

DOUBLE LOGARITHMIC INEQUALITY WITH A SHARP CONSTANT

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ABSTRACT. We prove a Log Log inequality with a sharp constant. We also show that the constant in the Log estimate is “almost” sharp. These estimates are applied to prove a Moser-Trudinger type inequality for solutions of a 2D wave equation.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

By the Sobolev embedding theorem, it is well known that the Sobolev space $H^1(\mathbb{R}^2)$ is embedded in all Lebesgue spaces $L^p(\mathbb{R}^2)$ for $2 \leq p < +\infty$ but not in $L^\infty(\mathbb{R}^2)$. Moreover, H^1 functions are in a so-called Orlicz space i.e their exponential powers are integrable functions. Precisely, we have the following Moser-Trudinger inequality (see [1], [10], [11]).

Proposition 1.1. *There exists a universal positive constant C such that, for all $u \in H^1(\mathbb{R}^2)$, we have*

$$(1.1) \quad \|u\|_{H^1(\mathbb{R}^2)} \leq 1 \implies \int_{\mathbb{R}^2} (e^{4\pi u(x)^2} - 1) dx \leq C.$$

In this paper, we show that we can control the L^∞ norm by the H^1 norm and a stronger norm with a logarithmic growth or double logarithmic growth. The inequality is sharp for the double logarithmic growth.

Recall that H^1 is the usual Sobolev space endowed with the norm $\|u\|_{H^1}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2$. For any real number $\alpha \in]0, 1[$, we denote by \dot{C}^α the sub-space of α - Hölder continuous functions endowed with the semi-norm

$$\|u\|_{\dot{C}^\alpha} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Also, we denote $\|u\|_{C^\alpha} := \|u\|_{\dot{C}^\alpha} + \|u\|_{L^\infty}$ and define $N_\alpha(u)$ to be the ratio $N_\alpha(u) := \frac{\|u\|_{\dot{C}^\alpha}}{\|\nabla u\|_{L^2}}$.

For any bounded domain Ω in \mathbb{R}^2 , define $H_0^1(\Omega)$ to be the completion in the Sobolev space $H^1(\Omega)$ of smooth and compactly supported functions.

The main result of this paper is the following.

Theorem 1.2 (Double logarithmic inequality). *Let $\alpha \in]0, 1[$ and B_1 be the unit ball in \mathbb{R}^2 . Any function in $H_0^1(B_1) \cap \dot{C}^\alpha(B_1)$ is bounded. Moreover, a positive constant C_0 exists such that for any function $u \in H_0^1(B_1) \cap \dot{C}^\alpha(B_1)$, one has*

$$(1.2) \quad \|u\|_{L^\infty}^2 \leq \frac{1}{2\pi\alpha} \|\nabla u\|_{L^2}^2 \log \left[e^3 + C_0 N_\alpha(u) \sqrt{\log(2e + N_\alpha(u))} \right]$$

and, the constant $\frac{1}{2\pi\alpha}$ in (1.2) is sharp.

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Note that $\log(e) = 1$. Our second result concerns the following logarithmic inequality.

Theorem 1.3 (Logarithmic inequality). *Let α be in $]0, 1[$. For any real number $\lambda > \frac{1}{2\pi\alpha}$, a constant C_λ exists such that, for any function $u \in H_0^1(B_1) \cap \dot{C}^\alpha(B_1)$, we have*

$$(1.3) \quad \|u\|_{L^\infty}^2 \leq \lambda \|\nabla u\|_{L^2}^2 \log(C_\lambda + N_\alpha(u)).$$

Moreover, the above inequality does not hold for $\lambda = \frac{1}{2\pi\alpha}$.

