Well-posedness of compressible Euler equations in a physical vacuum

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Abstract

An important problem in gas and fluid dynamics is to understand the behavior of vacuum states, namely the behavior of the system in the presence of vacuum. In particular, physical vacuum, in which the boundary moves with a nontrivial finite normal acceleration, naturally arises in the study of the motion of gaseous stars or shallow water. Despite its importance, there are only few mathematical results available near vacuum. The main difficulty lies in the fact that the physical systems become degenerate along the vacuum boundary. In this paper, we establish the local-in-time well-posedness of three-dimensional compressible Euler equations for polytropic gases with physical vacuum by considering the problem as a free boundary problem.

1 Introduction

Compressible Euler equations of isentropic, ideal gas dynamics take the form in Eulerian coordinates as follows:

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) = 0\,, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \operatorname{grad} p = 0\,. \end{aligned} \tag{1}$$

Here ρ , u and p denote respectively the density, velocity, and pressure of gas. In considering the polytropic gases, the constitutive relation, which is also called the equation of state, is given by

$$p = K\rho^\gamma \tag{2}$$

where K is an entropy constant and $\gamma > 1$ is the adiabatic gas exponent.

Compressible Euler system (1) is a fundamental example of a system of hyperbolic conservation laws. The first and second equations express respectively conservation of mass and momentum. It is well-known [15] that the system (1) is strictly hyperbolic if the density is bounded below from zero: $\rho > 0$. When the initial density function contains a vacuum, the vacuum boundary Γ is defined as

$$\Gamma = cl\{(t,x) : \rho(t,x) > 0\} \cap cl\{(t,x) : \rho(t,x) = 0\}$$

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where cl denotes the closure. It is convenient to introduce the sound speed c of the Euler equations (1)

$$c = \sqrt{\frac{d}{d\rho}p(\rho)} \quad (= \sqrt{K\gamma}\rho^{\frac{\gamma-1}{2}} \text{ for polytropic gases})$$

We recall that for one-dimensional flows, $u \pm c$ are the characteristic speeds of the system. If Γ is nonempty, the Euler system fails to be strictly hyperbolic along Γ, namely degenerate hyperbolic.

A vacuum boundary Γ is called *physical* if the normal acceleration near the boundary is bounded from below and above:

$$-\infty < \frac{\partial c^2}{\partial n} < 0 \tag{3}$$

in a small neighborhood of the boundary, where n is the outward unit normal to Γ. In other words, the pressure or the enthalpy ($= c^2/(\gamma - 1)$) accelerates the boundary in the normal direction. This physical vacuum can be realized by some self-similar solutions and stationary solutions for different physical systems such as Euler equations with damping, Euler-Poisson or Navier-Stokes-Poisson systems for gaseous stars [27, 28, 46, 63].

Recently, the physical vacuum has gotten a great deal of attention among mathematical community (see [1] and the review paper [30]). Despite its physical importance, the local existence theory of classical solutions featuring the physical vacuum boundary even for one-dimensional flows was only established recently. This is because if the physical vacuum boundary condition (3) is assumed, the classical theory of hyperbolic systems can not be directly applied [46, 63]: the characteristic speeds $u \pm c$ become singular with infinite spatial derivatives near the vacuum boundary and this singularity creates an analytical difficulty in standard Sobolev spaces. Local existence for the physical vacuum states was given by the authors [29]. In [29], the authors consider the one-dimensional Euler equations in mass Lagrangian coordinates. Existence was proved using a new structure lying upon the physical vacuum in the framework of free boundary problems. The nonlinear energy spaces in [29] are designed to guarantee such *minimal* regularity that the physical vacuum (3) is realized. Coutand and Shkoller [12] constructed H^2-type solutions with moving boundary in Lagrangian coordinates based on Hardy inequalities and degenerate parabolic regularization.

For multi-dimensional flows, parallel to a lot of activity and important progress in free surface boundary problems, in particular regarding incompressible fluids [2, 3, 4, 11, 20, 35, 58, 61, 64], more recently, there are some works trying to prove the local well-posedness of physical vacuum in 3D. Coutand, Lindblad and Shkoller [14] established a priori estimates based on time differentiated energy estimates and elliptic estimates for normal derivatives for $\gamma = 2$. As noted in [14], in order to carry out the additional elliptic estimates, sufficient smoothness of solutions was assumed and its justification by their energy should require additional work. The purpose of this article is to provide a new analysis of physical vacuum based on a hyperbolic type of weighted energy estimates of tangential and normal derivatives and to establish the local well-posedness of the three-dimensional Euler system with free surface boundary which moves with nonzero finite acceleration towards normal to the boundary. Independently of this work, Coutand and Shkoller [13] extended their one-dimensional methodology and combined it with the a priori estimates given in [14] to construct smooth solutions of the three-dimensional Euler equations. We were also informed that Lindblad [39] has a similar result using the linearized compressible Euler equations with a Nash-Moser iteration. The methods are very different.

1.1 Existence theories of compressible Euler flows

Before we formulate our problem, we briefly review some existence theories of compressible flows with vacuum states from various aspects. We will not attempt to address exhaustive references in this paper. In the absence of vacuum, namely when the system is strictly hyperbolic everywhere, one can use the theory of symmetric hyperbolic systems developed by Friedrichs-Lax-Kato [19, 33, 36] to construct smooth solutions; for instance, see Majda [48]. The breakdown of classical solutions was demonstrated by Sideris [59].

When the initial datum is compactly supported, there are at least three ways of looking at the problem. The first consists in solving the Euler equations in the whole space and requiring that the system (1) holds in the sense of distribution for all $x \in \mathbb{R}^d$ and $t \in [0, T]$. This is in particular the strategy used to construct global weak solutions (see for instance Diperna [16] and [10, 40]). The second way consists in symmetrizing the system first and then solving it using the theory of symmetric hyperbolic system. Again the symmetrized form has to be solved in the whole space. The third way is to require the Euler equations to hold on the set $\{(t, x) : \rho(t, x) > 0\}$ and write an equation for Γ. Here, the vacuum boundary Γ is part of the unknown: this is a free boundary problem and in this case, an appropriate boundary condition at vacuum is necessary.

In the first and second ways, there is no need of knowing exactly the position of the vacuum boundary. DiPerna used the theory of compensated compactness to pass to the limit weakly in a parabolic approximation of the system and recovered a weak solution of the Euler system (see also [40] where a kinetic formulation of the system was also used). Makino, Ukai and Kawashima [52] wrote the system in a symmetric hyperbolic form which allows the density to vanish. The system they get is not equivalent to the Euler equations when the density vanishes. This special symmetrization was also used for the Euler-Poisson system. This formulation was also used by Chemin [9] to prove the local existence of regular solutions in the sense that $c, u \in C([0, T); H^m(\mathbb{R}^d))$ for some $m > 1 + d/2$ and d is the space dimension (see also Serre [56] and Grassin [23], for some global existence result of classical solutions under some special conditions on the initial data, by extracting a dispersive effect after some invariant transformation). However, it was noted in [49, 50] that the requirement that c is continuously differentiable excludes many interesting solutions such as the stationary solutions of the Euler-Poisson system which have a behavior of the type $\rho \sim |x - x_0|^{\frac{1}{\gamma-1}}$, namely $c^2 \sim |x - x_0|$ near the vacuum boundary. Indeed, Nishida in [55] suggested to consider a free boundary problem which includes this kind of singularity caused by vacuum, not shock wave singularity.

For the third way: the free boundary problem, we divide into a few cases according to the initial behavior of the sound speed c. For simplicity, let the origin be the initial vacuum contact point ($x_0 = 0$). And let $c \sim |x|^h$. When $h \geq 1$, namely initial contact to vacuum is sufficiently smooth, Liu and Yang [45] constructed the local-in-time solutions to one-dimensional Euler equations with damping by using the energy method based on the adaptation of the theory of symmetric hyperbolic system. They also prove that c^2 can not be smooth across Γ after a finite time. We note that in these regimes there is no acceleration along the vacuum boundary. For $0 < h < 1$, the initial contact to vacuum is only Holder continuous. In particular, the corresponding behavior to $h = 1/2$ is the case of physical vacuum [12, 13, 14, 29]. For $0 < h < 1/2$ and $1/2 < h < 1$, its boundary behavior is believed to be ill-posed; indeed, we conjecture that it should instantaneously change into the physical vacuum. However, there is no mathematical justification available so far.

The case $h = 0$ is when there is no continuous initial contact of the density with vacuum.

It can be considered as either Cauchy problem or free boundary problem. An example of Cauchy problem when $h = 0$ is the Riemann problem for genuinely discontinuous initial datum (see for instance [7, 24]). An example of a free boundary problem when $h = 0$ is the work by Lindblad [38] where the density is positive at the vacuum boundary.

Having the local existence theory of vacuum states, the next important question is whether such a local solution exists globally in time or how it breaks down. The study of vacuum free boundary automatically excludes the breakdown of solutions caused by vacuum, which is one possible scenario of the breakdown for positive solutions to compressible Euler equations (1). It was shown in [43] that the shock waves vanish at the vacuum and the singular behavior is similar to the behavior of the centered rarefaction waves corresponding to the case when c is regular [46], which indicates that vacuum has a regularizing effect. Therefore it would be very interesting to investigate the long time behavior of vacuum states.

When there is damping, based on self-similar behavior, Liu conjectured [42] that time asymptotically, solutions to Euler equations with damping should behave like the ones to the porus media equation, where the canonical boundary is characterized by the physical vacuum condition (3). This conjecture was established by Huang, Marcati and Pan [26] in the framework of the entropy solution where the method of compensated compactness yields a global weak solution in L^∞. But in their work, there is no way of tracking the vacuum boundary. It would be interesting to investigate the asymptotic relationship between smooth solutions obtained by solving a free boundary problem of Euler equations with damping and smooth solutions of the porus media equation.

1.2 Other interesting vacuum states

Physical vacuum as well as other vacuum states appear in the theory of other physical systems, where we believe the methodology of this article could be applied. Here we briefly present some of them; see [30] for more detailed discussion.

The study of vacuum is important in understanding viscous flows [18, 41]. When vacuum appears initially, studying Cauchy problems for compressible Navier-Stokes equations with constant viscosity coefficients yields somewhat negative results: for instance, a finite time blow-up for nontrivial compactly supported initial density [62] and a failure of continuous dependence on initial data [25]. There are some existence theories available with the physical vacuum boundary for one-dimensional Navier-Stokes free boundary problems [47], for three-dimensional Navier-Stokes-Poisson equations with radial symmetry [28], and for other related models [17, 54]. On the other hand, to resolve the issue of no continuous dependence on initial data in [25] for constant viscosity coefficient, a density-dependent viscosity coefficient was introduced in [44]. Since then, there has been a lot of studies on global weak solutions for various models and stabilization results under gravitation and external forces: see [37, 65] and the references therein. Despite significant progress over the years, many interesting and important questions are still unanswered especially for general multi-dimensional flows.

The relativistic Euler equations are known to be symmetric hyperbolic away from vacuum [51]. A particular interest is compactly supported relativistic flows which for instance can be applied to the dynamics of stars in the context of special relativity. Whether one can extend the theory of free boundary problems including physical vacuum developed for non-relativistic Euler equations to relativistic case is an open problem. It turns out that vacuum states also arise in the theory of magnetohydrodynamics (MHD), another interesting system of hyperbolic conservation laws arising from electromechanical phenomena [15, 22]. Due to the interplay between the scalar pressure of the fluid and the anisotropic magnetic stress, vacuum states are richer than in hydrodynamics and their rigorous study in the context of nonlinear partial differential equations seems to be widely open.

Lastly, it is interesting to point out the connection between vacuum states of degenerate hyperbolic systems and degenerate elliptic and parabolic equations. One of the main difficulty of studying the Euler system with a free boundary is that it leads to a degenerate hyperbolic equation due to the fact that the density vanishes at the free boundary. For this we need techniques coming from degenerate elliptic equations; for instance, see [5]. Also, similar problems arise in degenerate parabolic equations for instance, in porous medium equations [8], in thin film equations [21], in the study of polymeric flows [53].

In the next section, we formulate the problem, introduce notations and state the main result.

2 Lagrangian Formulation and Main Result

2.1 Derivation of the system in Lagrangian coordinates

The boundary moves with a finite normal acceleration under the physical vacuum condition (3) and it is part of the unknown. The vacuum free boundary problem is studied in Lagrangian coordinates where the free boundary is fixed.

For smooth solutions, the Euler equations (1) can be written in Eulerian coordinates as :

$$\begin{aligned} (\partial_t + u\cdot\nabla)\rho + \rho(\nabla\cdot u) = 0\,, \\ \rho(\partial_t + u\cdot\nabla)u + K\nabla\rho^\gamma = 0\,. \end{aligned} \tag{4}$$

Let $\eta(t,x)$ be the position of the gas particle x at time t so that

$$\eta_t = u(t,\eta(t,x)) \text{ for } t>0 \text{ and } \eta(0,x) = x \text{ in } \Omega\,. \tag{5}$$

As in [14], we define the following Lagrangian quantities:

$$\begin{aligned} v(t,x) &\equiv u(t,\eta(t,x)) \text{ (Lagrangian velocity)} \\ f(t,x) &\equiv \rho(t,\eta(t,x)) \text{ (Lagrangian density)} \\ A &\equiv [D\eta]^{-1} \text{ (inverse of deformation tensor)} \\ J &\equiv \det D\eta \text{ (Jacobian determinant)} \\ a &\equiv JA \text{ (transpose of cofactor matrix)} \end{aligned}$$

We use Einstein's summation convention and the notation $F,_k$ to denote the k^{th}-partial derivative of F: $\partial_k F$. Both expressions will be used throughout the paper. We use i,j,k,l,r,s to denote 1, 2, 3. The Euler equations (4) read as follows:

$$\begin{aligned} f_t + fA^j_i v^i,_j = 0\,, \\ fv^i_t + KA^k_i f^\gamma,_k = 0\,. \end{aligned} \tag{6}$$

Since

$$J_t = JA^j_i v^i,_j \text{ and } J(0) = 1\,,$$

together with the equation for f, we find that

$$fJ = \rho_0$$

where ρ_0 is given initial density function. Thus, using $A^k_i = J^{-1}a^k_i$, (6) reduce to the following:

$$\rho_0 v^i_t + K a^k_i ({\rho_0}^\gamma J^{-\gamma}),_k = 0 \tag{7}$$

along with

$$\eta^i_t = v^i\,. \tag{8}$$

Now let w be

$$w \equiv K\rho_0^{\gamma-1}\,. \tag{9}$$

Note that $\frac{\gamma}{\gamma-1}w$ is the initial enthalpy. We are interested in smooth initial enthalpy profiles satisfying the physical vacuum condition (3):

$$\begin{gathered} w = 0 \text{ on } \partial\Omega\,(=\Gamma(0))\,,\ w > 0 \text{ in } \Omega\,, \\ \frac{1}{C}d(x) \le w(x) \le Cd(x) \text{ in } \Omega\,, \end{gathered} \tag{10}$$

where $d(x)$ is the distance function to the boundary. The equation (7) now takes the form

$$w^\alpha v^i_t + (w^{1+\alpha}\, A^k_i J^{-1/\alpha}),_k = 0\,, \tag{11}$$

where

$$\alpha \equiv \frac{1}{\gamma-1}\,.$$

We have used the Piola identity (16) to get (11) from (7). Note that $\alpha > 0$ and $\alpha \to \infty$ as $\gamma \to 1$. Since $\eta^i_t = v^i$, the equation (11) reads as an η equation:

$$w^\alpha \eta^i_{tt} + (w^{1+\alpha}\, A^k_i J^{-1/\alpha}),_k = 0\,. \tag{12}$$

Thus the equation (12) can be viewed as a degenerate nonlinear acoustic (wave) equation for η. Multiply (12) by η^i_t and integrate over Ω: the zeroth order energy estimates formally lead to the following energy conservation

$$\frac{d}{dt}\int_\Omega \left\{\frac{1}{2}w^\alpha|v|^2 + \alpha w^{1+\alpha}J^{-1/\alpha}\right\}dx = 0 \text{ denoted by } \frac{dE}{dt} = 0\,. \tag{13}$$

Note that due to the vanishing factor $w^{1+\alpha}$ at the free boundary, all the boundary terms from the integration by parts disappeared. This energy conservation is equivalent to the more familiar form in Eulerian coordinates:

$$\frac{d}{dt}\int_{\Omega(t)} \left\{\frac{1}{2}\rho|u|^2 + \frac{p}{\gamma-1}\right\}dx = 0$$

which is the conservation of the physical energy.

