GLOBAL EXISTENCE OF WEAK SOLUTIONS TO THE FENE DUMBBELL MODEL OF POLYMERIC FLOWS

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Dedicated to those who died for their dignity

Key words: Nonlinear Fokker-Planck equations, Navier-Stokes equations, FENE model, micro-macro interactions, defect measure, global existence.
AMS subject classification: 35Q30, 82C31, 76A05.

Abstract

Systems coupling fluids and polymers are of great interest in many branches of sciences. One of the most classical models to describe them is the FENE (Finite Extensible Nonlinear Elastic) dumbbell model. We prove global existence of weak solutions to the FENE dumbbell model of polymeric flows. The main difficulty is the passage to the limit in a nonlinear term that has no obvious compactness properties. The proof uses many weak convergence techniques. In particular it is based on the control of the propagation of strong convergence of some well chosen quantity by studying a transport equation for its defect measure. In addition, this quantity controls a rescaled defect measure of the gradient of the velocity.

1. INTRODUCTION

Systems coupling fluids and polymers are of great interest in many branches of applied physics, chemistry and biology. They are of course used in many industrial and medical applications such as food processing, blood flows... Although a polymer molecule may be a very complicated object, there are simple theories to model it. One of these models is the FENE (Finite Extensible Nonlinear Elastic) dumbbell model. In this model, a polymer is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring, which can be represented by a vector R (see Bird, Curtis, Amstrong and Hassager [11, 12], Doi and Edwards [27] for some physical introduction to the model, Ottinger [74] for a more mathematical treatment (in particular the stochastic point of view) and Owens and Phillips [76] for the computational aspect). In the FENE model (1), the polymer elongation R cannot exceed a limit R_0. This yields some nice mathematical problems near the boundary, namely when $|R|$ approaches R_0. At the level of the polymeric liquid, we get a system coupling the Navier-Stokes equation for the fluid velocity with a Fokker-Planck equation describing the evolution of the polymer density. This density depends on t, x and R. The coupling comes from an extra stress term in the fluid equation due to the microscopic effect of the polymer molecules. This is the micro-macro interaction. There is also a drift term in the Fokker-Planck equation that depends on the spatial gradient of the velocity. This is a macro-micro term. The coupling satisfies the fact that the free-energy dissipates, which is important from the physical point of view. Mathematically, this is also important in order to get uniform bounds and hence prove global existence of weak solutions.

The system obtained attempts to describe the behavior of this complex mixture of polymer molecules and fluid, and as such, it presents numerous challenges, simultaneously at the level of their derivation [22], the level of their numerical simulation [76, 46], the level of their physical properties (rheology) and that of their mathematical treatment (see references below). In this paper we concentrate on the mathematical treatment and more precisely the global existence of weak solutions to the FENE dumbbell model (1). These solutions are the generalization of the Leray weak solutions [55, 54] of the incompressible Navier-Stokes system to the FENE model. The method of proof has some similarities with the proof of global existence of renormalized solutions to the Boltzmann equation [25] and to the proof of global existence of weak solutions to the compressible Navier-Stokes system [61, 31].

An approximate closure of the linear Fokker-Planck equation reduces the description to a closed viscoelastic equation for the extra stress tensor. This leads to well-known non-Newtonian fluid models such as the Oldroyd B model or the FENE-P model (see for instance [28, 22]). These models have been studied extensively. Guillopé and Saut [38, 39] proved the existence of local strong solutions, Fernández-Cara, Guillén and Ortega [33], [32] and [34] proved local well posedness in Sobolev spaces. In Chemin and Masmoudi [14], local and global well-posedness in critical Besov spaces were given. For global existence of weak solutions, we refer to Lions and Masmoudi [63]. We also mention Lin, Liu and Zhang [57] where a formulation based on the deformation tensor is used to study the Oldroyd-B model. Global existence for small data was proved in [53, 51] and some non-blow-up criteria were given in [52, 72].