2. A LITTLEWOOD-PALEY PROOF

To prove the fundamental theorems, we start by showing that inequality (1.3) can easily be obtained with an unknown absolute constant C instead of $\frac{1}{2\pi\alpha}$. To do so, we give a brief recall of the Littlewood-Paley theory and we refer the reader to [4] for a thorough treatment. Denote by \mathcal{C}_0 the annular ring defined by

$$\mathcal{C}_0 = \left\{ \xi \in \mathbb{R}^2 \text{ such that } \frac{3}{4} < |\xi| < \frac{8}{3} \right\},$$

and choose two non-negative radial functions χ and φ belonging respectively to $\mathcal{D}(B(0, 4/3))$ and $\mathcal{D}(\mathcal{C}_0)$ such that for all $\xi \in \mathbb{R}^2$

$$\chi(\xi) + \sum_{k \in \mathbb{N}} \varphi(2^{-k}\xi) = 1.$$

Denote by $h = \mathcal{F}^{-1}\varphi$ and define the frequency projector Δ_k by, for all $u \in \mathcal{S}'(\mathbb{R}^2)$,

$$\Delta_k u = \varphi(2^{-k}D)u = 2^{2k} \int_{\mathbb{R}^2} h(2^k y) u(x - y) dy,$$

and

$$\tilde{\Delta}_0 = \sum_{k \leq 0} \Delta_k.$$

Recall that

$$\|\nabla u\|_{L^2} \sim \left(\sum_{k \in \mathbb{Z}} 2^{2k} \|\Delta_k u\|_{L^2}^2 \right)^{\frac{1}{2}}$$

and

$$\|u\|_{\dot{C}^\alpha} \sim \sup_k 2^{k\alpha} \|\Delta_k u\|_{L^\infty}.$$

We have the following result in the whole space.

Proposition 2.1. *Let α be in $]0, 1[$. A positive constant C exists such that for any function $u \in \mathcal{C}^\alpha(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, one has*

$$(2.4) \quad \|u\|_{L^\infty(\mathbb{R}^2)}^2 \leq C \|u\|_{L^2(\mathbb{R}^2)}^2 + C \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \log\left(e + \frac{\|u\|_{\dot{C}^\alpha(\mathbb{R}^2)}}{\|\nabla u\|_{L^2(\mathbb{R}^2)}}\right).$$

Proof. Write

$$u = \tilde{\Delta}_0 u + \sum_{j=1}^{\infty} \Delta_j u = \tilde{\Delta}_0 u + \sum_{j=1}^{N-1} \Delta_j u + \sum_{j=N}^{\infty} \Delta_j u,$$

where N is a non-negative integer which will be chosen later. Using Bernstein's inequality, we get

$$\begin{aligned} \|u\|_{L^\infty} &\leq C\|\tilde{\Delta}_0 u\|_{L^2} + C\sum_{j=1}^{N-1} 2^j \|\Delta_j u\|_{L^2} + \sum_{j=N}^{\infty} 2^{-j\alpha} (2^{j\alpha} \|\Delta_j u\|_{L^\infty}) \\ &\leq C\|u\|_{L^2} + C\sqrt{N} \left(\sum_{j=1}^{N-1} 2^{2j} \|\Delta_j u\|_{L^2}^2 \right)^{1/2} + \left(\sum_{j=N}^{\infty} 2^{-j\alpha} \right) \|u\|_{\dot{C}^\alpha} \\ &\leq C\|u\|_{L^2} + C\sqrt{N} \|\nabla u\|_{L^2} + \frac{2^{-\alpha N}}{1-2^{-\alpha}} \|u\|_{\dot{C}^\alpha}. \end{aligned}$$

So

$$\|u\|_{L^\infty}^2 \leq 2C^2 \|u\|_{L^2}^2 + 2C^2 N \|\nabla u\|_{L^2}^2 + 2 \frac{2^{-2\alpha N}}{(1-2^{-\alpha})^2} \|u\|_{\dot{C}^\alpha}^2.$$