Remark 2.1. *The degenerate nonlinear acoustic system* (12) *is in a sense equivalent to the Euler equation* (4) *with physical vacuum. Notice that the initial datum* $w = K\rho_0^{\gamma-1}$ *is a parameter in the equation and that one recovers the Lagrangian density by taking* $f = \rho_0 J^{-1}$.

2.2 Differentiation of A and J

In order to have sufficient regularity so that the flow map η is guaranteed to be non-degenerate and smooth and in particular the Jacobian determinant J is to be bounded away from zero and smooth, we need estimates of sufficiently high order derivatives of η or v. The minimal number of derivatives needed will be determined according to the strength of degeneracy, namely vanishing exponent α. Here we present the differentiation of A and J.

Differentiating the inverse of deformation tensor, since $A \cdot [D\eta] = I$, one obtains

$$\partial_t A^k_i = -A^k_r v^r,_s A^s_i \ ; \quad \partial_l A^k_i = -A^k_r \partial_l \eta^r,_s A^s_i \tag{14}$$

Differentiating the Jacobian determinant, one obtains

$$\partial_t J = J A^s_r v^r,_s \ ; \quad \partial_l J = J A^s_r \partial_l \eta^r,_s \tag{15}$$

For the cofactor matrix $a = JA$, from (14) and (15), one obtains the following Piola identity.

$$a^k_i,_k = 0 \tag{16}$$

2.3 Notation

For a given vector field F on Ω, we use DF, div$F$, curl$F$ to denote its full gradient, its divergence, and its curl:

$$\begin{aligned} [DF]^i_j &\equiv F^i,_j \\ \text{div}F &\equiv F^r,_r \\ [\text{curl}F]^i &\equiv \epsilon_{ijk} F^k,_j \end{aligned}$$

Here ϵ_{ijk} is the Levi-Civita symbol: it is 1 if (i,j,k) is an even permutation of $(1,2,3)$, -1 if (i,j,k) is an odd permutation of $(1,2,3)$, and 0 if any index is repeated.

We introduce the following Lie derivatives along the flow map η:

$$\begin{aligned} [D_\eta F]^i_r &\equiv A^s_r F^i,_s \\ \text{div}_\eta F &\equiv A^s_r F^r,_s \\ [\text{curl}_\eta F]^i &\equiv \epsilon_{ijk} A^s_j F^k,_s \end{aligned}$$

which indeed correspond to Eulerian full gradient, Eulerian divergence, and Eulerian curl written in Lagrangian coordinates. When η is the identity map – for instance, the initial state of the flow map is the identity map as in (5) – these Lie derivatives are the standard full gradient, divergence, and curl. In addition, it is convenient to introduce the anti-symmetric curl matrix $\text{Curl}_\eta F$

$$[\text{Curl}_\eta F]^i_j \equiv A^s_j F^i,_s - A^s_i F^j,_s$$

Note that $\text{Curl}_\eta F$ is a matrix version of a vector $\text{curl}_\eta F$ and that $|\text{Curl}_\eta F|^2 = 2|\text{curl}_\eta F|^2$ holds. We will use both curl_η and Curl_η.

2.4 Energy and main result

To state our result, we need to introduce some energy which behaves differently for normal and tangential derivatives close to the boundary. We first introduce the tangent vector fields and the normal vector field for the initial domain.

For given smooth initial domain Ω, we consider a smooth function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\{x, \phi(x) = 0\} = \partial\Omega$ and $\{x, \phi(x) > 0\} = \Omega$, and that $|\nabla\phi(x)| = 1$ for all $x \in \Omega_\delta = \{x \in \Omega,\, d(x) < \delta\}$ for some small $\delta > 0$. In particular $\phi(x) = d(x)$ in Ω_δ. Let $\mathcal{T}$ be a finite set of (smooth enough) vector fields in Ω which are tangent to the boundary and $\mathcal{N}$ the set formed by the vector field ζ which is normal to the boundary and such that $\zeta = \nabla\phi$ for $x \in \Omega_\delta$. We also assume that at each point x in Ω the vectors of $\mathcal{T}$ and $\mathcal{N}$ span the whole space $\mathbb{R}^3$ and that the second order differential operator $\sum_{\beta\in\mathcal{T}} \beta^*\beta + \zeta^*\zeta$ is strictly elliptic. In the simplified case where $\Omega = \mathbb{T}^2 \times (0,1)$, we can just take $\partial_\zeta = \partial_3$ and $\mathcal{T} = \{\partial_1, \partial_2\}$.

We define the energies $\overline{\mathcal{B}}^N$, $\overline{\mathcal{C}}^N$, $\overline{\mathcal{D}}^N$ and $\overline{\mathcal{E}}^N$ by

$$\begin{aligned}
\overline{\mathcal{B}}^N(v) &\equiv \frac{1}{2}\int_\Omega \sum_{|m|+n=0}^{N} d^{1+\alpha+n}|\mathrm{curl}_\eta \partial_\beta^m \partial_\zeta^n v|^2 dx \\
\overline{\mathcal{C}}^N(\eta) &\equiv \frac{1}{2}\int_\Omega \sum_{|m|+n=0}^{N} d^{1+\alpha+n}|\mathrm{curl}_\eta\, \partial_\beta^m \partial_\zeta^n \eta|^2 dx \\
\overline{\mathcal{D}}^N(\eta) &\equiv \frac{1}{2\alpha}\int_\Omega \sum_{|m|+n=0}^{N} d^{1+\alpha+n}|\mathrm{div}_\eta \partial_\beta^m \partial_\zeta^n \eta|^2 dx \\
\overline{\mathcal{E}}^N(\eta, v) &\equiv \frac{1}{2}\int_\Omega \sum_{|m|+n=0}^{N} d^{\alpha+n}|\partial_\beta^m \partial_\zeta^n v|^2 + d^{1+\alpha+n}|D_\eta \partial_\beta^m \partial_\zeta^n \eta|^2 dx
\end{aligned} \tag{17}$$

In the above formulae, the sum is made over $m \in \mathbb{N}^I$ and ∂_β^m denotes the derivative $\partial_{\beta_1}^{m_1}\partial_{\beta_2}^{m_2}...\partial_{\beta_I}^{m_I}$ where $I = \mathrm{Card}\mathcal{T}$. The total energy is defined by

$$\overline{\mathcal{TE}}^N \equiv \overline{\mathcal{TE}}^N(\eta, v) \equiv \overline{\mathcal{E}}^N(\eta, v) + \overline{\mathcal{B}}^N(v) \tag{18}$$

Since $\eta(0, x) = x$, the total energy for initial data $v(0, x) = u_0$ is given by

$$\overline{\mathcal{TE}}^N(0) = \frac{3}{2}\int_\Omega d^{1+\alpha}dx + \sum_{|m|+n=0}^{N} \frac{1}{2}\int_\Omega d^{\alpha+n}|\partial_\beta^m \partial_\zeta^n u_0|^2 + d^{1+\alpha+n}|\mathrm{curl}\, \partial_\beta^m \partial_\zeta^n u_0|^2 dx \tag{19}$$

We also introduce the function spaces $\overline{X}^{\alpha,b}$, $\overline{Y}^{\alpha,b}$

$$\begin{aligned}
\overline{X}^{\alpha,b} &\equiv \{d^{\frac{\alpha}{2}} F \in L^2(\Omega) : \int_\Omega d^{\alpha+n}|\partial_\beta^m \partial_\zeta^n F|^2 dx < \infty\,,\; 0 \le |m| + n \le b\} \\
\overline{Y}^{\alpha,b} &\equiv \{d^{\frac{1+\alpha}{2}} DF \in L^2(\Omega) : \int_\Omega d^{1+\alpha+n}|D\partial_\beta^m \partial_\zeta^n F|^2 dx < \infty\,,\; 0 \le |m| + n \le b\}
\end{aligned} \tag{20}$$

equipped with the following norms:

$$\|F\|^2_{\overline{X}^{\alpha,b}} \equiv \sum_{|m|+n=0}^{b} \int_\Omega d^{\alpha+n}|\partial_\beta^m \partial_\zeta^n F|^2 dx\,; \quad \|F\|^2_{\overline{Y}^{\alpha,b}} \equiv \sum_{|m|+n=0}^{b} \int_\Omega d^{1+\alpha+n}|D\partial_\beta^m \partial_\zeta^n F|^2 dx$$

Note that

$$\overline{\mathcal{E}}^N \sim \|v\|^2_{\overline{X}^{\alpha,N}} + \|\eta\|^2_{\overline{Y}^{\alpha,N}}\,.$$

We are now ready to state the main result of this article.

Theorem 2.2. *Let $\alpha > 0$ be fixed and $N \geq 2[\alpha] + 9$ be given. Suppose that $Dw \in \overline{X}^{\alpha,N}$ and that the initial energy (19) is bounded: $\overline{\mathcal{TE}}^N(0) < \infty$. Then there exist a time $T > 0$ depending only on $\overline{\mathcal{TE}}^N(0)$ and $\|Dw\|_{\overline{X}^{\alpha,N}}$ a unique solution $(v, \eta) \in C([0,T]; \overline{X}^{\alpha,N} \times \overline{Y}^{\alpha,N})$ to the Euler equation (8) and (12) on the time interval $[0,T]$ satisfying*

$$\overline{\mathcal{TE}}^N(\eta, v) \leq 2\overline{\mathcal{TE}}^N(0) \quad \text{and} \quad \|A - I\|_\infty \leq 1/8\,.$$

In particular, $2/3 \leq J \leq 2$.

Theorem 2.2 indicates that the minimal number of derivatives needed to capture the physical vacuum (3) depends on the value of the adiabatic exponent γ. The smaller γ is, the more derivatives are required to overcome stronger degeneracy caused by physical vacuum. Indeed, this phenomenon was captured in our one-dimensional result [29] where the analysis is carried out in mass Lagrangian coordinates. While there is some similarity between our new analysis and the previous one-dimensional analysis, besides boundary geometry, there is another critical difference regarding the energy functionals: In one-dimensional V, V^* framework, the energy space is very nonlinear in that it is not clear at all to deduce some equivalence to the standard weighted Sobolev spaces and moreover, the number of V, V^* to define the energy space is rigid. On the other hand, in the current analysis, the energy functionals given in (24) are indeed equivalent to the standard linear weighted Sobolev spaces as discussed in Section 3.1 and also the higher regularity can be readily established.

The method of the proof is based on a hyperbolic type of new energy estimates which consist in the instant energy estimates and the curl estimates. As soon as we linearize the Euler system, we start to see geometric structures: the full gradient, divergence and curl of flow map η in the equations. The new key is to extract *right algebraic weighted structure* for the linearization in the normal direction such that we can directly estimate normal derivatives via the energy estimates: each time we take normal derivative, we obtain more singular (degenerate) weight while the main structure of the equation remains the same for tangential derivatives. The effect of taking one normal derivative is worth gaining a half derivative. The new energy estimates provide a unified, systematic way of treating all the spatial derivatives. In the instant energy estimates, the curl part comes with undesirable negative sign. This can be absorbed by adding the curl estimates which will be obtained separately from the curl equation. For the construction of solutions, we implement the linear approximate schemes for $G = [\sum_{\beta\in\mathcal{T}}(\partial_\beta)^*\partial_\beta - w^{-\alpha}\partial_\zeta w^{1+\alpha}\partial_\zeta + \lambda]\eta$ with λ big enough and we use a relaxed curl of G. To build the well-posedness of linear approximate systems, we employ the duality argument as done in [29]. Finally, η is found by solving the above degenerate elliptic equation.

In the sequel we will give a detailed proof in the simplified case $\Omega = \mathbb{T}^2 \times (0,1)$ and then indicate the changes to be done in the general case. The rest of the paper is organized as follows. In section 3, we rewrite the energies in the simplified case $\Omega = \mathbb{T}^2 \times (0,1)$ and prove some Hardy type inequalities. In Section 4, we establish the a priori energy estimates. In Section 5, an approximate scheme is provided and Theorem 2.2 is proven. In Section 6, the general domain case is discussed. In Section 7, we conclude the article with a few remarks.

3 A simplified domain

For simplicity of the presentation, we will present the full proof in the case when the initial domain is taken as

$$\Omega = \mathbb{T}^2 \times (0,1)$$

where $\mathbb{T}^2$ is a two-dimensional period box in x_1, x_2. We will then present the main changes to be done in the general case. The initial boundary is given as

$$\Gamma(0) = \{x_3 = 0\} \cup \{x_3 = 1\} \text{ as the reference vacuum boundary.}$$

The moving vacuum boundary is given by $\Gamma(t) = \eta(t)(\Gamma(0))$.

We use Latin letters $i, j, k, ...$ to denote 1, 2, 3 and that we use Greek letters β, κ, σ to denote 1, 2 only. Recall the weight w – initial enthalpy (9) satisfying (10). We now rewrite the various energy functionals. We use ∂_β^m to denote $\partial_1^{m_1}\partial_2^{m_2}$ and $|m|$ to denote $|m| = m_1 + m_2$.

We also introduce η dependent energies. These energies are equivalent to the ones given in (17) with the difference that d is replaced by w and that a factor $J^{-1/\alpha}$ is added. It turns out that these versions are more adapted to the energy estimates and have better cancellation properties.

The curl energies are defined by

$$\begin{aligned}
\mathcal{B}^N(v) &\equiv \sum_{|m|+n=1}^{N} \frac{1}{2}\int_\Omega w^{1+\alpha+n} J^{-1/\alpha} |\mathrm{curl}_\eta \partial_\beta^m \partial_3^n v|^2 dx \equiv \sum_{|m|+n=1}^{N} \mathcal{B}^{m,n} \\
\mathcal{C}^N(\eta) &\equiv \sum_{|m|+n=1}^{N} \frac{1}{2}\int_\Omega w^{1+\alpha+n} J^{-1/\alpha} |\mathrm{curl}_\eta \partial_\beta^m \partial_3^n \eta|^2 dx \equiv \sum_{|m|+n=1}^{N} \mathcal{C}^{m,n}
\end{aligned} \tag{21}$$

The divergence energy is defined by

$$\mathcal{D}^N(\eta) \equiv \sum_{|m|+n=1}^{N} \frac{1}{2\alpha}\int_\Omega w^{1+\alpha+n} J^{-1/\alpha} |\mathrm{div}_\eta \partial_\beta^m \partial_3^n \eta|^2 dx \equiv \sum_{|m|+n=1}^{N} \mathcal{D}^{m,n} \tag{22}$$

The instant energy is defined by

$$\begin{aligned}
\mathcal{E}^N(\eta, v) &\equiv E + \sum_{|m|+n=1}^{N} \frac{1}{2}\int_\Omega w^{\alpha+n} |\partial_\beta^m \partial_3^n v|^2 dx + \frac{1}{2}\int_\Omega w^{1+\alpha+n} J^{-1/\alpha} |D_\eta \partial_\beta^m \partial_3^n \eta|^2 dx \\
&\equiv E + \sum_{|m|+n=1}^{N} \mathcal{E}^{m,n}
\end{aligned} \tag{23}$$

The total energy is defined by

$$\mathcal{TE}^N \equiv \mathcal{TE}^N(\eta, v) \equiv \mathcal{E}^N(\eta, v) + \mathcal{B}^N(v) \tag{24}$$

Since $\eta(0,x) = x$, the total energy for initial data $v(0,x) = u_0$ is given by

$$\mathcal{TE}^N(0) = \mathcal{TE}^N(x, u_0) = E + \sum_{|m|+n=1}^{N} \frac{1}{2}\int_\Omega w^{\alpha+n} |\partial_\beta^m \partial_3^n u_0|^2 + w^{1+\alpha+n} |\mathrm{curl}\, \partial_\beta^m \partial_3^n u_0|^2 dx \tag{25}$$

Next we introduce the function spaces $X^{\alpha,b}$, $Y^{\alpha,b}$, $Z^{\alpha,b}$, which will be used in the construction of solutions to approximate scheme in Section 5.1, associated to our various energy functionals (21) – (23):

$$
\begin{aligned}
X^{\alpha,b} &\equiv \{w^{\frac{\alpha}{2}}F \in L^2(\Omega) : \int_\Omega w^{\alpha+n}|\partial_\beta^m \partial_3^n F|^2 dx < \infty\,,\ 0 \le |m|+n \le b\} \\
Y^{\alpha,b} &\equiv \{w^{\frac{1+\alpha}{2}}D_\eta F \in L^2(\Omega) : \int_\Omega w^{1+\alpha+n} J^{-\frac{1}{\alpha}}|D_\eta \partial_\beta^m \partial_3^n F|^2 dx < \infty\,,\ 0 \le |m|+n \le b\} \\
Z^{\alpha,b} &\equiv \{w^{\frac{1+\alpha}{2}}F \in L^2(\Omega) : \int_\Omega w^{1+\alpha+n} J^{-\frac{1}{\alpha}}|\partial_\beta^m \partial_3^n F|^2 dx < \infty\,,\ 0 \le |m|+n \le b\}
\end{aligned} \tag{26}
$$

3.1 Imbedding of a weighted Sobolev space

For any given $\alpha > 0$ and given nonnegative integer b, we define the weighted Sobolev spaces $H^{\alpha,b}(\Omega)$ by

$$H^{\alpha,b}(\Omega) \equiv \{d^{\frac{\alpha}{2}}F \in L^2(\Omega) : \int_\Omega d^\alpha |D^k F|^2 dx < \infty\,,\ 0 \le k \le b\}$$

with the norm

$$\|F\|^2_{H^{\alpha,b}} \equiv \sum_{k=0}^{b} \int_\Omega d^\alpha |D^k F|^2 dx$$

We denote the standard Sobolev spaces by H^s. Then for $b \ge \alpha/2$, the weighted spaces $H^{\alpha,b}$ satisfy the following Hary type embedding [34]:

$$H^{\alpha,b}(\Omega) \hookrightarrow H^{b-\frac{\alpha}{2}}$$

with

$$\|F\|_{H^{b-\alpha/2}} \lesssim \|F\|_{H^{\alpha,b}}$$

As an application of the above embedding of weighted Sobolev spaces, we first obtain the embedding of $X^{\alpha,b}$ into Sobolev spaces for sufficiently smooth w.