At the micro-macro level, there are also several contributions. Indeed, from the mathematical point of view, the FENE model and some simplifications of it were studied by several authors. In particular Renardy [77] proved the local existence in Sobolev spaces, where the potential $\mathcal{U}$ is given by $\mathcal{U}(R) = (1 - |R|^2)^{1-\sigma}$ for some $\sigma > 1$. W. E, Li and Zhang [29] proved local existence when R is taken in the whole space and under some growth condition on the potential. Also, Jourdain, Lelievre and Le Bris [45] proved local existence in the case $b = 2k > 6$ for a Couette flow by solving a stochastic differential equation (see also [43] for the use of entropy inequality methods to prove exponential convergence to equilibrium). Zhang and Zhang [82] proved local well-posedness for the FENE model when $b > 76$. Local well-posedness was also proved in [68] when $b = 2k > 0$ (see also [48]). One of the main ingredients of [68] is the use of Hardy type inequalities to control the extra stress tensor by the H^1 norm in R which comes from the diffusion in R. In particular no regularity in R is necessary for the initial data. Moreover, Lin, Liu and Zhang [58] proved global existence near equilibrium under some restrictions on the potential (see also the related work [51]). Recently many other works have dealt with different aspects of the system. In particular the problem in a thin film was considered in [15], the problem of the long time behavior was considered in [79, 42, 4], the problem of global existence of smooth solutions in 2D for some simplified models (when there is a bound on τ in L^∞) was considered in [18, 59, 19, 72], the problem of non-blow-up was considered in [52], the stationary problem was considered in [15, 16], and the study of the boundary condition on ∂B was considered in [67, 66].

More related to this paper, the construction of global weak solutions for simplified (or regularized) models was considered in [6, 7, 8, 83, 79]. These papers dealt with the case when the system is regularized by introducing a smoothing operator in certain terms or by a microscopic cut-off in the drag term in the Fokker-Planck equation.

The existence and equilibration of global weak solutions to FENE-type models with center-of-mass diffusion was established in [9], and for Hookean-type models with center-of-mass diffusion in [10]. The case of the co-rotational model was considered in [65]. The co-rotational model preserves some of the compactness difficulties of the full model. It also allows one to get more integrability on ψ, which makes the compactness analysis simpler.

We end this introduction by mentioning other micro-macro models. Indeed, a principle based on an energy dissipation balance was proposed in [17], where the regularity of nonlinear Fokker-Planck systems coupled with Stokes equations in 3D was also proved. In particular the Doi model (or Rigid model) was considered in [75] where the linear Fokker-Planck system is coupled with a stationary Stokes equations. The nonlinear Fokker-Planck equation driven by a time averaged Navier-Stokes system in 2D was studied in [18] (see also [19]). Recently, there have been several review papers dealing with different mathematical aspects of these models [78, 56, 50]. In particular we refer to [50] for an exhaustive list of references dealing with the numerical point of view.

1.1. **The FENE model.** A macro-molecule is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring, which can be modeled by a vector R (see [12]). Before writing our main system (1), let us discuss the main physical assumptions that lead to it:

- The polymer molecules are described by their density at each time t, position x and elongation R. This is a *kinetic description* of the polymer molecules.
- The inertia of the polymer molecules is neglected and hence the sum of the forces applied on each polymer vanishes. We refer to [23] where inertia is taken into account and where the limit m going to zero it studied, m being the mass of the beads.
- The polymer solution is supposed to be dilute and hence the interaction between different polymers is neglected. This is why we get a linear Fokker-Planck equation. Let us also mention that there are models for polymer melts such as the reptation model (see for instance [74]).
- The polymer molecule is described by one vector R in $B(0, R_0)$. Let us mention that there are models where each polymer molecule is described by one vector R such that $|R| = 1$ (the rigid case, see [19]) or by K vectors R_i, $1 \leq i \leq K$ (the bead-spring chain model, see [9]). Usually the difference between these models comes from the length of the polymer molecules as well as their electric properties.
- In the Fokker-Planck equation an upper-convected derivative is used. This can be seen as the most physical one. Other used derivatives are the lower-convected and the co-rotational ones (see [11, 12]). The co-rotational one has the mathematical advantage that one has better a priori estimates (see [65]).
- We neglect the diffusion in x in the Fokker-Planck equation, namely the center-of-mass diffusion. Indeed, this diffusion is much smaller than the diffusion in R. Actually, its presence makes the mathematical problem simpler.

Under these assumptions, the micro-macro approach consists in writing a coupled multi-scale system :

$$(1) \qquad \begin{cases} \partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = \mathrm{div}\tau, \quad \mathrm{div} u = 0, \\ \\ \partial_t \psi + u.\nabla\psi = \mathrm{div}_R\Big[-\nabla u \, R\psi + \beta\nabla\psi + \nabla\mathcal{U}\psi\Big] \\ \\ \tau_{ij}(\psi) = \int_B (R_i \, \nabla_j \mathcal{U})\psi(t,x,R)dR \qquad (\nabla\mathcal{U}\psi + \beta\nabla\psi).\mathbf{n} = 0 \text{ on } \partial B(0,R_0). \end{cases}$$

In (1), $\psi(t, x, R)$ denotes the distribution function for the internal configuration and $F(R) = \nabla_R\mathcal{U}$ is the spring force, which derives from a potential $\mathcal{U}$ and $\mathcal{U}(R) = -k\log(1 - |R|^2/|R_0|^2)$ for some constant $k > 0$. Moreover, $\tau(\psi)$ is the extra stress tensor coming from the effect of the polymers on the fluid. It is given by the Kramers expression. Besides, β is related to the temperature of the system and $\nu > 0$ is the viscosity of the fluid.