Denoting by $]x[$ the integer part of the real number x and choosing

$$N := \text{Max}(1, 1 +]2 \log_2 \frac{\|u\|_{\dot{C}^\alpha}^2}{\|\nabla u\|_{L^2}^2} [),$$

the proof of Proposition 2.1 is achieved. ■

Clearly, if u is supported in B_1 then using the Poincaré inequality, we get

$$(2.5) \quad \|u\|_{L^\infty}^2 \leq C \|\nabla u\|_{L^2}^2 \log(C_0 + N(u)).$$

3. PROOF OF THEOREM 1.2

To prove (1.2) and the fact that the constant is sharp, it is sufficient to show that

$$(3.6) \quad 2\pi\alpha = \inf_{u \in H_0^1(B_1) \cap \dot{C}^\alpha(B_1)} \frac{\|\nabla u\|_{L^2}^2 \log \left[e^3 + C_0 N_\alpha(u) \sqrt{\log(2e + N_\alpha(u))} \right]}{\|u\|_{L^\infty}^2}.$$

for any C_0 big enough. Let us start by proving the sharpness of the constant. Defining $u_k(x) = f_k(-2 \log |x|)$, where for any non-negative integer k

$$\begin{aligned} f_k(t) &= \sqrt{\frac{k}{2\pi} \frac{t}{k}} \text{ if } t \leq k \\ f_k(t) &= \sqrt{\frac{k}{2\pi}} \text{ if not.} \end{aligned}$$

An easy computation shows that

$$\|\nabla u_k\|_{L^2}^2 = 2, \quad \|u_k\|_{\dot{C}^\alpha} = C k^{\frac{1}{2}-\alpha} \exp \frac{\alpha k}{2}$$

and therefore, after taking the limit as $k \rightarrow \infty$, we deduce that

$$2\pi\alpha \geq \inf_{u \in H_0^1(B_1) \cap \dot{C}^\alpha(B_1)} \frac{\|\nabla u\|_{L^2}^2 \log \left[e^3 + C_0 N_\alpha(u) \sqrt{\log(2e + N_\alpha(u))} \right]}{\|u\|_{L^\infty}^2}.$$

These functions was introduced in [1] and [9] to show the optimality of the exponent 4π in Trudinger-Moser inequality (see [10]).

To prove (1.2), we start by noticing that for any function u , the norms $\|\nabla u\|_{L^2}$ and $\|u\|_{\dot{C}^\alpha}$ are non-increasing under symmetric non-increasing rearrangements, while $\|u\|_{L^\infty}$ remains unchanged.

Using the fact that for all $C > 0$

$$t \rightarrow f(t) := t^2 \log \left[e^3 + \frac{C}{t} \sqrt{\left[\log \left(2e + \frac{1}{t} \right) \right]} \right]$$

is increasing, it is sufficient to check the minimizer figured in (3.6) in the class of non-negative, non-increasing and radially symmetric functions.

Without loss of generality, we can normalize $\|u\|_{L^\infty}$ to be equal to 1. Moreover, we will assume that $\|u\|_{\dot{C}^\alpha} \geq 1$ because in the contrary case, the proof is similar.

Let $H_{0,rad}^1(B_1)$ be the space of all non-increasing and radially symmetric functions in $H_0^1(B_1)$. For any parameter $D \geq 1$, we denote by K_D the closed convex subset of $H_{0,rad}^1(B_1)$ defined by

$$(3.7) \quad K_D = \{u \in H_{0,rad}^1(B_1) : u(r) \geq 1 - Dr^\alpha, \quad r \in [0, 1]\}.$$