Lemma 3.1. *For $b \ge \lceil \alpha \rceil$,*

$$\|F\|_{H^{\frac{b-\alpha}{2}}} \lesssim \|F\|_{X^{\alpha,b}}$$

In particular, for $b \ge [\alpha] + 4$,

$$\|F\|_\infty \lesssim \|F\|_{X^{\alpha,b}}$$

If $A = [D\eta]^{-1}$ is close to the identity – for instance, if $\|A - I\|_\infty \le 1/8$ which implies $2/3 \le J \le 2$ – then $Y^{\alpha,b}$ and $Z^{\alpha,b}$ are isomorphic to standard linear weighted Sobolev spaces like $X^{\alpha,b}$. Therefore, Lemma 3.1 dictates the similar embeddings for $Y^{\alpha,b}$ and $Z^{\alpha,b}$: for $b \ge \lceil \alpha \rceil + 1$,

$$\|DF\|_{H^{\frac{b-\alpha-1}{2}}} \lesssim \|F\|_{Y^{\alpha,b}} \quad \text{and} \quad \|F\|_{H^{\frac{b-\alpha-1}{2}}} \lesssim \|F\|_{Z^{\alpha,b}}$$

4 A Priori Energy Estimates

In this section, we will prove the following a priori estimates.

Proposition 4.1. *Let $\alpha > 0$ be given and let $N \ge 2[\alpha] + 9$ and $Dw \in X^{\alpha,N}$. Suppose η and v solve* (8) *and* (12) *for $t \in [0,T]$ with $\mathcal{T}\mathcal{E}^N(\eta, v) < \infty$ and $1/C_0 \le J \le C_0$ for some $C_0 \ge 1$.*

We further assume that η and v enjoy the a priori bound: for any $s = 1, 2,$ and 3,

$$\sum_{|p|+q=0}^{[N/2]} |w^{q/2}\partial_\beta^p\partial_3^q\eta^r,_s| + \sum_{|p|+q=0}^{[N/2]-1} |w^{q/2}\partial_\beta^p\partial_3^q v^r,_s| < \infty \tag{27}$$

Then we obtain the following a priori estimates:

$$\frac{d}{dt}\mathcal{E}^N(\eta, v) \leq \mathcal{F}(\mathcal{E}^N(\eta, v), \mathcal{B}^N(u_0), C_0) \tag{28}$$

where $\mathcal{F}(\mathcal{E}^N(\eta, v), \mathcal{B}^N(u_0), C_0)$ is a continuous function of $\mathcal{E}^N(\eta, v)$, $\mathcal{B}^N(u_0)$, and C_0. In addition, we have the curl energy $\mathcal{B}^N$ bounded

$$\mathcal{B}^N(v) \leq \mathcal{B}^N(u_0) + \mathcal{G}(\mathcal{E}^N(\eta, v), C_0, T) \tag{29}$$

where $\mathcal{G}(\mathcal{E}^N(\eta, v), C_0, T)$ is a continuous function of $\mathcal{E}^N(\eta, v)$, C_0 and T. Moreover, the a priori assumption (27) can be justified.

The proof of Proposition 4.1 is based on the following two key lemmas.

Lemma 4.2. *Assume as in Proposition 4.1. Then we obtain the following*

$$\frac{d}{dt}\left\{\mathcal{E}^N(\eta, v) + \mathcal{D}^N(\eta) - \mathcal{C}^N(\eta)\right\} \leq \mathcal{F}_1(\mathcal{E}^N(\eta, v), C_0)\,. \tag{30}$$

Lemma 4.3. *Assume as in Proposition 4.1. Then we obtain the following*

$$\frac{d}{dt}\mathcal{C}^N(\eta) \leq \mathcal{F}_2(\mathcal{E}^N(\eta, v), \mathcal{B}^N(u_0), C_0)\,. \tag{31}$$

Moreover, (29) is satisfied.

By adding (30) and (31), the main estimate (28) follows. Moreover, the a priori assumption (27) can be verified within $\mathcal{E}^N$ by Lemma 3.1 and the standard continuity argument.

As a preparation of higher order energy estimates, we first compute the spatial derivative of $A_i^k J^{-1/\alpha}$.

$$\begin{aligned}
\partial_l(A_i^k J^{-1/\alpha}) &= J^{-1/\alpha}\partial_l A_i^k - \tfrac{1}{\alpha} J^{-(1+\alpha)/\alpha} A_i^k \partial_l J \\
&= -J^{-1/\alpha} A_r^k A_i^s \partial_l \eta^r,_s - \tfrac{1}{\alpha} J^{-1/\alpha} A_i^k A_r^s \partial_l \eta^r,_s \\
&= -J^{-1/\alpha} A_r^k A_r^s \partial_l \eta^i,_s - \tfrac{1}{\alpha} J^{-1/\alpha} A_i^k A_r^s \partial_l \eta^r,_s \\
&\quad - J^{-1/\alpha} A_r^k [A_i^s \partial_l \eta^r,_s - A_r^s \partial_l \eta^i,_s]
\end{aligned}$$

Thus

$$\partial_l(A_i^k J^{-1/\alpha}) = -J^{-1/\alpha} A_r^k\,[D_\eta \partial_l \eta]_r^i - \tfrac{1}{\alpha} J^{-1/\alpha} A_i^k\,\mathrm{div}_\eta \partial_l \eta - J^{-1/\alpha} A_r^k\,[\mathrm{Curl}_\eta \partial_l \eta]_i^r \tag{32}$$

Similarly, the time derivative of $A_i^k J^{-1/\alpha}$ is given by

$$\partial_t(A_i^k J^{-1/\alpha}) = -J^{-1/\alpha} A_r^k\,[D_\eta v]_r^i - \tfrac{1}{\alpha} J^{-1/\alpha} A_i^k\,\mathrm{div}_\eta v - J^{-1/\alpha} A_r^k\,[\mathrm{Curl}_\eta v]_i^r \tag{33}$$

We remark that after taking derivatives, namely *after linearization*, we start to see structures: in (32) and (33), the first term corresponds to the full gradient, the second to the divergence, the last term to the curl. For $n \geq 1$, we write $\partial_l^n[A_i^k J^{-1/\alpha}]$ as follows.

$$\begin{aligned}\partial_l^n[A_i^k J^{-1/\alpha}] = & -J^{-1/\alpha} A_r^k\,[D_\eta \partial_l^n \eta]_r^i - \tfrac{1}{\alpha} J^{-1/\alpha} A_i^k\,\mathrm{div}_\eta \partial_l^n \eta - J^{-1/\alpha} A_r^k\,[\mathrm{Curl}_\eta \partial_l^n \eta]_i^r \\ & - \sum_{p=1}^{n-1}\{\partial_l^p[J^{-1/\alpha} A_r^k A_i^s]\,\partial_l^{n-p}\eta^r,_s + \tfrac{1}{\alpha}\partial_l^p[J^{-1/\alpha} A_i^k A_r^s]\,\partial_l^{n-p}\eta^r,_s\}\end{aligned} \tag{34}$$

It turns out that the curl comes with bad sign in the main energy estimates, and we will obtain the estimates of the curl separately from the curl equation (40) or (41) as if it were an ordinary differential equation.

4.1 The proof of Lemma 4.2 : Energy estimates for $\mathcal{E}^N$

We first claim that when taking ∂_3^n of the equation (12), we obtain the following higher order equations of different structures depending on n:

$$w^{\alpha+n}\partial_3^n \eta_{tt}^i + (w^{1+\alpha+n}\,\partial_3^n[A_i^k J^{-1/\alpha}]),_k + w^{\alpha+n} I^{0,n} = 0 \tag{35}$$

where $I^{0,n}$ are essentially lower-order terms given inductively as follows:

$$\begin{aligned}I^{0,0} = 0\,;\; I^{0,n} = & \,\partial_3 I^{0,n-1} + \partial_3 w \cdot \partial_3^{n-1}[A_i^\kappa J^{-1/\alpha}],_\kappa - w,_\kappa \cdot \partial_3^n[A_i^\kappa J^{-1/\alpha}] \\ & + (\alpha+n)\partial_3 w,_k \cdot \partial_3^{n-1}[A_i^k J^{-1/\alpha}] \text{ for } n \geq 1\end{aligned} \tag{36}$$

In order to see that, it is convenient to write the equation (35) in the following form:

$$\begin{aligned}&\partial_3^n \eta_{tt}^i + w\partial_3^{n+1}[A_i^3 J^{-1/\alpha}] + (1+\alpha+n)\partial_3 w\,\partial_3^n[A_i^3 J^{-1/\alpha}] \\ &\quad + w\partial_3^n[A_i^\kappa J^{-1/\alpha}],_\kappa + (1+\alpha+n)w,_\kappa\,\partial_3^n[A_i^\kappa J^{-1/\alpha}] + I^{0,n} = 0\end{aligned}$$

This form can be readily verified by the induction on n starting from (12). The equations for general mixed derivatives $\partial_\beta^m \partial_3^n \eta$ read as follows.

$$w^{\alpha+n}\partial_\beta^m \partial_3^n \eta_{tt}^i + (w^{1+\alpha+n}\,\partial_\beta^m \partial_3^n[A_i^k J^{-1/\alpha}]),_k + w^{\alpha+n} I^{m,n} = 0 \tag{37}$$

where $I^{m,n}$ is given inductively as follows: for $|m| \geq 1$

$$I^{m,n} = \partial_\beta I^{m-1,n} + \partial_\beta w \cdot \partial_\beta^{m-1}\partial_3^n[A_i^k J^{-1/\alpha}],_k + (1+\alpha+n)\partial_\beta w,_k \cdot \partial_\beta^{m-1}\partial_3^n[A_i^k J^{-1/\alpha}] \tag{38}$$

The degeneracy of the equation lies along the normal direction x_3. It is interesting to point out that the more normal derivatives we take, the more degeneracy we have, while the structure of the equation stays the same for the tangential derivatives ∂_β.

We now perform the energy estimates of (37) for $\partial_\beta^m \partial_3^n v$ and $D_\eta \partial_\beta^m \partial_3^n \eta$. By using (34), we rewrite (37) as follows: for $1 \leq |m| + n \leq N$,

$$
\begin{aligned}
&w^{\alpha+n}\partial_\beta^m\partial_3^n\eta_{tt}^i - (w^{1+\alpha+n}\, J^{-1/\alpha}A_r^k\,[D_\eta\partial_\beta^m\partial_3^n\eta]_r^i),_k - \tfrac{1}{\alpha}(w^{1+\alpha+n}\, J^{-1/\alpha}A_i^k\,\mathrm{div}_\eta\partial_\beta^m\partial_3^n\eta),_k \\
&- (w^{1+\alpha+n}\, J^{-1/\alpha}A_r^k\,[\mathrm{Curl}_\eta\partial_\beta^m\partial_3^n\eta]_i^r),_k \\
&- \sum_{|p|+q=1}^{|m|+n-1}(w^{1+\alpha+n}\,\{\partial_\beta^p\partial_3^q[J^{-1/\alpha}A_r^kA_i^s]\,\partial_\beta^{m-p}\partial_3^{n-q}\eta^r,_s + \tfrac{1}{\alpha}\partial_\beta^p\partial_3^q[J^{-1/\alpha}A_i^kA_r^s]\,\partial_\beta^{m-p}\partial_3^{n-q}\eta^r,_s\}),_k \\
&+ w^{\alpha+n}I^{m,n} = 0
\end{aligned}
$$

Multiplying by $\partial_\beta^m\partial_3^n\eta_t^i$ and integrating over Ω, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int w^{\alpha+n}|\partial_\beta^m\partial_3^n v|^2dx + \int w^{1+\alpha+n}\partial_\beta^m\partial_3^n\eta_t^i,_k\cdot J^{-1/\alpha}A_r^k\,[D_\eta\partial_\beta^m\partial_3^n\eta]_r^i dx& \\
+\tfrac{1}{\alpha}\int w^{1+\alpha+n}\partial_\beta^m\partial_3^n\eta_t^i,_k\cdot J^{-1/\alpha}A_i^k\,\mathrm{div}_\eta\partial_\beta^m\partial_3^n\eta dx& \\
+\int w^{1+\alpha+n}\partial_\beta^m\partial_3^n\eta_t^i,_k\cdot J^{-1/\alpha}A_r^k\,[\mathrm{Curl}_\eta\partial_\beta^m\partial_3^n\eta]_i^r dx& \\
-\sum_{|p|+q=1}^{|m|+n-1}\int\partial_\beta^m\partial_3^n\eta_t^i\cdot(w^{1+\alpha+n}\,\{\partial_\beta^p\partial_3^q[J^{-1/\alpha}A_r^kA_i^s]\,\partial_\beta^{m-p}\partial_3^{n-q}\eta^r,_s& \\
+\tfrac{1}{\alpha}\partial_\beta^p\partial_3^q[J^{-1/\alpha}A_i^kA_r^s]\,\partial_\beta^{m-p}\partial_3^{n-q}\eta^r,_s\}),_k\,dx& \\
+\int w^{\alpha+n}\partial_\beta^m\partial_3^n\eta_t^i\cdot I^{m.n}dx = 0&
\end{aligned}
\tag{39}
$$

We will estimate line by line. The first line in (39) gives rise to the L^2 integral of the full gradient $D_\eta\partial_\beta^m\partial_3^n\eta$ with the weight $w^{1+\alpha+n}J^{-1/\alpha}$ and commutators involving $\partial_tA_r^k$ and ∂_tJ.

