frac{\phi_2^{2k}}{\xi^k}\right)\frac{V_2^*(\xi^ku_2)}{\xi^{k+1}} \end{aligned}$$

For t derivative difference terms, use the equation to convert into u terms and apply the same argument. For instance, the second term can be rewritten as

$$\begin{aligned} (\star\star) &= \frac{V_2V_2^*(\xi^ku_2)}{\xi^k}\left\{\frac{V_1^*(\xi^ku_1)}{\phi_1} - \frac{V_2^*(\xi^ku_2)}{\phi_2}\right\} \\ &= \frac{V_2V_2^*(\xi^ku_2)}{\xi^k}\frac{V_1^*(\xi^ku_1) - V_2^*(\xi^ku_2)}{\phi_1} + \frac{V_2V_2^*(\xi^ku_2)}{\xi^k}\frac{V_2^*(\xi^ku_2)}{\xi^{k+1}}\left\{\frac{\xi^{k+1}}{\phi_1} - \frac{\xi^{k+1}}{\phi_2}\right\}. \end{aligned}$$

It is easy to deduce that $(\star\star)$ bounded by A and $\mathcal{D}^{\frac{1}{2}}$. Other terms can be similarly estimated. This finishes the proof of the uniqueness and Theorem 2.1.

7 Duality argument

Here we would like to prove the existence for the linear problem (60). This is a consequence of the following proposition.

Proposition 7.1. *For f and g in $L^1(0, T; L^2)$, there exists a unique solution (F, G) to the linear system*

$$\begin{cases} \partial_t F - V^*G = f \\ \partial_t G + VF = g \\ G(\xi = 1) = 0, F(t = 0) = G(t = 0) = 0 \end{cases} \quad (72)$$

on $(0, T)$ which satisfies

$$\|(F, G)\|_{C([0, T]; L^2)} \leq C\|(f, g)\|_{L^1 L^2}. \quad (73)$$

Moreover, if $(f, g) \in L^1(0, T; X^{k,j} \times Y^{k,j})$ for some integer $j \leq [k]$ and for $0 \leq 2i \leq j - 1$, we have $(V^*)^{2i}g = 0$ at $\xi = 1$ and for $1 \leq 2i + 1 \leq j - 1$, we have $(V)^{2i+1}f = 0$ at $\xi = 1$, then

$$\|(F, G)\|_{C([0, T]; X^{k,j} \times Y^{k,j})} \leq C \|(f, g)\|_{L^1(X^{k,j} \times Y^{k,j})} \quad (74)$$

for some constant C which only depends on $\|\xi^k \phi\|_{X^{k, [k]+3}}$.

The proof is based on a duality argument. Let \mathcal{A} denote the set

$$\mathcal{A} = \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in C^\infty((0, \infty) \times (0, 1]), \text{ such that, } \psi(\xi = 1) = 0, (\phi, \psi)_{t=T} = 0 \right\}.$$

Hence, (F, G) solves (72) on a time interval $(0, T)$ if and only if for each test function $(\phi, \psi) \in \mathcal{A}$, we have

$$\begin{aligned} \int_0^T \int -F \partial_t \phi - GV \phi &= \int_0^T \int f \phi \\ \int_0^T \int -G \partial_t \psi + FV^* \psi &= \int_0^T \int g \psi. \end{aligned} \quad (75)$$

We denote $\mathcal{L} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} \partial_t F - V^* G \\ \partial_t G + V F \end{pmatrix}$ defined on the core

$$\left\{ \begin{pmatrix} F \\ G \end{pmatrix}, |\partial_t \begin{pmatrix} F \\ G \end{pmatrix} \in L_t^2 L_\xi^2, (F, G) \in L_t^2(\mathcal{D}(V), \mathcal{D}(V^*)) \right\}.$$

Hence, \mathcal{L} can be extended in a unique way to a closed operator. Moreover, $\mathcal{A} \subset \mathcal{D}(\mathcal{L}^*)$, the dual of \mathcal{L} and $\mathcal{L}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} -\partial_t \phi + V^* \psi \\ -\partial_t \psi - V \phi \end{pmatrix}$. Hence, (75) holds for each $(\phi, \psi) \in \mathcal{A}$ if and only if for each $(\phi, \psi) \in \mathcal{A}$, we have

$$\int_0^T \int \begin{pmatrix} F \\ G \end{pmatrix} \cdot \mathcal{L}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \int_0^T \int \begin{pmatrix} f \\ g \end{pmatrix} \cdot \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \quad (76)$$