To get the result, it is sufficient to prove that

$$\begin{aligned} 2\pi\alpha &\leq \inf_{D \geq 1} \inf_{\{u \in K_D\}} \|\nabla u\|_{L^2}^2 \log \left[e^3 + \frac{C_0 D}{\|\nabla u\|_{L^2}} \sqrt{\log \left(2e + \frac{D}{\|\nabla u\|_{L^2}} \right)} \right] \\ &\leq \inf_{D \geq 1} \inf_{\{u \in K_D, \|u\|_{L^\infty} = 1, \|u\|_{\dot{C}^\alpha} = D\}} \|\nabla u\|_{L^2}^2 \log \left[e^3 + \frac{C_0 D}{\|\nabla u\|_{L^2}} \sqrt{\log \left(2e + \frac{D}{\|\nabla u\|_{L^2}} \right)} \right]. \end{aligned}$$

Consider the following problem of minimizing

$$(3.8) \quad I[u] := \|\nabla u\|_{L^2(B_1)}^2,$$

among all the functions belonging to the set K_D . This is a variational problem with obstacle. It is well known (see for example, Kinderlehrer-Stampacchia [8] and L. C. Evans [5]) that it has a unique minimizer u^* which is variationally characterized by

$$(3.9) \quad \int_{B_1} \nabla u^* \cdot \nabla v \, dx \geq \|\nabla u^*\|_{L^2(B_1)}^2,$$

for any $v \in K_D$. Moreover u^* is in the Sobolev space $W^{2,\infty}(B_1)$. Hence the following radially symmetric set

$$\mathcal{O} := \{x \in B_1 : u^*(x) > 1 - D|x|^\alpha\}$$

is open and u^* is harmonic in \mathcal{O} . On the other hand, note that any radially symmetric harmonic functions in \mathbb{R}^2 can only coincide in a unique tangent point with the function $r \rightarrow 1 - Dr^\alpha$. Note also that because of the boundary condition at $r = 1$, u^* cannot start to be harmonic near $r = 0$. Therefore there exists, a unique $a \in]0, 1[$ such that

$$(3.10) \quad \begin{aligned} u^*(r) &= 1 - Dr^\alpha \text{ if } r \in [0, a] \\ u^*(r) &= (1 - Da^\alpha) \frac{\log r}{\log a} \text{ if } r \in [a, 1], \end{aligned}$$

satisfying also the tangent condition

$$(3.11) \quad a^\alpha = \frac{1 - Da^\alpha}{D|\log(a^\alpha)|}.$$

Note that if $D \rightarrow 1$ then $a \rightarrow 1$ and therefore (3.11) still makes sense in the limit case. Also, because of the regularity of u^* at $r = 0$ it is necessary that $a \neq 1$. In particular, note that $\|u^*\|_{L^\infty} = 1$, $\|u^*\|_{\dot{C}^\alpha} = D$, and

$$(3.12) \quad \|\nabla u^*\|_{L^2}^2 = \pi\alpha D^2 a^{2\alpha} - 2\pi \left(\frac{1 - Da^\alpha}{\log(a)}\right)^2 \log(a).$$

Substituting D from (3.11) into (3.12), we get the following

$$\|\nabla u^*\|_{L^2}^2 = 2\pi\alpha \frac{1/2 - \log(a^\alpha)}{(1 - \log(a^\alpha))^2}.$$

Denoting by $x := a^\alpha \in]0, 1[$, then we have

$$(3.13) \quad \|\nabla u^*\|_{L^2}^2 = 2\pi\alpha \frac{1/2 - \log(x)}{(1 - \log(x))^2}$$

and

$$(3.14) \quad \|u^*\|_{\dot{C}^\alpha} = \frac{1}{x(1 - \log(x))}.$$

Setting

$$g(x) := \frac{1}{x\sqrt{2\pi\alpha}(1/2 - \log(x))},$$

and

$$F_C(x) := \frac{\frac{1}{2} - \log(x)}{(1 - \log(x))^2} \log \left[e^3 + Cg(x)\sqrt{\log(2e + g(x))} \right],$$

it is sufficient to show that a constant C_0 exists such that for all $0 < x \leq 1$, the function F_{C_0} satisfies

$$(3.15) \quad F_{C_0}(x) \geq 1.$$

First, observe that for every $0 < x \leq 1$

$$\frac{\frac{1}{2} - \log(x)}{(1 - \log(x))^2} \geq \frac{1}{(2 - \log(x))}.$$

Hence for any $C > 0$, (3.15) holds if $2 - \log x \leq 3$, namely if $x \geq 1/e$.