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\{\int w^{\alpha+n}|\partial_\beta^m\partial_3^n v|^2dx + \int w^{1+\alpha+n}J^{-1/\alpha}|D_\eta\partial_\beta^m\partial_3^n\eta|^2dx\} \\
&-\int w^{1+\alpha+n}\partial_\beta^m\partial_3^n\eta^i,_k\,J^{-1/\alpha}A_{r\,t}^k\cdot[D_\eta\partial_\beta^m\partial_3^n\eta]_r^idx + \frac{1}{2\alpha}\int w^{1+\alpha+n}J^{-\frac{1+\alpha}{\alpha}}J_t\,|D_\eta\partial_\beta^m\partial_3^n\eta|^2dx \\
&= \frac{d}{dt}\mathcal{E}^{m,n} + (i)
\end{aligned}
$$

For the second line in (39), $\mathrm{div}_\eta\partial_\beta^m\partial_3^n\eta$ and commutators come out.

$$
\begin{aligned}
&\frac{1}{2\alpha}\frac{d}{dt}\int w^{1+\alpha+n}J^{-1/\alpha}|\mathrm{div}_\eta\partial_\beta^m\partial_3^n\eta|^2dx \\
&-\tfrac{1}{\alpha}\int w^{1+\alpha+n}\partial_\beta^m\partial_3^n\eta^i,_k\,J^{-1/\alpha}A_{i\,t}^k\cdot\mathrm{div}_\eta\partial_\beta^m\partial_3^n\eta dx + \tfrac{1}{2\alpha^2}\int w^{1+\alpha+n}J^{-\frac{1+\alpha}{\alpha}}J_t\,|\mathrm{div}_\eta\partial_\beta^m\partial_3^n\eta|^2dx \\
&= \frac{d}{dt}\mathcal{D}^{m,n} + (ii)
\end{aligned}
$$

The third line can be written as $\mathrm{Curl}_\eta \partial_\beta^m \partial_3^n \eta$ plus commutators.

$$
\begin{aligned}
&-\sum_{i>r}\int w^{1+\alpha+n}J^{-1/\alpha}[A_i^k\partial_\beta^m\partial_3^n\eta_t^r,_k - A_r^k\partial_\beta^m\partial_3^n\eta_t^i,_k]\cdot[\mathrm{Curl}_\eta\partial_\beta^m\partial_3^n\eta]_i^r\,dx \\
&= -\frac{1}{4}\frac{d}{dt}\int w^{1+\alpha+n}J^{-1/\alpha}|\mathrm{Curl}_\eta\partial_\beta^m\partial_3^n\eta|^2dx - \frac{1}{4\alpha}\int w^{1+\alpha+n}J^{-\frac{1+\alpha}{\alpha}}J_t\,|\mathrm{Curl}_\eta\partial_\beta^m\partial_3^n\eta|^2dx \\
&\quad+\sum_{i>r}\int w^{1+\alpha+n}J^{-1/\alpha}[A^k_{i\,t}\,\partial_\beta^m\partial_3^n\eta^r,_k - A^k_{r\,t}\,\partial_\beta^m\partial_3^n\eta^i,_k]\cdot[\mathrm{Curl}_\eta\partial_\beta^m\partial_3^n\eta]_i^r\,dx \\
&= -\frac{d}{dt}\mathcal{C}^{m,n} + (iii)
\end{aligned}
$$

Note that we have used the following anti-symmetrization to obtain the curl structure. For any $(1,1)$ tensors E and F

$$
\begin{aligned}
&\sum_{i,r=1}^{3} E_r^i\,(F_i^r - F_r^i) = \left\{\sum_{i>r} + \sum_{i<r}\right\} E_r^i\,(F_i^r - F_r^i) \\
&= \sum_{i>r}(E_r^i - E_i^r)\,(F_i^r - F_r^i) + \underbrace{\sum_{i>r} E_i^r\,(F_i^r - F_r^i) + \sum_{i<r} E_r^i\,(F_i^r - F_r^i)}_{=0} \\
&= -\sum_{i>r}(E_i^r - E_r^i)\,(F_i^r - F_r^i)
\end{aligned}
$$

Since $|A^k_{r\,t}|$ and $|J_t|$ are bounded by the energy $\mathcal{E}^N$ given in (24), the commutators (i), (ii), (iii) are bounded by a continuous function of $\mathcal{E}^N$:

$$
|(i)| + |(ii)| + |(iii)| \le \mathcal{F}_3(\mathcal{E}^N)
$$

Next we turn into the fourth and fifth lines in (39) which consist of lower order nonlinear terms. We present the detail only for the first term and the other can be estimated in the same way. We consider two cases: $|m| = 0$ and $|m| \ge 1$. For $|m| = 0$, the index p does not appear and the first term reads as follows: for $1 \le q \le n-1$

$$
\begin{aligned}
&\int \partial_3^n\eta_t^i\cdot(w^{1+\alpha+n}\,\partial_3^q[J^{-1/\alpha}A_r^kA_i^s]\,\partial_3^{n-q}\eta^r,_s),_k\,dx \\
&= \int \partial_3^n\eta_t^i\cdot(w^{1+\alpha+n}\partial_3^q[J^{-1/\alpha}A_r^\kappa A_i^s]\,\partial_3^{n-q}\eta^r,_s),_\kappa\,dx \\
&\quad+ \int \partial_3^n\eta_t^i\cdot(w^{1+\alpha+n}\partial_3^q[J^{-1/\alpha}A_r^3A_i^s]\,\partial_3^{n-q}\eta^r,_s),_3\,dx \\
&\equiv (iv) + (v)
\end{aligned}
$$

For (iv) where $\kappa = 1,\,2$

$$
\begin{aligned}
(iv) &= \int \partial_3^n\eta_t^i\cdot w^{1+\alpha+n}\partial_\kappa\partial_3^q[J^{-1/\alpha}A_r^\kappa A_i^s]\,\partial_3^{n-q}\eta^r,_s\,dx \\
&\quad+ \int \partial_3^n\eta_t^i\cdot w^{1+\alpha+n}\partial_3^q[J^{-1/\alpha}A_r^\kappa A_i^s]\,\partial_\kappa\partial_3^{n-q}\eta^r,_s\,dx
\end{aligned}
$$

Since $|w^{q/2}\partial_3^q \eta^r,_s|$ and $|w^{(q-1)/2}\partial_\beta\partial_3^{q-1}\eta^r,_s|$ for each $q \le [n/2]$ are bounded due to the assumption (27), (iv) is bounded by

$$\mathcal{E}^{0,n} + \sum_{q=1}^{n-1}(\mathcal{E}^{0,q} + \mathcal{E}^{1,q}) \le \mathcal{F}_4(\mathcal{E}^N)\,.$$

For (v):

$$\begin{aligned}(v) = (1+\alpha+n)&\int w'\partial_3^n \eta_t^i \cdot w^{\alpha+n}\partial_3^q[J^{-1/\alpha}A_r^3 A_i^s]\,\partial_3^{n-q}\eta^r,_s\,dx \\ &+ \int \partial_3^n\eta_t^i \cdot w^{1+\alpha+n}\partial_3^{q+1}[J^{-1/\alpha}A_r^3A_i^s]\,\partial_3^{n-q}\eta^r,_s\,dx \\ &+ \int \partial_3^n\eta_t^i \cdot w^{1+\alpha+n}\partial_3^{q}[J^{-1/\alpha}A_r^3A_i^s]\,\partial_3^{n+1-q}\eta^r,_s\,dx\end{aligned}$$

Since $|w^{q/2}\partial_3^q \eta^r,_s|$ and $|w^{(q-1)/2}\partial_\beta\partial_3^{q-1}\eta^r,_s|$ for each $q \le [n/2]$ are bounded due to (27), (v) is bounded by

$$\sum_{q=1}^{n}\mathcal{E}^{0,q} \le \mathcal{F}_5(\mathcal{E}^N)\,.$$

Now consider $|m| \ge 1$. Recall that $|m| + n \le N$, $1 \le |p| + q \le |m| + n - 1$, $0 \le |p| \le |m|$ and $0 \le q \le n$. We first integrate by parts in each x_k direction of the second factor and then integrate by parts in x_β ($\beta = 1$ or 2) direction of the first factor.

$$\begin{aligned}&\int \partial_\beta^m\partial_3^n\eta_t^i \cdot (w^{1+\alpha+n}\,\partial_\beta^p\partial_3^q[J^{-1/\alpha}A_r^kA_i^s]\,\partial_\beta^{m-p}\partial_3^{n-q}\eta^r,_s),_k\,dx \\ &= \int \partial_\beta^{m-1}\partial_3^n\eta_t^i,_k \cdot w^{1+\alpha+n}\,\partial_\beta^{p+1}\partial_3^q[J^{-1/\alpha}A_r^kA_i^s]\,\partial_\beta^{m-p}\partial_3^{n-q}\eta^r,_s\,dx \\ &+ \int \partial_\beta^{m-1}\partial_3^n\eta_t^i,_k \cdot w^{1+\alpha+n}\,\partial_\beta^{p}\partial_3^q[J^{-1/\alpha}A_r^kA_i^s]\,\partial_\beta^{m+1-p}\partial_3^{n-q}\eta^r,_s\,dx\end{aligned}$$

Note that for $k = 1$ or 2, the first factor together with the weight $w^{(\alpha+n)/2}$ is bounded by $\mathcal{E}^{m,n}$ and for $k = 3$, the first factor with the weight $w^{(\alpha+n+1)/2}$ is bounded by $\mathcal{E}^{m-1,n+1}$. Thus from (27) again, the above is bounded by

$$\mathcal{E}^{m,n} + \underbrace{\mathcal{E}^{m-1,n+1}}_{(*)} + \sum_{\substack{p+q=1;\\ 0\le p\le m, 0\le q\le n}}^{m+n-1} \mathcal{E}^{p+1,q} \ \le\ \mathcal{F}_6(\mathcal{E}^N)\,.$$

Note that since $|m| \ge 1$, $n + 1 \le N$ and $|m| + n \le N$, $(*)$ is bounded by the energy $\mathcal{E}^N$.

For the last line in (39), first note that $I^{m,n}$ in (36) and (38) consist of terms having at most $|m| + n$ derivatives of $D\eta$ with appropriate weights. Hence as done for the previous terms, by using (27) and the fact of $Dw \in X^{\alpha,N}$, it is easy to see that it is bounded by

$$\mathcal{E}^{m,n} + \mathcal{E}^{m+1,n-1} + \mathcal{E}^{m-1,n} + \mathcal{E}^{m,n-1}$$

This completes the proof of Lemma 4.2.

4.2 The proof of Lemma 4.3 : Energy estimates for $\mathcal{C}^N$

It is well known that in Eulerian coordinates, the curl of a gradient vector is zero. Here is the Lagrangian version.

Claim 4.4. *If* $\omega^k = A^r_k h,_r$,

$$curl_\eta \omega = 0$$

Proof.

$$\begin{aligned}[\text{curl}_\eta \omega]^i &= \epsilon_{ijk} A^s_j\, \omega^k,_s = \epsilon_{ijk} A^s_j (A^r_k,_s\, h,_r + A^r_k h,_{rs}) \\ &= \epsilon_{ijk}(-A^s_j A^r_m \eta^m,_{sn}\, A^n_k h,_r + A^s_j A^r_k h,_{rs})\end{aligned}$$

Since both terms are symmetric in j, k, the conclusion follows. □

The equation for v in (6) is equivalent to

$$v^i_t + (1+\alpha) A^k_i (w J^{-1/\alpha}),_k = 0$$

and therefore, by the above claim, the curl equation is written as follows:

$$\text{curl}_\eta \partial_t v = 0 \quad (\epsilon_{ijk} A^s_j\, \partial_t v^k,_s = 0) \tag{40}$$

It can be also written

$$\partial_t [\text{curl}_\eta v]^i = \epsilon_{ijk} \partial_t A^s_j\, v^k,_s \tag{41}$$

In order to estimate $\mathcal{C}^{m,n}$, we first integrate (41) in time:

$$\text{curl}_\eta \partial_t \eta = \text{curl}\, u_0 + \epsilon_{ijk} \int_0^t \partial_t A^s_j\, v^k,_s\, d\tau$$

For $0 < |m| + n \le N$, we take $\partial^m_\beta \partial^n_3$ first to get

$$\begin{aligned}\partial_t[\text{curl}_\eta \partial^m_\beta \partial^n_3 \eta]^i &= [\text{curl}\, \partial^m_\beta \partial^n_3 u_0]^i \\ &\quad + \epsilon_{ijk} \partial_t A^s_j \cdot \partial^m_\beta \partial^n_3 \eta^k,_s - \sum_{|p|+q\ge 1} \epsilon_{ijk} \partial^p_\beta \partial^q_3 A^s_j \cdot \partial^{m-p}_\beta \partial^{n-q}_3 \partial_t \eta^k,_s \\ &\quad + \sum_{|p|+q\ge 0} \epsilon_{ijk} \int_0^t \partial^p_\beta \partial^q_3 \partial_t A^s_j \cdot \partial^{m-p}_\beta \partial^{n-q}_3 v^k,_s\, d\tau\end{aligned} \tag{42}$$

The last term in (42) seems to contain more derivatives than the total energy. However, since each term is integrated in time, by the fundamental theorem of Calculus, we can move a time derivative of one factor to the other. If $|p| + q \ge [N/2]$, we write it as

$$\begin{aligned}\int_0^t \partial^p_\beta \partial^q_3 \partial_t A^s_j \cdot \partial^{m-p}_\beta \partial^{n-q}_3 v^k,_s\, d\tau &= \partial^p_\beta \partial^q_3 A^s_j \cdot \partial^{m-p}_\beta \partial^{n-q}_3 v^k,_s - \partial^p_\beta \partial^q_3 A^s_j \cdot \partial^{m-p}_\beta \partial^{n-q}_3 v^k,_s (0) \\ &\quad - \int_0^t \partial^p_\beta \partial^q_3 A^s_j \cdot \partial^{m-p}_\beta \partial^{n-q}_3 \partial_t v^k,_s\, d\tau\end{aligned}$$

and if $|p| + q \le [N/2]$,

$$\int_0^t \partial_\beta^p \partial_3^q \partial_t A_j^s \cdot \partial_\beta^{m-p} \partial_3^{n-q} v^k,_s \, d\tau = \partial_\beta^p \partial_3^q \partial_t A_j^s \cdot \partial_\beta^{m-p} \partial_3^{n-q} \eta^k,_s - \partial_\beta^p \partial_3^q \partial_t A_j^s \cdot \partial_\beta^{m-p} \partial_3^{n-q} \eta^k,_s (0) - \int_0^t \partial_\beta^p \partial_3^q \partial_t^2 A_j^s \cdot \partial_\beta^{m-p} \partial_3^{n-q} \eta^k,_s \, d\tau$$

Note that $\partial_\beta^p \partial_3^q A_j^s(0) = 0$ for $|p| + q \ge [N/2]$ and $\partial_\beta^{m-p} \partial_3^{n-q} \eta^k,_s (0) = 0$ for $|p| + q \le [N/2]$ since $[D\eta](0) = I$, and thus there will be no contribution of initial data. Hence, the curl equation (42) can be written as

$$\begin{aligned}
\partial_t [\mathrm{curl}_\eta \partial_\beta^m \partial_3^n \eta]^i &= [\mathrm{curl}\, \partial_\beta^m \partial_3^n u_0]^i + 2\epsilon_{ijk} \partial_t A_j^s \cdot \partial_\beta^m \partial_3^n \eta^k,_s \\
&+ \sum_{0<|p|+q<[N/2]} \epsilon_{ijk} (\partial_\beta^p \partial_3^q \partial_t A_j^s \cdot \partial_\beta^{m-p} \partial_3^{n-q} \eta^k,_s - \partial_\beta^p \partial_3^q A_j^s \cdot \partial_\beta^{m-p} \partial_3^{n-q} \partial_t \eta^k,_s) \\
&- \sum_{|p|+q \ge [N/2]} \epsilon_{ijk} \int_0^t \partial_\beta^p \partial_3^q A_j^s \cdot \partial_\beta^{m-p} \partial_3^{n-q} \partial_t^2 \eta^k,_s \, d\tau \\
&- \sum_{|p|+q<[N/2]} \epsilon_{ijk} \int_0^t \partial_\beta^p \partial_3^q \partial_t^2 A_j^s \cdot \partial_\beta^{m-p} \partial_3^{n-q} \eta^k,_s \, d\tau
\end{aligned} \tag{43}$$

Note that each factor in the right-hand side has no more than $|m| + n + 1$ derivatives of η. Hence, in multiplying (43) by $w^{1+\alpha+n} J^{-1/\alpha} [\mathrm{curl}_\eta \partial_\beta^m \partial_3^n \eta]^i$, integrating over Ω, and using the a priori assumption (27) as done in the previous energy estimates, we obtain the desired energy inequality (31) in Lemma 4.3.