Here, R is in a bounded open ball $B(0, R_0)$ of radius R_0; which means that the extensibility of the polymer molecules is finite and $x \in \Omega$ where Ω is a smooth bounded domain of $\mathbb{R}^D$ where $D \geq 2$ or $\Omega = \mathbb{T}^D$ or $\Omega = \mathbb{R}^D$. In the case when Ω has a boundary, we add the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. We also have to add a boundary condition to insure the conservation of ψ, namely $(-\nabla uR\psi + \nabla\mathcal{U}\psi + \beta\nabla\psi).\mathbf{n} = 0$ on $\partial B(0, R_0)$ where $\mathbf{n} = \frac{R}{R_0}$ is the exterior normal vector to $B(0, R_0)$. This condition is actually implied by

$$(2) \qquad (\nabla\mathcal{U}\psi + \beta\nabla\psi).\mathbf{n} = 0 \quad \text{on} \quad \partial B(0, R_0)$$

as we will see later. The reason is that (2) combined with the fact that $\sqrt{\frac{\psi}{\psi_\infty}} \in \mathcal{H}^1_k$ (see the definition in (14)) implies necessarily that ψ has to vanish on $\partial B(0, R_0)$. The boundary condition on $\partial B(0, R_0)$ insures the conservation of the polymer density and should be understood in the weak sense, namely for any function $\phi(R) \in C^1(\overline{B})$, we have

$$\partial_t \int_B \phi\psi dR + u.\nabla_x \int_B \phi\psi dR = - \int_B \nabla_R \phi. \Big[- \nabla u\, R\, \psi + \beta \nabla\psi + \nabla\mathcal{U}\psi \Big] dR. \tag{3}$$

In particular, if we take $\phi(R) = 1$, we deduce that $\int_B \psi dR$ is transported by u, namely $\partial_t \int_B \psi dR + u.\nabla_x \int_B \psi dR = 0$. In the sequel, we will take $\beta = 1$ and $R_0 = 1$.

When performing numerical simulation on the FENE model, it is usually better to think of the distribution function ψ as the density of a random variable R, which solves (see [74])

$$dR + u.\nabla R dt = (\nabla u R - \nabla_R \mathcal{U}(R))dt + \sqrt{2} dW_t \tag{4}$$

where the stochastic process W_t is the standard Brownian motion in $\mathbb{R}^D$ and the additional stress tensor is given by the following expectation $\tau_{ij} = \mathbb{E}(R_i\, \nabla_j \mathcal{U})$. Of course, we may need to add a boundary condition for (4) if R reaches the boundary of B. This is done by requiring that R stays in $\overline{B}$ (see [44]). Using this stochastic formulation has the advantage of replacing the second equation of (2.1), which has $2D + 1$ variables by (4). Of course one has to solve (4) several times to get the expectation τ, which is the only information needed in the fluid equation. This strategy was used for instance by Keunings [47] (see also [35]) and by Öttinger [74] (see also [37]).

We would like to explain (at least formally for now) why (2) implies that $\psi = 0$ on $\partial B(0, R_0)$. Indeed, if we assume that $\psi(R)$ behaves like $c(\frac{R}{|R|}) + b(\frac{R}{|R|})(1 - |R|)^\alpha + o((1 - |R|)^\alpha)$ for some $\alpha > 0$ when R approaches $\partial B(0, 1)$ then, (2) becomes

$$\lim_{|R| \to 1} \frac{2k(c + b(1 - |R|)^\alpha)}{1 - |R|^2} - b\alpha(1 - |R|)^{\alpha - 1} = 0. \tag{5}$$

Hence, necessarily, $c = 0$ and $\alpha = k$ or $\alpha > 1$. Of course, to make this argument rigorous, one has to put ψ in some function space that comes from our notion of weak solution. As we can deduce from the first statement in Lemma 3.2, the fact that $\sqrt{\frac{\psi}{\psi_\infty}} \in L^2(\mathbb{R}_+; L^2(\Omega; \dot{H}^1_R(\psi_\infty dR)))$ and $\psi \in L^\infty(\mathbb{R}_+; L^1(\Omega \times B))$ implies that ψ vanishes when $|R|$ goes to 1 (see also Corollary 3.5 for a more precise behavior of ψ and subsection B.1 for more about the boundary condition).