In the sequel, we suppose that $x \leq 1/e$, hence

$$(3.16) \quad \begin{aligned} F(x) &\geq \frac{1}{(2 - \log(x))} \left[-\log(x) + \log\left(\frac{C_0}{\sqrt{2\pi\alpha}}\right) - \frac{1}{2}\log(1/2 - \log(x)) + \frac{1}{2}\log(\log(2e + g(x))) \right] \\ &\geq 1 + \frac{1}{(2 - \log(x))} \left[\log\left(\frac{C_0}{e^2\sqrt{2\pi\alpha}}\right) + \frac{1}{2}\log\left(\frac{\log(2e + g(x))}{(1/2 - \log(x))}\right) \right]. \end{aligned}$$

The function $h(x) = \frac{\log(2e+g(x))}{(1/2-\log(x))}$ is bounded away from zero on $(0, 1/e)$. Hence, we can find C_0 big enough such that the second term on the right hand side of (3.16) is non negative. This achieves the proof of Theorem 1.2. \blacksquare

4. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is similar to that of Theorem 1.2. Indeed, consider u^* the minimizer of the Dirichlet norm (3.8) among all functions in K_D defined in (3.7). Note that according to (3.13) and (3.14), we have

$$\|\nabla u^*\|_{L^2}^2 \log(C_\lambda + N_\alpha(u^*)) := H(x),$$

where

$$H(x) = 2\pi\alpha \frac{1/2 - \log(x)}{(1 - \log(x))^2} \log\left(C_\lambda + \frac{1}{x\sqrt{2\pi\alpha}(1/2 - \log(x))}\right).$$

Taking $C_\lambda = e$ in $H(x)$, we see that $H(x)$ goes to $2\pi\alpha$ as x goes to 0. Hence, for any $\lambda > \frac{1}{2\pi\alpha}$, there exists $x_\lambda > 0$ such that $\lambda H(x) \geq 1$, for any $0 < x < x_\lambda$ and $C_\lambda \geq e$. Now, if $x \in [x_\lambda, 1]$, choosing the constant $C_\lambda > e$ big enough such that

$$\frac{1/2}{(1 - \log(x_\lambda))^2} \log(C_\lambda) \geq 1,$$

we see that $\lambda H(x) \geq 1$. Hence, by this choice of C_λ , we see that $\lambda H(x) \geq 1$ for all $0 < x \leq 1$. This achieves the proof of (1.3).

Now, let us prove that (1.3) does not hold for $\lambda = \frac{1}{2\pi\alpha}$. More precisely, we will prove that a sequence of functions $(u_n)_n$ exists such that $u_n \in H_0^1(B_1) \cap \dot{C}^\alpha(B_1)$ and for n big enough the following holds

$$(4.17) \quad \|u_n\|_{L^\infty}^2 > \frac{1}{2\pi\alpha} \|\nabla u_n\|_{L^2}^2 \log(n^{1/4} + n^{1/4} N_\alpha(u_n)).$$

Let u_n be the radially symmetric function defined by

$$u_n(r) = 1 - e^n r^\alpha \text{ if } r \in [0, a_n], \text{ and } u_n(r) = (1 - e^n a_n^\alpha) \frac{\log r}{\log a_n} \text{ if } r \in [a_n, 1],$$

where a_n is chosen such that $a_n^\alpha := x_n$ is the unique solution in $(0, 1)$ of the equation $x = \frac{1 - e^n x}{e^n |\log(x)|}$. Notice indeed, that the function $h(x) = e^n(x + x|\log(x)|)$ is increasing on $(0, 1)$. Hence, we see easily that