The energy bound (29) for $\mathcal{B}^N(v)$ is obtained directly from (43) since $\partial_t [\mathrm{curl}_\eta \partial_\beta^m \partial_3^n \eta]^i = [\mathrm{curl}_\eta \partial_\beta^m \partial_3^n v]^i + \epsilon_{ijk} \partial_t A_j^s \cdot \partial_\beta^m \partial_3^n \eta^k,_s$ and each term in the right-hand side of (43) is bounded by $\mathcal{E}^N(\eta, v)$ and C_0. This completes the proof of Lemma 4.3.

5 Existence proof

5.1 Iteration Scheme

In this section, we implement the linear approximate scheme and prove that the linear system is well-posed in some energy space.

Let the initial data $\eta(0, x) = \eta_0 = x$ and $\partial_t \eta(0, x) = u_0(x)$ of the Euler equation (12) be given so that $\mathcal{T}\mathcal{E}^N(0) \le B$ for a constant $B > 0$ in (25). We will construct approximate solutions $\eta_\nu(t, x)$ and $\partial_t \eta_\nu(t, x)$ for each nonnegative integer ν, by induction satisfying the following properties:

$$\eta_\nu(0, x) = x\,, \ \partial_t \eta_\nu(0, x) = u_0(x)\,, \ \|A_\nu - I\|_\infty \le C_\nu \text{ for } C_\nu \le 1/8 \tag{44}$$

as well as

$$\sum_{|p|+q=0}^{[N/2]} |w^{q/2} \partial_\beta^p \partial_3^q \eta_\nu{}^r,_s | + \sum_{|p|+q=0}^{[N/2]-1} |w^{q/2} \partial_\beta^p \partial_3^q \partial_t \eta_\nu{}^r,_s | < \infty \tag{45}$$

where the approximate inverse of deformation tensor and Jacobian determinant are defined by

$$A_\nu \equiv [D\eta_\nu]^{-1}\ ;\ J_\nu \equiv \det D\eta_\nu\,. \tag{46}$$

The condition $\|A_\nu - I\|_\infty \le C_\nu$ for $C_\nu \le 1/8$ in (44) also guarantees the non-degeneracy of the approximate flow map:

$$2/3 \le \frac{1}{1+3C_\nu+6C_\nu^2+6C_\nu^3} \le J_\nu \le \frac{1}{1-3C_\nu-6C_\nu^2-6C_\nu^3} \le 2$$

The differentiation of A_ν and J_ν, which is similar to (14) and (15), is given by

$$\begin{gathered}\partial_t {A_\nu}_i^k = -{A_\nu}_r^k \partial_t {\eta_\nu}^r,_s {A_\nu}_i^s\ ;\ \partial_l {A_\nu}_i^k = -{A_\nu}_r^k \partial_l {\eta_\nu}^r,_s {A_\nu}_i^s \\ \partial_t J_\nu = J_\nu {A_\nu}_r^s \partial_t {\eta_\nu}^r,_s\ ;\ \partial_l J_\nu = J_\nu {A_\nu}_r^s \partial_l {\eta_\nu}^r,_s\end{gathered} \tag{47}$$

Note that Piola identity (16) also holds for A_ν: ${a_\nu}_i^k,_k = 0$. Having A_ν and J_ν defined, we now introduce the following vector-valued, non-negative and weighted second order linear operators $\mathcal{L}_\nu^e$, $\mathcal{L}_\nu^d$ for a given vector G and a weighted first order linear operator $\mathcal{L}_\nu^c$ for a given anti-symmetric matrix H as follows:

$$\begin{aligned}[]
[\mathcal{L}_\nu^e\, G]^i &\equiv -(w^{1+\alpha}\, {J_\nu}^{-1/\alpha} {A_\nu}_r^k {A_\nu}_r^s G^i,_s),_k \\
[\mathcal{L}_\nu^d\, G]^i &\equiv -\tfrac{1}{\alpha}(w^{1+\alpha}\, {J_\nu}^{-1/\alpha} {A_\nu}_i^k {A_\nu}_r^s G^r,_s),_k \\
[\mathcal{L}_\nu^c\, H]^i &\equiv -(w^{1+\alpha}\, {J_\nu}^{-1/\alpha} {A_\nu}_r^k H_i^r),_k
\end{aligned} \tag{48}$$

For the iteration scheme, instead of η, we approximate $[-\partial_1^2 - \partial_2^2 - w^{-\alpha}\partial_3\, w^{1+\alpha}\partial_3 + \lambda]\,\eta \equiv G$ for a sufficiently large positive constant $\lambda > 0$ and $\mathrm{Curl}_\eta G = H$. For each $\nu \ge 0$, consider the following approximate system for $G_{\nu+1}$ and $H_{\nu+1}$:

$$\begin{aligned} w^\alpha \partial_t^2 G_{\nu+1} + \mathcal{L}_\nu^e\, G_{\nu+1} + \mathcal{L}_\nu^d\, G_{\nu+1} + \mathcal{L}_\nu^c\, H_{\nu+1} &= w^\alpha \mathcal{R}_\nu \\ \partial_t H_{\nu+1} &= \mathcal{S}_\nu \end{aligned} \tag{49}$$

where $\mathcal{R}_\nu$ and $\mathcal{S}_\nu$ are obtained by replacing η by η_ν in $\mathcal{R}$ and $\mathcal{S}$ given in (66) and (67). For the convenience of readers, we put the formal derivation of G and H in Section 5.5. Note that $\mathcal{R}_\nu$ and $\mathcal{S}_\nu$ consist of lower order terms and moreover,

$$\|\mathcal{R}_\nu\|_{X^{\alpha,N-2}} < \infty \text{ and } \|\mathcal{S}_\nu\|_{Z_\nu^{\alpha,N-2}} < \infty \text{ if } \mathcal{E}^N(\eta_\nu, \partial_t\eta_\nu) < \infty.$$

And also note that $[\mathcal{S}_\nu]_j^k = -[\mathcal{S}_\nu]_k^j$ and thus $H_{\nu+1}$ is anti-symmetric.

The initial data for $G_{\nu+1}$ and $H_{\nu+1}$ are inherited from the original system:

$$\begin{aligned} &{G_{\nu+1}}^i(0,x) = -(1+\alpha)\partial_3 w \delta_3^i + \lambda x^i, \\ &\partial_t G_{\nu+1}(0,x) = [-\partial_1^2 - \partial_2^2 - w^{-\alpha}\partial_3 w^{1+\alpha}\partial_3 + \lambda]u_0(x)\,,\ H_{\nu+1}(0,x) = 0\,. \end{aligned} \tag{50}$$

The unique $\eta_{\nu+1}$ is to be found by solving a degenerate elliptic equation

$$[-\partial_1^2 - \partial_2^2 - w^{-\alpha}\partial_3\, w^{1+\alpha}\partial_3 + \lambda]\,\eta_{\nu+1} = G_{\nu+1} \tag{51}$$

The same goes for $\partial_t G_{\nu+1}$. $A_{\nu+1}$ and $J_{\nu+1}$ are defined as in (46). We remark that the approximate systems (49) converge to the equations for G and $\mathrm{Curl}_\eta G$ respectively in the formal limit $\nu \to \infty$.

We define the approximate energy functional $\tilde{\mathcal{E}}_{\nu+1}$ at the ν^{th} step:

$$
\begin{aligned}
\tilde{\mathcal{E}}_{\nu+1}(t) \equiv \sum_{|m|+n=0}^{N-2} \{ & \frac{1}{2}\int w^{\alpha+n}|\partial_\beta^m \partial_3^n \partial_t G_{\nu+1}|^2 dx + \frac{1}{2}\int w^{1+\alpha+n} J_\nu^{-\frac{1}{\alpha}} |D_{\eta_\nu}\partial_\beta^m\partial_3^n G_{\nu+1}|^2 dx \\
& + \frac{1}{2\alpha}\int w^{1+\alpha+n} J_\nu^{-\frac{1}{\alpha}} |\mathrm{div}_{\eta_\nu}\partial_\beta^m\partial_3^n G_{\nu+1}|^2 dx \\
& - \frac{1}{2}\int w^{1+\alpha+n} J_\nu^{-\frac{1}{\alpha}} \mathrm{Curl}_{\eta_\nu}\partial_\beta^m\partial_3^n G_{\nu+1}\cdot \partial_\beta^m\partial_3^n H_{\nu+1} dx \\
& + 2\int w^{1+\alpha+n} J_\nu^{-\frac{1}{\alpha}} |\partial_\beta^m\partial_3^n H_{\nu+1}|^2 dx \} \\
\equiv \ & \frac{1}{2}\|\partial_t G_{\nu+1}\|^2_{X^{\alpha,N-2}} + \frac{1}{2}\|G_{\nu+1}\|^2_{Y_\nu^{\alpha,N-2}} + 2\|H_{\nu+1}\|^2_{Z_\nu^{\alpha,N-2}} \\
& + \sum_{|m|+n=0}^{N-2}\{\mathcal{D}_\nu^{m,n}(G_{\nu+1}) - \frac{1}{2}\int w^{1+\alpha+n} J_\nu^{-\frac{1}{\alpha}} \mathrm{Curl}_{\eta_\nu}\partial_\beta^m\partial_3^n G_{\nu+1}\cdot\partial_\beta^m\partial_3^n H_{\nu+1} dx\}
\end{aligned}
\tag{52}
$$

where $Y_\nu^{\alpha,N-2}$ and $Z_\nu^{\alpha,N-2}$ denote $Y^{\alpha,N-2}$ and $Z^{\alpha,N-2}$ in (26) induced by η_ν. Note that by Cauchy-Schwartz inequality,

$$
\begin{aligned}
& |\frac{1}{2}\int w^{1+\alpha+n} J_\nu^{-\frac{1}{\alpha}} \mathrm{Curl}_{\eta_\nu}\partial_\beta^m\partial_3^n G_{\nu+1}\cdot\partial_\beta^m\partial_3^n H_{\nu+1} dx| \\
& \leq \frac{1}{3}\int w^{1+\alpha+n} J_\nu^{-\frac{1}{\alpha}} |D_{\eta_\nu}\partial_\beta^m\partial_3^n G_{\nu+1}|^2 dx + \frac{3}{2}\int w^{1+\alpha+n} J_\nu^{-\frac{1}{\alpha}} |\partial_\beta^m\partial_3^n H_{\nu+1}|^2 dx
\end{aligned}
$$

and therefore

$$
\begin{aligned}
& \frac{1}{2}\|\partial_t G_{\nu+1}\|^2_{X^{\alpha,N-2}} + \frac{1}{6}\|G_{\nu+1}\|^2_{Y_\nu^{\alpha,N-2}} + \frac{1}{2}\|H_{\nu+1}\|^2_{Z_\nu^{\alpha,N-2}} \leq \tilde{\mathcal{E}}_{\nu+1} \\
& \qquad \leq \frac{1}{2}\|\partial_t G_{\nu+1}\|^2_{X^{\alpha,N-2}} + (\frac{5}{6}+\frac{1}{2\alpha})\|G_{\nu+1}\|^2_{Y_\nu^{\alpha,N-2}} + \frac{7}{2}\|H_{\nu+1}\|^2_{Z_\nu^{\alpha,N-2}}
\end{aligned}
\tag{53}
$$

We now state and prove that the approximate system (49) is well-posed in the energy space generated by $\mathcal{E}_\nu$ under the following induction hypotheses:

(HP1) $\tilde{\mathcal{E}}_\nu < \infty$ and η_ν and $\partial_t\eta_\nu$ satisfy (44).
(HP2) The left-hand side of (45) is bounded by $\tilde{\mathcal{E}}_\nu$ and $\tilde{\mathcal{E}}_{\nu-1}$.
(HP3) $\mathcal{E}^N(\eta_\nu, \partial_t\eta_\nu)$ is bounded by $\tilde{\mathcal{E}}_\nu$ and $\tilde{\mathcal{E}}_{\nu-1}$.

Proposition 5.1 (Well-Posedness of Approximate system and Regularity)**.** *Under hypotheses (HP1) - (HP3), linear system* (49) *admits a unique solution* $(\partial_t G_{\nu+1}, G_{\nu+1}, H_{\nu+1})$ *in* $X^{\alpha,N-2}, Y_\nu^{\alpha,N-2}, Z_\nu^{\alpha,N-2}$. *Furthermore, we obtain the following energy bounds:*

$$
\tilde{\mathcal{E}}_{\nu+1}(t) \leq \tilde{\mathcal{E}}_{\nu+1}(0) + \int_0^t \mathcal{F}_7(\tilde{\mathcal{E}}_{\nu+1}, \tilde{\mathcal{E}}_\nu, \tilde{\mathcal{E}}_{\nu-1}, B^N(u_0))(\tilde{\mathcal{E}}_{\nu+1})^{\frac{1}{2}} d\tau
$$

where $\mathcal{F}_7(\tilde{\mathcal{E}}_{\nu+1}, \tilde{\mathcal{E}}_\nu, \tilde{\mathcal{E}}_{\nu-1}, B^N(u_0))$ *is a continuous function of* $\tilde{\mathcal{E}}_{\nu+1}$, $\tilde{\mathcal{E}}_\nu$, $\tilde{\mathcal{E}}_{\nu-1}$, $B^N(u_0)$.

Note that $\mathcal{R}_\nu \in Y_\nu^{\alpha,N-2}$ and $\mathcal{S}_\nu \in Z_\nu^{\alpha,N-2}$ under (HP3). Proposition 5.1 directly follows from Proposition 5.5 given in Section 5.4.

In order to complete the induction procedure of approximate schemes, it now remains to verify the induction hypotheses (HP1), (HP2), and (HP3) for $\nu+1$.

By the following Elliptic Regularity lemma, which will be proven in Section 5.3, we obtain a unique $\eta_{\nu+1}$ to the degenerate elliptic equation (51).

Lemma 5.2. *Let $k \geq 0$ be given. For each $G \in X^{\alpha,k}$, there exists a unique solution $u \in X^{\alpha,k+2}$ to the following degenerate elliptic equation*

$$[-\partial_1^2 - \partial_2^2 - w^{-\alpha}\partial_3 w^{1+\alpha}\partial_3 + \lambda]u = G$$

Moreover, we have

$$\|u\|_{X^{\alpha,k+2}} \lesssim \|G\|_{X^{\alpha,k}} .$$

First (HP1). By Lemma 5.2, $\eta_{\nu+1}$ and $\partial_t\eta_{\nu+1}$ constructed in the above satisfy the following

$$\|\partial_t\eta_{\nu+1}\|_{X^{\alpha,N}} + \|\eta_{\nu+1}\|_{Y_\nu^{\alpha,N}} \lesssim \|\partial_t G_{\nu+1}\|_{X^{\alpha,N-2}} + \|G_{\nu+1}\|_{Y_\nu^{\alpha,N-2}}$$

as well as the initial boundary conditions in (44). The boundedness of $J_{\nu+1}$ will follow from the continuity argument by using the estimate of $\partial_t J_{\nu+1}$. Note that from Jacobi's formula

$$\partial_t J_{\nu+1} = \partial_t \det D\eta_{\nu+1} = \mathrm{tr}(\mathrm{adj}(D\eta_{\nu+1})\partial_t D\eta_{\nu+1})$$

where $\mathrm{adj}(\cdot)$ denotes the adjugate of a given matrix. Since $\partial_t\eta_{\nu+1} \in X^{\alpha,N}$ and $\eta_{\nu+1} \in Y_\nu^{\alpha,N}$, by Lemma 3.1, $|D\eta_{\nu+1}|$ and $|\partial_t D\eta_{\nu+1}|$ are bounded by their $X^{\alpha,N}$ and $Y_\nu^{\alpha,N}$ norms and hence bounded by $\tilde{\mathcal{E}}_{\nu+1}$, $\tilde{\mathcal{E}}_\nu$. Thus

$$|\partial_t J_{\nu+1}| \leq \tilde{C}_{\nu+1}$$

where $\tilde{C}_{\nu+1}$ depends only on $\tilde{\mathcal{E}}_{\nu+1}$, $\tilde{\mathcal{E}}_\nu$, and initial data. Since $J_{\nu+1}(t) = J_{\nu+1}(0) + \int_0^t \partial_t J_{\nu+1} d\tau$, we get

$$1 - \tilde{C}_{\nu+1}T \leq J_{\nu+1} \leq 1 + \tilde{C}_{\nu+1}T$$

Now since $J_{\nu+1}$ is bounded away from zero for sufficiently small T, $D\eta_{\nu+1}$ is invertible and thus $A_{\nu+1}$ is well-defined and the formulation in (47) is also well-defined. To verify the last condition in (44) for $(\nu+1)^{\text{th}}$, we use (47). Since $A_{\nu+1}$ and $\partial_t D\eta_{\nu+1}$ are bounded by $\tilde{\mathcal{E}}_{\nu+1}$, $\tilde{\mathcal{E}}_\nu$ and since $A_{\nu+1}(t) = A_{\nu+1}(0) + \int_0^t \partial_t A_{\nu+1} d\tau$,

$$\|A_{\nu+1} - I\|_\infty \leq \tilde{\tilde{C}}_{\nu+1}T$$

and thus for sufficiently small T, $C_{\nu+1} \leq 1/8$ can be found.