We take $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{A}$ and denote $\mathcal{L}^* \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$. The energy estimate written for \mathcal{L}^* yields that

$$\sup_{0 \leq t \leq T} \|\phi\|_{L^2}^2 + \|\psi\|_{L^2}^2 \leq C \int_0^T \|\Phi\|_{L^2}^2 + \|\Psi\|_{L^2}^2.$$

Hence, the operator \mathcal{L}^* defines a bijection between \mathcal{A} and $\mathcal{L}^*(\mathcal{A})$. Let S_0 be the inverse. Hence

$$\begin{aligned} S_0 : \mathcal{L}^*(\mathcal{A}) &\rightarrow \mathcal{A} \\ \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} &\rightarrow \begin{pmatrix} \phi \\ \psi \end{pmatrix} \end{aligned}$$

and we have

$$\|S_0 \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}\|_{C([0, T]; L^2)} \leq C \left\| \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\|_{L^1 L^2}.$$

We extend this operator by density to $\overline{\mathcal{L}^*(\mathcal{A})}^{L^1 L^2}$ and to $L^1 L^2$ by Hahn-Banach. We denote this extension by S . Hence

$$\begin{aligned} S : L^1 L^2 &\rightarrow C([0, T]; L^2) \\ \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} &\rightarrow \begin{pmatrix} \phi \\ \psi \end{pmatrix} \end{aligned}$$

Now, we want to solve (72), namely $\mathcal{L}\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$ with $\begin{pmatrix} F \\ G \end{pmatrix}_{t=0} = 0$. This is, of course, equivalent to the fact that (76) holds for each $(\phi, \psi) \in \mathcal{A}$.

Hence it is enough that for all $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \in L^1L^2$, we have

$$\int_0^T \int \begin{pmatrix} F \\ G \end{pmatrix} \cdot \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \int_0^T \int \begin{pmatrix} f \\ g \end{pmatrix} \cdot S \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \quad (77)$$

Therefore, it is enough to take $\begin{pmatrix} F \\ G \end{pmatrix} = S^* \begin{pmatrix} f \\ g \end{pmatrix}$ where S^* is the dual of S which satisfies

$$S^* : \mathcal{M}(0, T; L^2) \rightarrow L^\infty(0, T; L^2).$$

In particular it maps L^1L^2 into $L^\infty L^2$. Hence (73) holds with $C([0, T]; L^2)$ replaced by $L^\infty([0, T]; L^2)$. At this stage we do not know whether $\begin{pmatrix} F \\ G \end{pmatrix}$ is continuous. This will actually follow from the regularity.

The uniqueness of $\begin{pmatrix} F \\ G \end{pmatrix}$ follows from the fact that if $f = g = 0$ in (72), then $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the unique solution to (72) in $L^\infty([0, T]; L^2)$. To prove this consider a solution $\begin{pmatrix} F \\ G \end{pmatrix} \in L^\infty([0, T]; L^2)$ to (72) with $f = g = 0$. We will also use the duality argument. Indeed, arguing as above and changing the roles of \mathcal{L} and \mathcal{L}^* , we can prove the existence of a solution $\begin{pmatrix} \phi \\ \psi \end{pmatrix} \in L^\infty([0, T]; L^2)$ to the dual problem

$$\begin{cases} -\partial_t \phi + V^* \psi = \Phi \\ -\partial_t \psi - V \phi = \Psi \\ \psi(\xi = 1) = 0, \\ \phi(t = T) = \psi(t = 0) = 0. \end{cases} \quad (78)$$

for each $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \in L^1L^2$.

Then, for each $\begin{pmatrix} \Phi \\ \Psi \end{pmatrix} \in L^1L^2$, we consider $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ a solution to (78). Hence, we can write (76) with the solution $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$. This yields

$$\int_0^T \int \begin{pmatrix} F \\ G \end{pmatrix} \cdot \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = 0, \quad (79)$$

which gives rise to the fact that $F = G = 0$.