$$(4.18) \quad \frac{e^{-n}}{n \log(n)} \leq x_n \leq \frac{e^{-n}}{n}.$$

Obviously, this construction is inspired from the minimizer of the variational problem with obstacle described in Section 3 where we have chosen $D_n = e^n$. Hence, according to (3.13) and (3.14), we have

$$\|\nabla u_n\|_{L^2}^2 = 2\pi\alpha \frac{1/2 - \log(x_n)}{(1 - \log(x_n))^2}$$

and

$$\|u_n\|_{\dot{C}^\alpha} = \frac{1}{x_n(1 - \log(x_n))}.$$

Now to prove (4.17), it is sufficient to prove that for n big enough we have

$$h_n := \frac{\frac{1}{2} - \log(x_n)}{(1 - \log(x_n))^2} \log \left[n^{1/4} + \frac{n^{1/4}}{x_n \sqrt{2\pi\alpha(1/2 - \log(x_n))}} \right] < 1.$$

Note that using (4.18), we have

$$h_n < \frac{\frac{1}{2} + n + \log(n) + \log \log n}{(1 + \log(n) + n)^2} \log \left[n^{1/4} + \frac{n^{1/4} e^n n \log n}{\sqrt{2\pi\alpha n}} \right]$$

Hence $h_n < 1 - \frac{1}{4} \frac{\log n}{n} + o(\frac{\log n}{n})$ which is strictly less than 1 if n is sufficiently large. The proof of (4.17) is achieved. \blacksquare

5. CASE OF THE WHOLE SPACE

Theorems 1.2 and 1.3 were stated in the ball of radius one. If the function u is supported in a bigger ball $B_R = B(0, R)$ then a simple scaling argument shows that

$$\|u\|_{L^\infty(B_R)}^2 \leq \frac{1}{2\pi\alpha} \|\nabla u\|_{L^2(B_R)}^2 \log \left[e^3 + C_0 R^\alpha N_\alpha(u) \sqrt{\log(2e + R^\alpha N_\alpha(u))} \right].$$

Remark 5.1. Using symmetric non-increasing rearrangement of functions, the results of Theorem 1.2 and Theorem 1.3 remain true for any bounded and regular domain Ω of \mathbb{R}^2 . Precisely, if $f \in H_0^1(\Omega) \cap \dot{C}^\alpha(\Omega)$ then, its corresponding symmetric non-increasing function, usually denoted by f^* , is in $f^* \in H_0^1(B_R) \cap \dot{C}^\alpha(B_R)$, where $R = \sqrt{\frac{|\Omega|}{2\pi}}$. We refer to [12], [2] for the definition, the properties and applications of rearrangements of functions. Applying Theorem 1.2 and Theorem 1.3 results to f^* and using the fact that

$$\begin{aligned} \|f^*\|_{L^\infty} &= \|f\|_{L^\infty} \\ \|\nabla f^*\|_{L^2} &\leq \|\nabla f\|_{L^2}, \quad \|f^*\|_{\dot{C}^\alpha} \leq \|f\|_{\dot{C}^\alpha} \end{aligned}$$

we get the result for general domain.

Note that this estimate can not be extended to the whole space since R^α diverges. Instead, we have the following result concerning the whole space.

Corollary 5.2. *Let $\alpha \in]0, 1[$. For any $\lambda > \frac{1}{2\pi\alpha}$ and any $0 < \mu \leq 1$, a constant $C_\lambda > 0$ exists such that, for any function $u \in H^1(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2)$*

$$(5.19) \quad \|u\|_{L^\infty}^2 \leq \lambda (\|\nabla u\|_{L^2}^2 + \mu^2 \|u\|_{L^2}^2) \log \left(C_\lambda + \frac{8^\alpha \mu^{-\alpha} \|u\|_{C^\alpha}}{\sqrt{\|\nabla u\|_{L^2}^2 + \mu^2 \|u\|_{L^2}^2}} \right),$$

Proof. Let u be a function in $H^1(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2)$, $\lambda > \frac{1}{2\pi\alpha}$ and $0 < \mu \leq 1$. Fix a radially symmetric function φ in $C_0^\infty(B_4)$ satisfying $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ for r near 0, $|\partial_r \varphi| \leq 1$ and $|\Delta \varphi| \leq 1$. Define φ_μ by $\varphi_\mu(x) = \varphi(\frac{\mu}{2}|x|)$.