We move onto (HP2). This directly follows from the embedding as in Lemma 3.1, since N is sufficiently large $N \geq 2[\alpha] + 9$.

For (HP3), it suffices to show that for each $m+n \leq N$, $\int_\Omega w^{1+\alpha+n} J_{\nu+1}^{-1/\alpha} |D_{\eta_{\nu+1}}\partial_\beta^m \partial_3^n \eta_{\nu+1}|^2 dx$ is bounded by $\tilde{\mathcal{E}}_{\nu+1}$ and $\tilde{\mathcal{E}}_\nu$. Note that

$$\begin{aligned}
&\int_\Omega w^{1+\alpha+n} J_{\nu+1}^{-1/\alpha} |D_{\eta_{\nu+1}}\partial_\beta^m \partial_3^n \eta_{\nu+1}|^2 dx \\
&= \int_\Omega w^{1+\alpha+n} \left(\frac{J_{\nu+1}}{J_\nu}\right)^{-1/\alpha} J_\nu^{-1/\alpha} |A_{\nu+1}{}_r^s \eta_\nu,{}_s^k\, A_\nu{}_k^s \partial_\beta^m \partial_3^n \eta_{\nu+1}{}^i,{}_s|^2 dx
\end{aligned}$$

Because $J_{\nu+1}$ is bounded away from zero and $A_{\nu+1}$ and $D\eta_\nu$ are bounded by $\tilde{\mathcal{E}}_{\nu+1}$ and $\tilde{\mathcal{E}}_\nu$, (HP3) at $(\nu+1)^{\text{th}}$ step readily follows.

5.2 Convergence of the iteration scheme and uniqueness of solution

In order to prove Theorem 2.2, it now remains to show that η_ν and $\partial_t\eta_\nu$ converge, the limit functions solve Euler equations (8) and (12), and they are unique.

First, by applying the Gronwall inequality to the energy inequality obtained in Proposition 5.1, we can deduce the following Claim:

Claim 5.3. *Suppose that the initial data $\eta(0,x)=x$ and $\partial_t\eta(0,x)=u_0(x)$ of Euler equations* (12) *are given such that $\mathcal{TE}^N(0)\le B$ for a constant $B>0$. Then there exist $T>0$ such that if for all $\iota\le\nu$, $\tilde{\mathcal{E}}_\iota\le 3B/2$ for $t\le T$, then $\tilde{\mathcal{E}}_{\nu+1}\le 3B/2$ for $t\le T$ and in addition, for all ν, $\|A_\nu-I\|_\infty\le 1/8$ and $2/3\le J_\nu\le 2$.*

Thus we get the uniform bound of $\tilde{\mathcal{E}}_\nu$ as well as the uniform upper and lower bounds of J_ν. In order to take the limit $\nu\to\infty$, in view of (53), we define the homogenous energy functional $\overline{\mathcal{E}}_{\nu+1}$ for $G_{\nu+1}$ and $H_{\nu+1}$:

$$\begin{aligned}\overline{\mathcal{E}}_{\nu+1}\equiv\sum_{|m|+n=0}^{N-2}\{&\frac{1}{2}\int w^{\alpha+n}|\partial_\beta^m\partial_3^n\partial_t G_{\nu+1}|^2dx+\frac{1}{6}\int w^{1+\alpha+n}|\partial_\beta^m\partial_3^n DG_{\nu+1}|^2dx\\&+\frac{1}{2}\int w^{1+\alpha+n}|\partial_\beta^m\partial_3^n H_{\nu+1}|^2dx\}\end{aligned}\tag{54}$$

Then due to (53) and since $\|A_\nu-I\|_\infty\le 1/8$ and $2/3\le J_\nu\le 2$, $\tilde{\mathcal{E}}_{\nu+1}$ and $\overline{\mathcal{E}}_{\nu+1}$ are equivalent:

$$\frac{1}{1+M_\nu}\tilde{\mathcal{E}}_{\nu+1}\le\overline{\mathcal{E}}_{\nu+1}\le(1+M_\nu)\tilde{\mathcal{E}}_{\nu+1}$$

where M_ν depends only on C_ν and $\tilde{\mathcal{E}}_\nu$. Now by Claim 5.3, $\tilde{\mathcal{E}}_{\nu+1}$ and M_ν's have the uniform bound over $t\le T$. Therefore, there exists a sequence ν_ι such that G_{ν_ι}, H_{ν_ι}, and η_{ν_ι} converge strongly to some G,H,η. Due to the uniform energy bound, we also conclude that η and $\partial_t\eta$ solve (8) and (12) with the desired properties.

For uniqueness, let (η,v) and $(\overline{\eta},\overline{v})$ be two solutions to (8) and (12) with the same initial boundary conditions having the total energy bounds: $\mathcal{TE}^N(\eta,v)$, $\mathcal{TE}^N(\overline{\eta},\overline{v})\le 2B$. Define $\mathcal{Z}(t)$ by

$$\begin{aligned}\mathcal{Z}(t)\equiv&\frac{1}{2}\int w^\alpha|v-\overline{v}|^2dx+\alpha\int w^{1+\alpha}J^{-1/\alpha-2}|J-\overline{J}|^2dx\\&+\sum_{|m|+n=1}^{N-1}\frac{1}{2}\int w^{\alpha+n}|\partial_\beta^m\partial_3^n(v-\overline{v})|^2dx+\frac{1}{2}\int w^{1+\alpha+n}J^{-1/\alpha}|D_\eta\partial_\beta^m\partial_3^n(\eta-\overline{\eta})|^2dx\end{aligned}$$

From (37), the equations for $\eta-\overline{\eta}$ read as

$$w^{\alpha+n}\partial_\beta^m\partial_3^n(\eta-\overline{\eta})^i_{tt}+(w^{1+\alpha+n}\,\partial_\beta^m\partial_3^n[A_i^kJ^{-1/\alpha}-\overline{A}_i^k\overline{J}^{-1/\alpha}]),_k+w^{\alpha+n}[I^{m,n}(\eta)-I^{m,n}(\overline{\eta})]=0$$

In performing the energy estimates of $\eta-\overline{\eta}$ as done in the a priori estimates and noting that $|J^{-1/\alpha}-\overline{J}^{-1/\alpha}|\le C_{1B}|D\eta-D\overline{\eta}|$ and $|A-\overline{A}|\le C_{2B}|D\eta-D\overline{\eta}|$, one can obtain

$$\frac{d\mathcal{Z}}{dt}\le C\mathcal{Z}\text{ for }C\text{ depending only on }B\,.$$

Since $\mathcal{Z}(0)=0$, this immediately yields the uniqueness.

5.3 Proof of Lemma 5.2

Here we give a sketch of proof of Lemma 5.2. We refer to [5] and [6] for similar regularity results of degenerate elliptic problems. We introduce the spaces $H = X^{\alpha,0}$ and $V = X^{\alpha,1}$ defined by

$$V = \{u \in \mathcal{D}'(\Omega); \quad w^{\frac{\alpha}{2}}u,\, w^{\frac{\alpha}{2}}\partial_\beta u \in L^2(\Omega),\, w^{\frac{\alpha+1}{2}}\partial_3 u \in L^2(\Omega)\}.$$

We also define the following scalar product on H, $(u,v)_H = \int_\Omega w^\alpha uv$. We define on V the following norm

$$\|u\|_V^2 = \|w^{\frac{\alpha}{2}}u\|^2_{L^2(\Omega)} + \sum_{\beta=1}^{2} \|w^{\frac{\alpha}{2}}\partial_\beta u\|^2_{L^2(\Omega)} + \|w^{\frac{\alpha+1}{2}}\partial_3 u\|^2_{L^2(\Omega)}$$

Lemma 5.4. *$\mathcal{D}$ is dense in V.*

The proof is based on an explicit approximation. We define $\chi_n(t) = 1$ if $\frac{2}{n} \le t \le 1 - \frac{2}{n}$ and $\chi_n(t) = nt - 1$ if $\frac{1}{n} \le t \le \frac{2}{n}$ and $\chi_n(t) = n - 1 - nt$ if $1 - \frac{2}{n} \le t \le 1 - \frac{1}{n}$ and $\chi_n(t) = 0$ for $0 \le t \le \frac{1}{n}$ or $1 - \frac{1}{n} \le t \le 1$. For each $v \in V$, we define $u_n = \chi_n(x_3)v$ and $v^n = \rho_{\frac{1}{2n}} * u_n$ the convolution of u_n with the mollifier $\rho_{\frac{1}{2n}} = (2n)^3\rho(\frac{\cdot}{2n})$ where $\rho \in C_0^\infty(\mathbb{R}^3)$, $\rho \ge 0$, $\int \rho = 1$ and $\mathrm{supp}(\rho) \in B(0,1)$. Hence $v_n \in \mathcal{D}(\Omega)$ and v_n goes to v in V when n goes to infinity. To prove that v_n converges to v in V, we have to use the Hardy inequality, namely the fact that

$$\int_\Omega w^{\alpha-1}|v|^2 dx \le \int_\Omega w^{\alpha+1}|\partial_3 v|^2 dx.$$

This ends the proof of lemma 5.4.

We also define the following bilinear form

$$B[u,v] = \int_\Omega \lambda w^\alpha uv + \sum_{\beta=1}^{2} \partial_\beta u \partial_\beta(w^\alpha v) + w^{\alpha+1}\partial_3 u \partial_3 v$$

where λ is big enough. Notice that B is not symmetric if w depends on the tangential variable. If λ is chosen big enough then, B satisfies the hypotheses of the Lax-Milgram theorem, namely the fact that

$$|B[u,v]| \le C\|u\|_V\|v\|_V, \quad \text{and} \quad \frac{1}{C}\|v\|_V^2 \le B[v,v].$$

Hence, for each bounded linear functional f on V, namely $f : V \to \mathbb{R}$, there exists a unique element $u \in V$ such that $B[u,v] = f(v)$ for each $v \in V$. One can try to characterize the set V', but we do not need to do it here. For any $G \in H$, $f(v) = (v,G)_H$ defines a linear functional on V and hence there exists a unique $u \in V$ such that $B[u,v] = \int_\Omega w^\alpha Gv$ for all $v \in V$.

Using lemma 5.4 and using a density argument, it is easy to see that u solves

$$[-\partial_1^2 - \partial_2^2 - w^{-\alpha}\partial_3 w^{1+\alpha}\partial_3 + \lambda]u = G \tag{55}$$

if and only if $B[u,v] = \int_\Omega w^\alpha Gv$ for all $v \in V$. Hence, u is the unique solution of (55). To prove that u is more regular, we use again Lax-Milgram to construct U_β the solution of (55) with the right hand side $G_\beta = \partial_\beta G + \alpha\frac{\partial_\beta w}{w}w^{-\alpha}\partial_3[w^{1+\alpha}\partial_3 u] + (1+\alpha)w^{-\alpha}\partial_3[w^{\alpha+1}\frac{\partial_\beta w}{w}\partial_3 u]$.

Indeed, one has just to observe that $f_\beta(v) = (v, G_\beta)_H =$

$$-(w^{-\alpha}\partial_\beta(w^\alpha v), G)_H - \alpha(w^{1/2}\partial_3(\frac{\partial_\beta w}{w}v), w^{1/2}\partial_3 u)_H - (1+\alpha)(w^{1/2}\partial_3 v, w^{1/2}\frac{\partial_\beta w}{w}\partial_3 u)_H$$

defines a linear functional of V. Then by uniqueness of the solution to (55) with the right hand side G_β, we deduce that $U_\beta = \partial_\beta u$. To be more precise, one has to replace the partial derivative with respect to x_β by difference quotient and then pass to the limit to deduce that $U_\beta = \partial_\beta u \in V$ and that $\|\partial_\beta u\|_V \le C\|G\|_H$. Using the equation (55), we also see that $w^{-\alpha}\partial_3 w^{1+\alpha}\partial_3 u \in H$ and hence by Hardy inequality, we deduce that

$$\int_\Omega w^{-\alpha-2}|w^{1+\alpha}\partial_3 u|^2 dx \le C\int_\Omega w^{-\alpha}|\partial_3 w^{1+\alpha}\partial_3 u|^2 dx \le C\|G\|_H. \tag{56}$$

Using that $w^{-\alpha}\partial_3 w^{1+\alpha}\partial_3 u = (1+\alpha)\partial_3 w\partial_3 u + w\partial_3^2 u$ and the fact that (56) yields that $\partial_3 u \in H$, we deduce that $w\partial_3^2 u \in H$.

We now assume that G is more regular, namely that $G \in X^{\alpha,k}$ for some $k \ge 1$. We want to prove that $u \in X^{\alpha,k+2}$. From the previous argument, we know that $u \in X^{\alpha,2}$. Proving regularity in the tangential direction is very similar to the previous argument. Indeed, differentiating (55) with respect to β, we see that $\partial_\beta u$ solves (55) with the right hand side G_β and due to the extra regularity of G, we see that $G_\beta \in H$. Hence, we can apply $X^{\alpha,2}$ regularity property to $\partial_\beta u$ and deduce that it is in $X^{\alpha,2}$. Of course we can repeat this k times and deduce that the tangential derivatives $\partial_\beta^k u \in X^{\alpha,2}$. It remains to control the normal derivatives. Taking one normal derivative, we observe that $\partial_3 u$ solves

$$[-\partial_1^2 - \partial_2^2 - w^{-\alpha-1}\partial_3 w^{2+\alpha}\partial_3 + \lambda]\partial_3 u = G_3 \tag{57}$$

with the right hand side $G_3 = \partial_3 G + (1+\alpha)\partial_3^2 w\partial_3 u$. Moreover, it is clear that $\partial_3 u \in H$ and that $G_3 \in H$. We introduce two other Hilbert spaces $H_{\alpha+1}$ and $V_{\alpha+1}$ where for each $\gamma > 0$, H_γ and V_γ are given by the following norms

$$\|u\|_{H_\gamma}^2 = \|w^{\frac{\gamma}{2}}u\|_{L^2(\Omega)}^2$$

$$\|u\|_{V_\gamma}^2 = \|w^{\frac{\gamma}{2}}u\|_{L^2(\Omega)}^2 + \sum_{\beta=1}^2 \|w^{\frac{\gamma}{2}}\partial_\beta u\|_{L^2(\Omega)}^2 + \|w^{\frac{\gamma+1}{2}}\partial_3 u\|_{L^2(\Omega)}^2.$$

In particular, we notice that $H = H_\alpha$ and $V = V_\alpha$.

Now, we can apply the previous regularity argument with H replaced by $H_{\alpha+1}$ and $X^{\alpha,2}$ replaced by $X^{\alpha+1,2}$. We can also combine the x_3 derivatives with tangential derivatives and prove that for all m, $0 \le m \le k-1$, we have $\partial_\beta^m\partial_3 u \in X^{\alpha+1,2}$.