To prove (74), we can argue by induction of j . We start with the the case $j = 1$ and we first argue formally. Applying V and V^* to (72), we get

$$\begin{cases} \partial_t V^* G + V^* V F = V^* g + V_t^* G \\ \partial_t V F - V V^* G = V f + V_t F \\ V F(\xi = 1) = 0, \\ V F(t = 0) = V^* G(t = 0) = 0. \end{cases} \quad (80)$$

Notice that the boundary condition $V F(\xi = 1) = 0$, comes from the fact that $g = 0$ at $\xi = 1$. Hence, we deduce formally that

$$\|(V F, V^* G)\|_{L^\infty([0, T]; L^2)} \leq C \|(V^* g + V_t^* G, V f + V_t F)\|_{L^1 L^2}.$$

To make this rigorous, we define first $\begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix}$, the solution of (80) with the right hand side replaced by $\begin{pmatrix} V^* g \\ V f \end{pmatrix}$. Hence, $\mathcal{L}\begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} = \begin{pmatrix} V^* g \\ V f \end{pmatrix}$. Then, we define for each integer i , $\begin{pmatrix} Y_i \\ Z_i \end{pmatrix}$, the

solution of (80) with the right hand side replaced by $\begin{pmatrix} V_t^* V^{*-1} Y_{i-1} \\ V_t V^{-1} Z_{i-1} \end{pmatrix} = \begin{pmatrix} \frac{\phi_t}{\phi} Y_{i-1} \\ V_t V^{-1} Z_{i-1} \end{pmatrix}$. Hence,

$$\|(Y_0, Z_0)\|_{L^\infty([0,T];L^2)} \leq C \|(V^* g, V f)\|_{L^1 L^2}$$

and using the fact that $V_t V^{-1}$ is bounded from L^2 to L^2 , we get that

$$\begin{aligned} \|(Y_i, Z_i)\|_{L^\infty([0,T];L^2)} &\leq C \|(Y_{i-1}, Z_{i-1})\|_{L^1 L^2} \leq CT \|(Y_{i-1}, Z_{i-1})\|_{L^\infty L^2} \\ &\leq C(CT)^i \|(V^* g, V f)\|_{L^\infty L^2}. \end{aligned}$$

Hence, denoting

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \sum_{i=0}^{\infty} \begin{pmatrix} Y_i \\ Z_i \end{pmatrix},$$

we see that $\begin{pmatrix} Y \\ Z \end{pmatrix}$ solves

$$\begin{cases} \partial_t Y + V^* Z = V^* g + V_t^* V^{*-1} Y \\ \partial_t Z - V Y = V f + V_t V^{-1} Z \\ Z(\xi = 1) = 0, \\ Z(t = 0) = Y(t = 0) = 0. \end{cases} \quad (81)$$

The operator V can be thought of as an operator from $\mathcal{D}(V) \rightarrow L_\xi^2$. It has a transpose ${}^t V$ which goes from $L_\xi^2 \rightarrow \mathcal{D}(V)'$ where $\mathcal{D}(V)'$ is the dual space of $\mathcal{D}(V)$. It satisfies : for $f \in \mathcal{D}(V)$ and $u \in L^2$,

$$(V(f), u)_{L^2, L^2} = (f, {}^t V(u))_{\mathcal{D}(V), \mathcal{D}(V)'}$$

Moreover, if we identify L^2 to a subspace of $\mathcal{D}(V)'$, then ${}^t V$ extends V^* to L^2 .

The same argument shows that we can also extend V to L^2 by considering ${}^t V^*$. Hence, one can also interpret the system (72) as equalities in \mathcal{V}' and \mathcal{V}' and the same remark holds for (81).

Using that $V[\partial_t(V^{-1}Z)] = \partial_t Z - V_t V^{-1}Z$, we deduce that $V[\partial_t(V^{-1}Z) - VY - Vf]$. Since, the kernel of V is $\{0\}$, we deduce that

$$\partial_t(V^{-1}Z) - Y = f.$$

We also have $V^*[\partial_t(V^{*-1}Y)] = \partial_t Y - V_t^* V^{*-1}Y$. Hence, the first equation of (81) can be written

$$V^*[\partial_t(V^{*-1}Y) + Z - g] = 0.$$

Hence since the kernel of V^* is $\{0\}$ when we add the vanishing of the boundary condition at $\xi = 1$, we deduce that

$$\partial_t(V^{*-1}Y) + Z - g = 0.$$

By uniqueness for (72), we deduce that $F = V^{-1}Z$ and $G = V^{*-1}Y$. Hence, we get that $(F, G) \in L^\infty(0, T; X^{k,1} \times Y^{k,1})$. In particular, this also shows that $(\partial_t F, \partial_t G) \in L^1(0, T; L^2)$.