Without loss of generality, we can assume that $\|u\|_{L^\infty} = |u(0)|$. Note that in particular one has

$$\|\varphi_\mu u\|_{\dot{C}^\alpha} \leq \|u\|_{C^\alpha}$$

$$\|\nabla(\varphi_\mu u)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \frac{\mu^2}{4}\|u\|_{L^2}^2 + 2 \int_{\mathbb{R}^2} \varphi_\mu u \nabla \varphi_\mu \nabla u dx.$$

Integrating by parts,

$$2 \int_{\mathbb{R}^2} \varphi_\mu u \nabla \varphi_\mu \nabla u dx = -\frac{1}{2} \int_{\mathbb{R}^2} \Delta \varphi_\mu^2 u^2 dx = -\frac{\mu^2}{8} \int_{\mathbb{R}^2} \Delta \varphi^2\left(\frac{\mu}{2}x\right) u^2 dx.$$

Hence,

$$\|\nabla(\varphi_\mu u)\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \mu^2 \|u\|_{L^2}^2.$$

Applying the result of Theorem 1.3 and using the fact that for any constant $C > 0$, the function $x \rightarrow x^2 \log(C_\lambda + \frac{C}{x})$ is increasing, the proof of Corollary 5.2 is achieved. \blacksquare

We also have the following result

Corollary 5.3. *Let $\alpha \in]0, 1[$. For any $\lambda > \frac{1}{2\pi\alpha}$, a constant $C_\lambda > 0$ exists such that, for any function $u \in H^1(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2)$*

$$(5.20) \quad \|u\|_{L^\infty} \leq \|u\|_{L^2} + \|\nabla u\|_{L^2} \sqrt{\lambda \log\left(e + C_\lambda \frac{\|u\|_{C^\alpha}}{\|\nabla u\|_{L^2}}\right)}.$$

For the proof of Corollary 5.3, we take the Littlewood-Paley decomposition of u , $u = \tilde{\Delta}_0 u + v$ where $v = \sum_{j=1}^{\infty} \Delta_j u$. Hence $\|v\|_{L^2} \leq C \|\nabla v\|_{L^2}$ and $\|v\|_{C^\alpha} \leq \|u\|_{C^\alpha}$. So

$$\|u\|_{L^\infty} \leq \|\tilde{\Delta}_0 u\|_{L^\infty} + \|v\|_{L^\infty}.$$

Then, we apply Corollary 5.2 to v with λ' and μ' such that $\lambda'(1 + C^2 \mu'^2) < \lambda$. \blacksquare

Of course, we have similar inequalities for the Log Log inequality (1.2) in \mathbb{R}^2 with the sharp constant $\frac{1}{2\pi\alpha}$.

6. APPLICATION TO THE WAVE EQUATION

Corollary 5.2 is useful in studying 2D-nonlinear wave equations with exponential nonlinearities, and the constant $\frac{1}{2\pi\alpha}$ is crucial for local wellposedness results (see [7] for further discussion). In particular from Corollary 5.2 we can derive a Moser-Trudinger type inequality for the solution of the linear Klein-Gordon. Precisely, let $(f, g) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that $\|f\|_{H^1}^2 + \|g\|_{L^2}^2 \leq 1$. Denote by v the solution of the 2D linear Klein-Gordon equation

$$\begin{aligned} \partial_t^2 v - \Delta v + v &= 0 \\ v(0, \cdot) &= f, \quad \partial_t v(0, \cdot) = g. \end{aligned}$$