By an induction argument on the number of x_3 derivatives, we can finally prove that for all m, $0 \le m \le k-n$, we have $\partial_\beta^m\partial_3^n u \in X^{\alpha+n,2}$. Hence, we deduce that

$$\|u\|_{X^{\alpha,k+2}} \le C\|G\|_{X^{\alpha,k}}.$$

5.4 Duality Argument: solvability for (49)

Proposition 5.5. *Let η and $\partial_t\eta$ be given such that $\mathcal{E}^N(\eta, \partial_t\eta) < \infty$. For $f = f(\eta)$ and $g = g(\eta, \partial_t\eta)$ where $f \in L^1(0,T; X^{\alpha,0})$, $g \in L^1(0,T; Z^{\alpha,0})$, and g is anti-symmetric, there* *exists a unique solution $(\partial_t G, G, H)$ on $(0,T)$ to the linear system*

$$\begin{cases} w^\alpha \partial_t^2 G + \mathcal{L}^e G + \mathcal{L}^d G + \mathcal{L}^c H = w^\alpha f \\ w^{1+\alpha} J^{-1/\alpha} \partial_t H = w^{1+\alpha} J^{-1/\alpha} g \\ G(t=0) = \partial_t G(t=0) = H(t=0) = 0 \end{cases} \tag{58}$$

and the solution satisfies

$$\|(\partial_t G, G, H)\|_{C([0,T];X^{\alpha,0}\times Y^{\alpha,0}\times Z^{\alpha,0})} \le C\|(f,g)\|_{L^1(X^{\alpha,0}\times Z^{\alpha,0})} \tag{59}$$

Moreover, if $f \in X^{\alpha,N-2}$, $g \in Z^{\alpha,N-2}$, then

$$\|(\partial_t G, G, H)\|_{C([0,T];X^{\alpha,N-2}\times Y^{\alpha,N-2}\times Z^{\alpha,N-2})} \le C\|(f,g)\|_{L^1(X^{\alpha,N-2}\times Z^{\alpha,N-2})} \tag{60}$$

for some constant C that depends only on $\mathcal{TE}^N(\eta, \partial_t\eta)$.

Let $\mathcal{A}$ denote the set

$$\mathcal{A} = \left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in C^\infty((0,\infty)\times\Omega) \text{ such that } (\phi, \partial_t\phi, \psi)_{t=T} = 0 \right\}$$

Hence, $(\partial_t G, G, H)$ solves (58) on a time interval $(0,T)$ if and only if for each test function $(\phi,\psi) \in \mathcal{A}$, we have

$$\begin{aligned} \int_0^T \int G\cdot(w^\alpha\partial_t^2\phi + \mathcal{L}^e\phi + \mathcal{L}^d\phi) - \frac{1}{2} w^{1+\alpha}J^{-1/\alpha}H : \mathrm{Curl}_\eta\phi \, dxdt &= \int_0^T \int w^\alpha f\phi\, dxdt \\ \int_0^T \int 4w^{1+\alpha}J^{-1/\alpha}H : (\partial_t\psi - \partial_t(J^{-1/\alpha})\psi) dxdt &= \int_0^T \int -4w^{1+\alpha}J^{-1/\alpha} g\psi dxdt \end{aligned} \tag{61}$$

We denote

$$\mathcal{V}\begin{pmatrix} G \\ H \end{pmatrix} = \begin{pmatrix} w^\alpha\partial_t^2 G + \mathcal{L}^e G + \mathcal{L}^d G + \mathcal{L}^c H \\ -w^{1+\alpha}J^{-1/\alpha}\partial_t H \end{pmatrix}$$

defined on the core

$$\left\{ \begin{pmatrix} G \\ H \end{pmatrix} \middle| \partial_t^2 G \in L_t^2 X^{\alpha,0}, \partial_t H \in L_t^2 Z^{\alpha,0}, G \in L_t^2(\mathcal{D}(\mathcal{L}^e)) \right\}$$

Hence $\mathcal{V}$ can be extended uniquely to a closed operator. Moreover, $\mathcal{A} \subset \mathcal{D}\mathcal{V}^*$, the dual of $\mathcal{V}$, and

$$\mathcal{V}^*\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} w^\alpha\partial_t^2\phi + \mathcal{L}^e\phi + \mathcal{L}^d\phi \\ w^{1+\alpha}J^{-1/\alpha}[4\partial_t\psi - 4\partial_t(J^{-1/\alpha})\psi - \frac{1}{2}\mathrm{Curl}_\eta\phi] \end{pmatrix}$$

Therefore, (61) holds for each $(\phi,\psi) \in \mathcal{A}$ if and only if for each $(\phi,\psi) \in \mathcal{A}$, we have

$$\int_0^T \int \begin{pmatrix} G \\ H \end{pmatrix} \cdot \mathcal{V}^*\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \int_0^T \int \begin{pmatrix} w^\alpha f \\ -4w^{1+\alpha}J^{-1/\alpha}g \end{pmatrix} \cdot \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{62}$$

We take $\binom{\phi}{\psi} \in A$ and denote

$$\mathcal{V}^*\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}$$

The energy estimates for $\mathcal{V}^*$ with (52) and (53) yield that

$$\sup_{0\le t\le T}\{\frac{1}{2}\|\partial_t\phi\|^2_{X^{\alpha,0}}+\frac{1}{6}\|\phi\|^2_{Y^{\alpha,0}}+\frac{1}{2}\|\psi\|^2_{Z^{\alpha,0}}\}\le C\int_0^T\|\Phi\|^2_{X^{\alpha,0*}}+\|\Psi\|^2_{Z^{\alpha,0*}}dt$$

where $X^{\alpha,0*}$ and $Z^{\alpha,0*}$ denote the dual spaces of $X^{\alpha,0}$ and $Z^{\alpha,0}$. Thus the operator $\mathcal{V}^*$ defines a bijection between $\mathcal{A}$ and $\mathcal{V}^*(\mathcal{A})$. Let S_0 be its inverse. Hence

$$S_0:\mathcal{V}^*(\mathcal{A})\to A \text{ given by } \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}|\to\begin{pmatrix}\phi\\ \psi\end{pmatrix}$$

and we have

$$\|(\partial_t\phi,\phi,\psi)\|_{C([0,T];X^{\alpha,0}\times Y^{\alpha,0}\times Z^{\alpha,0})}\le C\|\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}\|_{L^1(X^{\alpha,0*}\times Z^{\alpha,0*})}$$

We extend this operator by density to $\overline{\mathcal{V}^*(\mathcal{A})}^{L^1(X^{\alpha,0*}\times Z^{\alpha,0*})}$ and to $L^1(X^{\alpha,0*}\times Z^{\alpha,0*})$ by Hahn-Banach. We denote this extension by S:

$$S:L^1(X^{\alpha,0*}\times Z^{\alpha,0*})\to C([0,T];X^{\alpha,0}\times Y^{\alpha,0}\times Z^{\alpha,0})\;,\quad \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}|\to\begin{pmatrix}\phi\\ \psi\end{pmatrix}$$

Now we want to solve (58), namely $\mathcal{V}\binom{G}{H}=\binom{w^\alpha f}{w^{1+\alpha}J^{-1/\alpha}g}$ with $G(t=0)=\partial_t G(t=0)=H(t=0)=0$. This is equivalent to requiring that (62) holds for each $(\phi,\psi)\in\mathcal{A}$. Hence, it is enough to show that for all $\binom{\Phi}{\Psi}\in L^1(X^{\alpha,0*}\times Z^{\alpha,0*})$, we have

$$\int_0^T\int\begin{pmatrix}G\\ H\end{pmatrix}\cdot\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}=\int_0^T\int\begin{pmatrix}w^\alpha f\\ -4w^{1+\alpha}J^{-1/\alpha}g\end{pmatrix}\cdot S\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}$$

Therefore, it is enough to take $\binom{G}{H}=S^*\binom{w^\alpha f}{-4w^{1+\alpha}J^{-1/\alpha}g}$ where S^* is the dual of S, which satisfies

$$S^*:\mathcal{M}(0,T;X^{\alpha,0*}\times Z^{\alpha,0*})\to L^\infty(0,T;X^{\alpha,0}\times Y^{\alpha,0}\times Z^{\alpha,0})$$

and thus $\|(\partial_t G,G,H)\|_{L^\infty(0,T;X^{\alpha,0}\times Y^{\alpha,0}\times Z^{\alpha,0})}\le C\|(f,g)\|_{L^1(X^{\alpha,0}\times Z^{\alpha,0})}$. At this stage we do not know whether (G,H) is continuous in (59). This actually follows from the regularity and the density argument.

The uniqueness of (G,H) will follows from the fact that if $f=g=0$ in (58), then $(0,0)$ is the only solution to (58) in $L^\infty((0,T;X^{\alpha,0}\times Y^{\alpha,0}\times Z^{\alpha,0})$. To prove it, consider a solution $(\partial_t G,G,H)\in L^\infty((0,T;X^{\alpha,0}\times Y^{\alpha,0}\times Z^{\alpha,0})$ to (58) with $f=g=0$. We will also make use of the duality argument. Indeed, as changing the roles of $\mathcal{V}$ and $\mathcal{V}^*$ and arguing as above, we can prove the existence of a solution (ϕ,ψ) to the dual problem

$$\begin{cases}w^\alpha\partial_t^2\phi+\mathcal{L}^e\phi+\mathcal{L}^d\phi=\Phi\\ w^{1+\alpha}J^{-1/\alpha}[4\partial_t\psi-4\partial_t(J^{-1/\alpha})\psi-\frac{1}{2}\mathrm{Curl}_\eta\phi]=\Psi\\ \phi(t=0)=\partial_t\phi(t=0)=\psi(t=0)=0\end{cases}\tag{63}$$

for each $(\Phi,\Psi)\in L^1(X^{\alpha,0*}\times Z^{\alpha,0*})$. Then for each $(\Phi,\Psi)\in L^1(X^{\alpha,0*}\times Z^{\alpha,0*})$, we consider

(ϕ, ψ) a solution to (63). Hence, we can write (62) with the solution (ϕ, ψ). This yields

$$\int_0^T \int \begin{pmatrix} G \\ H \end{pmatrix} \cdot \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} dxdt = 0$$

which implies $G = H = 0$.

The proof of (60) is an easy modification of the above argument based on induction, duality and density arguments as done in [29]. The tangential regularity can be obtained in the same way. The only difference for the normal regularity is that the degeneracy of the linear operators $\mathcal{L}^e, \mathcal{L}^d$ in (58) will change according to the number of ∂_3 for G and thus the appropriate function space is $X^{\alpha,k}$ as in the same spirit of Section 5.3. We omit the details.

5.5 Formal derivation of the equation for G and $H = \mathbf{Curl}_\eta G$

We first derive the equation for $F \equiv w^{-\alpha}\partial_3(w^{1+\alpha}\partial_3\eta) = w\partial_3^2\eta + (1+\alpha)\partial_3 w\, \partial_3\eta$. From (35), we obtain

$$\begin{aligned}
&\partial_t^2 F^i + \underbrace{\frac{1}{w^{1+\alpha}}\Big(w^{2+\alpha}\underbrace{\{w\partial_3^2[A_i^k J^{-1/\alpha}] + (1+\alpha)\partial_3 w\, \partial_3[A_i^k J^{-1/\alpha}]\}}_{(\star\star)}\Big),_k}_{(\star)} \\
&- (1+\alpha) w\, \partial_3 w,_k \partial_3[A_i^k J^{-1/\alpha}] + w\, I^{0,2} + (1+\alpha)\partial_3 w\, I^{0,1} = 0
\end{aligned}$$

First we look at $(\star\star)$

$$\begin{aligned}
&w\partial_3^2[A_i^k J^{-1/\alpha}] + (1+\alpha)\partial_3 w\, \partial_3[A_i^k J^{-1/\alpha}] \\
&= -\{J^{-1/\alpha}A_r^k A_i^s + \tfrac{1}{\alpha}J^{-1/\alpha}A_i^k A_r^s\}\{w\partial_3^2\eta^r,_s + (1+\alpha)\partial_3 w\, \partial_3\eta^r,_s\} \\
&\quad - w\partial_3\{J^{-1/\alpha}A_r^k A_i^s + \tfrac{1}{\alpha}J^{-1/\alpha}A_i^k A_r^s\}\cdot \partial_3\eta^r,_s \\
&= -\{J^{-1/\alpha}A_r^k A_i^s + \tfrac{1}{\alpha}J^{-1/\alpha}A_i^k A_r^s\}F^r,_s \\
&\quad + \{J^{-1/\alpha}A_r^k A_i^s + \tfrac{1}{\alpha}J^{-1/\alpha}A_i^k A_r^s\}\{w,_s \partial_3^2\eta^r + (1+\alpha)\partial_3 w,_s \partial_3\eta^r\} \\
&\quad - w\partial_3\{J^{-1/\alpha}A_r^k A_i^s + \tfrac{1}{\alpha}J^{-1/\alpha}A_i^k A_r^s\}\cdot \partial_3\eta^r,_s
\end{aligned}$$

Now we rewrite $(\star)$ as follows:

$$\begin{aligned}
(\star) &= \frac{1}{w^{1+\alpha}}\Big(w\cdot w^{1+\alpha}\{w\partial_3^2[A_i^k J^{-1/\alpha}] + (1+\alpha)\partial_3 w\, \partial_3[A_i^k J^{-1/\alpha}]\}\Big),_k \\
&= \frac{1}{w^{\alpha}}\Big(w^{1+\alpha}\{w\partial_3^2[A_i^k J^{-1/\alpha}] + (1+\alpha)\partial_3 w\, \partial_3[A_i^k J^{-1/\alpha}]\}\Big),_k \\
&\quad + w,_k \{w\partial_3^2[A_i^k J^{-1/\alpha}] + (1+\alpha)\partial_3 w\, \partial_3[A_i^k J^{-1/\alpha}]\} \\
&= -\frac{1}{w^{\alpha}}\Big(w^{1+\alpha}\{J^{-1/\alpha}A_r^k A_i^s + \tfrac{1}{\alpha}J^{-1/\alpha}A_i^k A_r^s\}F^r,_s\Big),_k \\
&\quad + \underbrace{\frac{1}{w^{\alpha}}\Big(w^{1+\alpha}\{J^{-1/\alpha}A_r^k A_i^s + \tfrac{1}{\alpha}J^{-1/\alpha}A_i^k A_r^s\}\{w,_s \partial_3^2\eta^r + (1+\alpha)\partial_3 w,_s \partial_3\eta^r\}\Big),_k}_{(*)} \\
&\quad + \underbrace{w,_k \{w\partial_3^2[A_i^k J^{-1/\alpha}] + (1+\alpha)\partial_3 w\, \partial_3[A_i^k J^{-1/\alpha}]\}}_{(**)} \\
&\quad - \frac{1}{w^{\alpha}}(w^{2+\alpha}\partial_3\{J^{-1/\alpha}A_r^k A_i^s + \tfrac{1}{\alpha}J^{-1/\alpha}A_i^k A_r^s\}\cdot \partial_3\eta^r,_s),_k
\end{aligned}$$

We now show that the undesirable terms $w\partial_3^3\eta$ and $\partial_3^2\eta$ cancel out in $(*)+(**)$ and thus the dominant terms are of $w\partial_3^2\eta,_\sigma$ and $\partial_3\eta,_\sigma$. Indeed, $(*)+(**)$ can be written as follows:

$$
\begin{aligned}
&(*)+(**)\\
&= w^{-\alpha}\Big(w^{1+\alpha}\{J^{-1/\alpha}A^\kappa_r A^s_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^\kappa_i A^s_r\}\{w,_s\,\partial_3^2\eta^r+(1+\alpha)\partial_3 w,_s\,\partial_3\eta^r\}\Big),_\kappa\\
&+wJ^{-1/\alpha}\{[A^3_rA^\sigma_i+\tfrac{1}{\alpha}A^3_iA^\sigma_r]w,_\sigma\,\partial_3^3\eta^r-[A^\kappa_rA^s_i+\tfrac{1}{\alpha}A^\kappa_iA^s_r]w,_\kappa\,\partial_3^2\eta^r,_s-[A^k_rA^\sigma_i+\tfrac{1}{\alpha}A^k_iA^\sigma_r]\\
&\quad\cdot w,_k\,\partial_3^2\eta^r,_\sigma+[A^3_rA^s_i+\tfrac{1}{\alpha}A^3_iA^s_r][(2+\alpha)\partial_3 w,_s\,\partial_3^2\eta^r+(1+\alpha)\partial_3^2 w,_s\,\partial_3\eta^r]\}\\
&+(1+\alpha)w,_3\,J^{-1/\alpha}\{[A^3_rA^\sigma_i+\tfrac{1}{\alpha}A^3_iA^\sigma_r]w,_\sigma\,\partial_3^2\eta^r-[A^\kappa_rA^s_i+\tfrac{1}{\alpha}A^\kappa_iA^s_r]w,_\kappa\,\partial_3\eta^r,_s\\
&\quad-[A^k_rA^\sigma_i+\tfrac{1}{\alpha}A^k_iA^\sigma_r]w,_k\,\partial_3\eta^r,_\sigma+(1+\alpha)[A^3_rA^s_i+\tfrac{1}{\alpha}A^3_iA^s_r]\partial_3 w,_s\,\partial_3\eta^r\}\\
&+w\{\partial_3[J^{-1/\alpha}A^3_rA^\sigma_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^3_iA^\sigma_r]w,_\sigma\,\partial_3^2\eta^r-\partial_3[J^{-1/\alpha}A^\kappa_rA^s_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^\kappa_iA^s_r]w,_\kappa\,\partial_3\eta^r,_s\\
&-\partial_3[J^{-\frac{1}{\alpha}}A^k_rA^\sigma_i+\tfrac{1}{\alpha}J^{-\frac{1}{\alpha}}A^k_iA^\sigma_r]w,_k\,\partial_3\eta^r,_\sigma+(1+\alpha)\partial_3[J^{-\frac{1}{\alpha}}A^3_rA^s_i+\tfrac{1}{\alpha}J^{-\frac{1}{\alpha}}A^3_iA^s_r]\partial_3 w,_s\,\partial_3\eta^r\}
\end{aligned}
$$