Hence, integrating in time, we deduce that $(F, G) \in C([0, T]; L^2)$. In particular this shows that (73) holds if we have more regularity on (f, g) . By a density argument, this shows that (73) holds even if we only know that $(f, g) \in L^1 L^2$.

Arguing by induction on j , we prove that if $(f, g) \in L^1(0, T; X^{k,j} \times Y^{k,j})$, then $(F, G) \in$

$C([0, T]; X^{k,j} \times Y^{k,j})$.

8 Acknowledgement

N. M was partially supported by an NSF grant DMS-0703145. J. J was supported by an NSF grant DMS-0635607.

References

- [1] D. AMBROSE, N. MASMOUDI: Well-posedness of 3D vortex sheets with surface tension, *Commun. Math. Sci.* **5**, 391-430 (2007)
- [2] J.-Y. CHEMIN: Dynamique des gaz à masse totale finie. *Asymptotic Anal.*, **3**, 215-220 (1990)
- [3] M. GRASSIN: Global smooth solutions to Euler equations for a perfect gas. *Indiana Univ. Math. J.* **47**, 1397-1432 (1998)
- [4] F. HUANG, P. MARCATI, R. PAN: Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum, *Arch. Ration. Mech. Anal.* **176**, 1-24 (2005)
- [5] J. JANG: Local wellposedness of viscous gaseous stars, *preprint* (2007)
- [6] H. LINDBLAD: Well posedness for the motion of a compressible liquid with free surface boundary, *Comm. Math. Phys.* **260**, 319-392 (2005)
- [7] T.-P. LIU: Compressible flow with damping and vacuum, *Japan J. Appl. Math* **13**, 25-32 (1996)
- [8] T.-P. LIU, J. SMOLLER: On the vacuum state for isentropic gas dynamics equations, *Advances in Math.* **1**, 345-359 (1980)
- [9] T.-P. LIU, T. YANG: Compressible Euler equations with vacuum, *J. Differential Equations* **140**, 223-237 (1997)
- [10] T.-P. LIU, T. YANG: Compressible flow with vacuum and physical singularity, *Methods Appl. Anal.* **7**, 495-509 (2000)
- [11] T. LUO, Z. XIN, T. YANG: Interface behavior of compressible Navier-Stokes equations with vacuum, *SIAM J. Math. Anal.* **31**, 1175-1191 (2000)
- [12] A. MAJDA: *Compressible fluid flow and systems of conservation laws in several space variables*, volume 53 of *Applied Mathematical Sciences*. Springer-Verlag, New York (1984)
- [13] T. MAKINO: Blowing up solutions of the Euler-Poisson equation for the evolution of gaseous stars. In *Proceedings of the Fourth International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics (Kyoto, 1991)*, volume 21, 615-624 (1992)
- [14] T. MAKINO, S. UKAI: Sur l'existence des solutions locales de l'équation d'Euler-Poisson pour l'évolution d'étoiles gazeuses. *J. Math. Kyoto Univ.* **27**, 387-399 (1987)

- [15] T. MAKINO, S. UKAI, S. KAWASHIMA: Sur la solution à support compact de l'équations d'Euler compressible. *Japan J. Appl. Math.* **3**, 249-257 (1986)
- [16] D. SERRE: Solutions classiques globales des équations d'Euler pour un fluide parfait compressible, *Ann. Inst. Fourier (Grenoble)* **47**, 139-153 (1997)
- [17] T. C. SIDERIS: Formation of singularities in three-dimensional compressible fluids. *Comm. Math. Phys.*, **101**, 475-485 (1985)
- [18] C.-J. XU, T. YANG: Local existence with physical vacuum boundary condition to Euler equations with damping, *J. Differential Equations* **210**, 217-231 (2005)
- [19] T. YANG: Singular behavior of vacuum states for compressible fluids, *Comput. Appl. Math.* **190**, 211-231 (2006)