2. Statement of the results

This paper is devoted to the proof of global existence of free-energy weak solutions to the FENE model. The main difficulty of the construction is the passage to the limit in the nonlinear term $\nabla u^n \psi^n$ when looking at regularized systems. Indeed, we only have a uniform bound on ∇u^n in $L^2((0, T) \times \Omega)$ and ψ^n in $L^\infty((0, T) \times \Omega; L^1(B))$ for all $T > 0$ and so assuming that u^n and ψ^n converge weakly to u and ψ, it is not clear how to deduce that $\nabla u^n \psi^n$ converges weakly to $\nabla u \psi$.

Before mentioning our main result, let us recall that the construction of global weak solutions to simplified (or regularized) models was considered in [6, 7, 8, 79, 65, 83]. In particular in [6], Barrett, Schwab and Süli prove global existence of weak solution with an $x-$mollified gradient in the Fokker-Planck equation and an $x-$mollified probability density function ψ in the Kramers expression. In [7], the velocity field was mollified (with an anisotropic Friedrichs mollifier) in certain terms in the model. In [8], a “microscopic” cut-off function was inserted in the drag term in the Fokker-Planck equation.

Recently, Barrett and Süli [9] extended their results to the case of bead-spring chain models where each polymer is described by K springs R^i, $1 \leq i \leq K$, with center-of-mass diffusion (diffusion in the x variable). Mathematically, the diffusion in the x variable yields a bound on $\nabla_x\sqrt{\psi}$ in $L^2((0,T)\times\Omega\times B)$. But, since the space of functions ψ such that $\psi\geq 0$ and $\nabla_x\sqrt{\psi}$ is in $L^2((0,T)\times\Omega\times B)$ is not a linear space, the Lions-Aubin lemma does not apply in the passage to the limit in the product $\nabla u^n\psi^n$. The authors of [9] used a more general compactness result due to Dubinskii to pass to the limit. Let us note that this extra diffusion term is physically justifiable but it is much smaller than the diffusion in the R variable and this is why we did not include it here.

Also, in [65], the co-rotational model was considered. It allowed us to get additional a priori estimates on ψ^n, namely one can show that ψ^n belongs to all L^p spaces. An argument based on propagation of compactness similar to the one used in [63] allowed us to prove the existence of global weak solutions.

Here, we consider the noncorotational FENE model without center-of-mass diffusion. The system (1) has to be complemented with initial data $u(t=0)=u_0$ and $\psi(t=0)=\psi_0$.

Notice that $(u=0,\psi_\infty)$, where ψ_∞ is given by

$$\psi_\infty(R)=\frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R')}dR'}, \tag{6}$$

defines a stationary solution of (1). To state our result, we first impose some conditions on the initial data. We take $u_0(x)\in L^2(\Omega)$, $\mathrm{div}(u_0)=0$, $u_0.n=0$ on $\partial\Omega$ and $\psi_0(x,R)\geq 0$ such that $\rho_0(x)=\int\psi_0 dR\in L^\infty(\Omega)$. Here $\rho_0(x)$ is the initial density of polymer molecules at the position x. We also assume the following entropy bound : $\frac{\psi_0}{\rho_0\psi_\infty}\in L\log L(\Omega\times B, dx\rho_0(x)\psi_\infty dR))$ where

$$\|\frac{\psi_0}{\rho_0\psi_\infty}\|_{L\log L(\Omega\times B,\rho_0(x)\psi_\infty dRdx)}=\int\int_{\Omega\times B}(\frac{\psi_0}{\rho_0\psi_\infty}\log\frac{\psi_0}{\rho_0\psi_\infty}-\frac{\psi_0}{\rho_0\psi_\infty}+1)\rho_0(x)\psi_\infty dRdx. \tag{7}$$

Finally, we also assume the following $L^{1/2}_x L\log^2 L$ bound, that we will call "$\log^2$" bound:

$$\int_\Omega\frac{\int_B\psi_0\log^2\frac{\psi_0}{\rho_0\psi_\infty}\,dR}{1+\left[\int_B\psi_0\log^2\frac{\psi_0}{\rho_0\psi_\infty}\,dR\right]^{1/2}}dx<\infty. \tag{8}$$

Another way of writing (8) is to say that $\int_\Omega\phi_1(\int_B\psi_0\log^2\frac{\psi_0}{\rho_0\psi_\infty}\,dR)<\infty$ where $\phi_1(s)=\min(\sqrt{s},s)$. Notice that interpolating (8) with the L^∞ bound on ρ_0, we can deduce the $L\log L$ bound (7).