Since the energy $\|\nabla v(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \|v(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_t v(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2$ is conserved, $v(t, \cdot)$ remains in the unit ball of H^1 uniformly in time. So according to (1.1) we have

$$\sup_{t \in \mathbb{R}} \int_{\mathbb{R}^2} (e^{4\pi v(t, x)^2} - 1) dx \leq C$$

which means that $\exp(4\pi v^2(t, \cdot)) - 1 \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^2))$. To solve the 2D linear Klein-Gordon equation with an exponential nonlinearity, we would like that $\exp(4\pi v^2(t, \cdot)) - 1 \in L^1_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$. This is the object of the following result.

Proposition 6.1. *For any $T > 0$, a non-negative constant C_T exists such that*

$$\int_0^T \|\exp(4\pi v^2(t, \cdot)) - 1\|_{L^2(\mathbb{R}^2)} dt \leq C_T.$$

Proof. For any $\mu > 0$, denote by

$$E_\mu(t) := \|\nabla v(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + \mu^2 \|v(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2.$$

Recall that since $v \in \mathcal{C}(\mathbb{R}, H^1) \cap \mathcal{C}^1(\mathbb{R}, L^2)$, $E_\mu(t)$ is a continuous function of t . The energy conservation satisfied by v shows that

$$\|\partial_t v(t, \cdot)\|_{L^2(\mathbb{R}^2)}^2 + E_1(t) = E_1(0) + \|g\|_{L^2}^2 \leq 1.$$

Now, fix $\mu < 1$ and $T > 0$. There exists a time $\tau = \tau(\mu, T)$ such that

$$\sup_{t \in [0, T]} E_\mu(t) = E_\mu(\tau) < 1.$$

For almost every t we have

$$(6.21) \int_{\mathbb{R}^2} (\exp(4\pi v^2(t, x)) - 1)^2 dx \leq \|\exp(4\pi v^2(t, \cdot)) - 1\|_{L^1} \exp(4\pi \|v(t, \cdot)\|_{L^\infty}^2).$$

Note that, thanks to conservation of the energy and Moser-Trudinger inequality, the first factor in the above inequality is uniformly bounded. On the other hand, choosing $\alpha = \frac{1}{4}$ in (5.19) we obtain, for any $\lambda > \frac{2}{\pi}$

$$\exp(2\pi \|v(t, \cdot)\|_{L^\infty}^2) \leq \left(e + \frac{\|v(t, \cdot)\|_{\mathcal{C}^{1/4}}}{E_\mu(\tau)^{1/2}}\right)^{2\pi\lambda E_\mu(\tau)}.$$

Since $E_\mu(\tau) < 1$, one can choose $\lambda > \frac{2}{\pi}$ such that $\beta := 2\pi\lambda E_\mu(\tau) < 4$. Hence, we have

$$\begin{aligned} \int_0^T \exp(2\pi \|v(t, \cdot)\|_{L^\infty}^2) dt &\leq C \int_0^T \left(e + \frac{\|v(t, \cdot)\|_{\mathcal{C}^{1/4}}}{E_\mu(\tau)^{1/2}}\right)^\beta dt \\ &\leq CT^{1-\frac{\beta}{4}} \int_0^T \left(e + \frac{\|v(t, \cdot)\|_{\mathcal{C}^{1/4}}}{E_\mu(\tau)^{1/2}}\right)^4 dt. \end{aligned}$$

Now, thanks to the so-called Strichartz estimates (see [6]), we have $v \in L^4(\mathbb{R}, \mathcal{C}^{1/4}(\mathbb{R}^2))$ and therefore Proposition 6.1 is proved. \blacksquare

Remark 6.2. Recall that in [3], a similar result was proved in a particular setting, namely, $f = 0$ and g is radially symmetric with compact support.

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