Therefore, the equation for F reads as follows.

$$
\begin{aligned}
&\partial_t^2F^i-w^{-\alpha}(w^{1+\alpha}\{J^{-1/\alpha}A^k_rA^s_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^k_iA^s_r\}F^r,_s),_k\\
&-w^{-\alpha}(w^{2+\alpha}\partial_3\{J^{-1/\alpha}A^k_rA^s_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^k_iA^s_r\}\cdot\partial_3\eta^r,_s),_k\\
&+w^{-\alpha}\Big(w^{1+\alpha}\{J^{-1/\alpha}A^\kappa_rA^s_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^\kappa_iA^s_r\}\{w,_s\,\partial_3^2\eta^r+(1+\alpha)\partial_3 w,_s\,\partial_3\eta^r\}\Big),_\kappa\\
&+wJ^{-1/\alpha}\{[A^3_rA^\sigma_i+\tfrac{1}{\alpha}A^3_iA^\sigma_r]w,_\sigma\,\partial_3^3\eta^r-[A^\kappa_rA^s_i+\tfrac{1}{\alpha}A^\kappa_iA^s_r]w,_\kappa\,\partial_3^2\eta^r,_s-[A^k_rA^\sigma_i+\tfrac{1}{\alpha}A^k_iA^\sigma_r]\\
&\quad\cdot w,_k\,\partial_3^2\eta^r,_\sigma+[A^3_rA^s_i+\tfrac{1}{\alpha}A^3_iA^s_r][(2+\alpha)\partial_3 w,_s\,\partial_3^2\eta^r+(1+\alpha)\partial_3^2 w,_s\,\partial_3\eta^r]\}\\
&+(1+\alpha)w,_3\,J^{-1/\alpha}\{[A^3_rA^\sigma_i+\tfrac{1}{\alpha}A^3_iA^\sigma_r]w,_\sigma\,\partial_3^2\eta^r-[A^\kappa_rA^s_i+\tfrac{1}{\alpha}A^\kappa_iA^s_r]w,_\kappa\,\partial_3\eta^r,_s\\
&\quad-[A^k_rA^\sigma_i+\tfrac{1}{\alpha}A^k_iA^\sigma_r]w,_k\,\partial_3\eta^r,_\sigma+(1+\alpha)[A^3_rA^s_i+\tfrac{1}{\alpha}A^3_iA^s_r]\partial_3 w,_s\,\partial_3\eta^r\}\\
&+w\{\partial_3[J^{-1/\alpha}A^3_rA^\sigma_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^3_iA^\sigma_r]w,_\sigma\,\partial_3^2\eta^r-\partial_3[J^{-1/\alpha}A^\kappa_rA^s_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^\kappa_iA^s_r]\\
&\quad\cdot w,_\kappa\,\partial_3\eta^r,_s-\partial_3[J^{-\frac{1}{\alpha}}A^k_rA^\sigma_i+\tfrac{1}{\alpha}J^{-\frac{1}{\alpha}}A^k_iA^\sigma_r]w,_k\,\partial_3\eta^r,_\sigma\\
&\quad+(1+\alpha)\partial_3[J^{-\frac{1}{\alpha}}A^3_rA^s_i+\tfrac{1}{\alpha}J^{-\frac{1}{\alpha}}A^3_iA^s_r]\partial_3 w,_s\,\partial_3\eta^r\}\\
&-(1+\alpha)w\,\partial_3 w,_k\,\partial_3[A^k_iJ^{-1/\alpha}]+w\,I^{0,2}+(1+\alpha)\partial_3 w\,I^{0,1}=0
\end{aligned}
\tag{64}
$$

The first line is the main part, the second line has full derivative but with the desirable weight w^2, the rest of lines are either of lower order with respect to ∂_3 with appropriate weights. We denote the second line through the last line by R^i. Next by using the fact $A^k_i=\delta^k_sA^s_i=A^k_r\eta^r,_s A^s_i$, we write the equation (12) as follows:

$$
\begin{aligned}
\eta^i_{tt}-w^{-\alpha}(w^{1+\alpha}J^{-1/\alpha}A^k_rA^s_i\eta^r,_s),_k&-\tfrac{1}{\alpha}w^{-\alpha}(w^{1+\alpha}J^{-1/\alpha}A^k_iA^s_r\eta^r,_s),_k\\
&+(2+\tfrac{3}{\alpha})w^{-\alpha}(w^{1+\alpha}J^{-1/\alpha}A^k_i),_k=0
\end{aligned}
\tag{65}
$$

Now combining (64) and (65) with the equation for $[\partial_1^2+\partial_2^2]\eta$ in (37), the equation for $G=[-\partial_1^2-\partial_2^2-w^{-\alpha}\partial_3w^{1+\alpha}\partial_3+\lambda]\eta$ reads as follows:

$$
\begin{aligned}
&\partial_t^2G^i-w^{-\alpha}(w^{1+\alpha}\{J^{-1/\alpha}A^k_rA^s_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^k_iA^s_r\}G^r,_s),_k\\
&=-w^{-\alpha}(w^{1+\alpha}\partial_\beta[J^{-1/\alpha}A^k_rA^s_i+\tfrac{1}{\alpha}J^{-1/\alpha}A^k_iA^s_r]\cdot\partial_\beta\eta^r,_s),_k+I^{2,0,0}\\
&\quad+R^i-\lambda(2+\tfrac{3}{\alpha})w^{-\alpha}(w^{1+\alpha}J^{-1/\alpha}A^k_i),_k\equiv\mathcal{R}^i
\end{aligned}
\tag{66}
$$

We note that $\mathcal{R}^i$ consists of essentially lower-order terms and in particular,

$$\|\mathcal{R}^i\|_{X^{\alpha,N-2}} < \infty \text{ if } \mathcal{E}^N(\eta, \partial_t\eta) < \infty \text{ and } \|Dw\|_{X^{\alpha,N}} < \infty .$$

The equation for $H = \text{Curl}_\eta G$ can be derived in the same way from the curl equation (42).

$$\begin{aligned}
&\partial_t H = [-\partial_\beta^2 - w\partial_3^2 - (2+\alpha)\partial_3 w\, \partial_3 + \lambda][\text{Curl} u_0]_j^k \\
&- \partial_t A_j^s[\partial_\beta^2 + w\partial_3^2 + (2+\alpha)\partial_3 w \partial_3 + \lambda]\eta^k,_s - \partial_t A_k^s[\partial_\beta^2 + w\partial_3^2 + (2+\alpha)\partial_3 w\partial_3 + \lambda]\eta^j,_s \\
&+ \sum_{p=1}^{2}(\partial_\beta^p A_j^s \cdot \partial_t\partial_\beta^{2-p}\eta^k,_s + w\partial_3^p A_j^s \cdot \partial_t\partial_3^{2-p}\eta^k,_s \\
&\qquad - \partial_\beta^p A_k^s \cdot \partial_t\partial_\beta^{2-p}\eta^j,_s - w\partial_3^p A_k^s \cdot \partial_t\partial_3^{2-p}\eta^j,_s) \\
&+ (2+\alpha)\partial_3 w(\partial_3 A_j^s \cdot \partial_t\eta^k,_s - \partial_3 A_k^s \cdot \partial_t\eta^j,_s) + \lambda\int_0^t (\partial_t A_j^s \cdot \partial_t\eta^k,_s - \partial_t A_k^s \cdot \partial_t\eta^j,_s)d\tau \\
&- \sum_{p=0}^{2}\int_0^t (\partial_t\partial_\beta^p A_j^s \cdot \partial_t\partial_\beta^{2-p}\eta^k,_s + w\partial_t\partial_3^p A_j^s \cdot \partial_t\partial_3^{2-p}\eta^k,_s \\
&\qquad - \partial_t\partial_\beta^p A_k^s \cdot \partial_t\partial_\beta^{2-p}\eta^j,_s - w\partial_t\partial_3^p A_k^s \cdot \partial_t\partial_3^{2-p}\eta^j,_s)d\tau \\
&- \sum_{p=0}^{1}(2+\alpha)\partial_3 w\int_0^t (\partial_t\partial_3^p A_j^s \cdot \partial_t\partial_3^{1-p}\eta^k,_s - \partial_t\partial_3^p A_k^s \cdot \partial_t\partial_3^{1-p}\eta^j,_s)d\tau \\
&+ \partial_3 w\partial_t(A_j^\sigma \cdot \partial_3\eta^k,_\sigma - A_k^\sigma \cdot \partial_3\eta^j,_\sigma) - \partial_t(w,_\sigma A_j^\sigma \cdot \partial_3^2\eta^k - w,_\sigma A_k^\sigma \cdot \partial_3^2\eta^j) \\
&- (2+\alpha)\partial_t(\partial_3 w,_s A_j^s \cdot \partial_3\eta^k - \partial_3 w,_s A_k^s \cdot \partial_3\eta^j) \equiv [\mathcal{S}]_j^k
\end{aligned} \tag{67}$$

We note that

$$\|\mathcal{S}\|_{Z^{\alpha,N-2}} < \infty \text{ if } \mathcal{E}^N(\eta, \partial_t\eta) < \infty \text{ and } \|Dw\|_{X^{\alpha,N}} < \infty .$$

6 General smooth initial domain Ω

Here we would like to discuss the changes to be made to the argument to prove theorem 2.2 for general domains.

There are different ways of trying to extend the result to the general case. One can use K charts to cover the boundary of Ω and one chart for the interior. Then, one can use change of coordinates to straighten out the boundary for each chart. One can then prove a priori estimates. However, we think that one of the disadvantage of this method is that the proof of existence of approximate solutions is technical since one has to solve $K+1$ problems simultaneously at each step. Here, we will present a more geometric way motivated by Shatah and Zeng [58].

Recall the notations introduced in Section 2.4. The main difference between the tangential and normal derivatives is the fact that $|\partial_\beta^m w| \le Cw$ and that $\frac{1}{C} \le \partial_\zeta w \le C$.

The proof of the a priori estimates, namely Proposition 4.1 is identical. One has just to replace ∂_3 by ∂_ζ and ∂_1, ∂_2 by ∂_β for $\beta \in \mathcal{T}$.

When solving the iteration scheme, we replace the definition of G by

$$G = [\sum_{\beta\in\mathcal{T}}(\partial_\beta)^*\partial_\beta - w^{-\alpha}\partial_\zeta w^{1+\alpha}\partial_\zeta + \lambda]u = \mathcal{G}u \tag{68}$$

we recall that if $\partial_\beta = a_1^\beta(x)\partial_1 + a_2^\beta(x)\partial_2 + a_3^\beta(x)\partial_3$ then $(\partial_\beta)^*(.) = \partial_1(a_1^\beta(x).) + \partial_2(a_2^\beta(x).) + \partial_3(a_3^\beta(x).)$. Actually, a better choice of the operator $\mathcal{G}$ is to take it to be selfadjoint in $L^2(w^\alpha dx)$ and hence take

$$G = [\sum_{\beta\in\mathcal{T}} w^{-\alpha}(\partial_\beta)^* w^\alpha \partial_\beta - w^{-\alpha}(\partial_\zeta)^* w^{1+\alpha}\partial_\zeta + \lambda]u = \mathcal{G}u. \tag{69}$$

As in section 5.1, we solve the following approximate system for $G_{\nu+1}$ and $H_{\nu+1}$:

$$\begin{aligned} w^\alpha \partial_t^2 G_{\nu+1} + \mathcal{L}_\nu^e G_{\nu+1} + \mathcal{L}_\nu^d G_{\nu+1} + \mathcal{L}_\nu^c H_{\nu+1} &= w^\alpha \mathcal{R}_\nu \\ \partial_t H_{\nu+1} &= \mathcal{S}_\nu \end{aligned} \tag{70}$$

where $\mathcal{R}_\nu$ and $\mathcal{S}_\nu$ are given by by formulae that are similar to those in section 5.1 and consist of lower order terms. The rest of the proof is identical. We just mention that Lemma 5.2 was proved above in the flat case. Extending it to the case of the operator $\mathcal{G}$ defined in (68) or (69) can be easily done following the arguments in [5, 6].

7 Discussion

The study of vacuum states in gas and fluid dynamics can be tracked back at least to a conference, Problems of Cosmical Aerodynamics, held in Paris,1949 where one session chaired by J. von Neumann was devoted to the existence and uniqueness or multiplicity of solutions of the aerodynamical equations [60]. See also the review article by Serre [57]. The last question raised by von Neumann concerns the validity of the Euler system in the presence of a vacuum:

There is a further difficulty in the expansion case considered by Burgers. It was accepted that the front advances into a vacuum. It is evident that you cannot get the normal conditions of kinetic theory here either, because the density of the gas goes to zero at the front, which means that the mean free path of the molecules will go to infinity. This means that if we are in the expanding gas and approach the (theoretical) front, we will necessarily come to regions where the mean free path is larger than the distance from the front. In such regions one cannot use the hydrodynamical equations. But, as in the case of the shock wave, where ordinary conditions are reached at a distance of a few mean free paths from the shock itself, so in the case of expansion into a vacuum, at a short distance from the theoretical front, one comes into regions where the mean free path is considerably smaller than the distance from the front, and where again the classical hydrodynamical equations can be applied. If this is applied to expanding interstellar clouds, I think that in order that the classical theory be true down to 1/1000 of the density of the clouds, it is necessary that the distance from the theoretical front should be of the order of a percent of a parsec.

This statement led to interesting discussions, which are still enlightening at present, among von Neumann and several participants including Heisenberg. An important comment made by Heisenberg was the role of boundary conditions related to the boundary layer theory. And von Neumann answered:

The boundary layer theory for a fluid of low viscosity certainly furnishes a monumental warning. The naive and yet prima facie seemingly reasonable procedure would be to apply the ordinary equations of the ideal fluid and then to expect that viscosity will somehow take *care of itself in a narrow region along the wall. We have learned that this procedure may lead to great errors; a complete theory of the boundary layer may give you completely different conditions also for the flow in the bulk of the field. It is possible that the same discipline will be necessary for the boundary with a vacuum.*

It is amazing to learn that the difficulty and importance of the problem with vacuum was already discussed more than 60 years ago. Although the mathematical theory of vacuum and related subjects in the domain of kinetic theory and boundary layer theory is far from being complete even now, it is very exciting to make a first step towards a better understanding of the problem.

Our methodology developed in this paper turns out to be robust and it would shed some light on other vacuum free boundary problems in more general framework such as problems as discussed in Section 1.2. We believe that our energy method can be adapted to the relativistic Euler equations in a physical vacuum [32]. And in the upcoming paper [31], with this new approach, we will give different proofs of the well-posedness of compressible liquid in vacuum studied in [38] and the well-posedness of smoother vacuum states of compressible Euler flow considered in [45].

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