2.1. Notion of weak solutions.

Let us now define precisely the notion of weak solution (or just solution) (u,ψ) to (1). We require that $u\in L^2(\mathbb{R}_+;\dot{H}^1(\Omega))$ ($H^1_0(\Omega)$ in the case of a bounded domain with a Dirichlet boundary condition), $u\in L^\infty(\mathbb{R}_+;L^2(\Omega))\cap C([0,\infty);L^2_w(\Omega))$ where $L^2_w(\Omega)$ is the L^2 space endowed with its weak topology. We also require that $\psi\geq 0$, $\frac{\psi}{\rho\psi_\infty}\in L^\infty(\mathbb{R}_+;L\log L(\Omega\times B, dx\rho(t,x)\psi_\infty dR)))$ where $\rho(t,x)=\int_B\psi dR\in L^\infty(\mathbb{R}_+\times\Omega)$. Moreover, $\psi\in C([0,\infty);L^1_w(K\times B))$ for any bounded subset K of Ω where $L^1_w(K\times B)$ is the space $L^1(K\times B)$ endowed with its weak topology and $\sqrt{\frac{\psi}{\psi_\infty}}\in L^2(\mathbb{R}_+;L^2(\Omega;\dot{H}^1_R(\psi_\infty dR)))$. One can then deduce from (35) that $\tau\in L^2(\mathbb{R}_+\times\Omega)$. We also require that the free energy bound (50) holds with an inequality $\leq$ instead of the equality. Finally, (1) is understood in the weak sense, namely for each $v\in C^\infty([0,\infty)\times\Omega;\mathbb{R}^D)$ compactly supported in $[0,\infty)\times\Omega$ and such that $\mathrm{div}(v)=0$, we have

$$\int_0^\infty\int_\Omega u.(\partial_t v+u.\nabla v)-\nu\nabla u.\nabla v\,dxdt+\int_\Omega v(t=0).u_0\,dx=\int_0^\infty\int_\Omega\tau:\nabla v\,dxdt, \tag{9}$$

and for each $\phi\in C^\infty([0,\infty)\times\overline{\Omega}\times\overline{B};\mathbb{R})$ compactly supported in $[0,\infty)\times\overline{\Omega}\times\overline{B}$, we have

$$\begin{aligned}(10)\qquad &\int_0^\infty \int_\Omega \int_B \psi(\partial_t\phi + u.\nabla\phi)\, dRdxdt + \int_\Omega\int_B \phi(t=0)\psi_0\, dRdx \\ &= \int_0^\infty \int_\Omega\int_B (-\nabla u R\psi + \psi_\infty \nabla_R \frac{\psi}{\psi_\infty}).\nabla_R\phi\, dRdxdt.\end{aligned}$$

Of course in view of classical density results (see Temam [81]), we can relax the condition on the test function v. The same holds for (10) (see Appendix 2).

2.2. **Main result.** Now, we state our main result.

Theorem 2.1. *Take a divergence-free field $u_0(x) \in L^2(\Omega)$ and $\psi_0(x,R) \geq 0$ such that $\rho_0(x) = \int \psi_0 dR \in L^\infty(\Omega)$ and (7) and (8) hold. Then, (1) has a global weak solution (u,ψ) in the sense of Subsection 2.1, namely $u \in L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot H^1)$, $\psi \in C([0,\infty); L^1_w(K\times B))$ for any bounded subset K of Ω, $\frac{\psi}{\rho\psi_\infty} \in L^\infty(\mathbb{R}_+; L\log L(\Omega\times B, dx\rho(x)\psi_\infty dR)))$ where $\rho(x) = \int_B \psi dR$ and $\sqrt{\frac{\psi}{\psi_\infty}} \in L^2(\mathbb{R}_+; L^2(\Omega; \dot H^1_R(\psi_\infty dR)))$ and (50) holds with an inequality $\leq$ instead of the equality. Moreover, (57) holds (with Ω replaced by any compact K of Ω in the whole space case).*

Remark 2.2. *1) Of course u and ψ have also some time regularity in some negative Sobolev spaces in x and R. This allows to give a sense to the initial data (see [63] for more details) and one can then prove the continuity in time of u and ψ in $L^2_w(\Omega)$ and in $L^1_w(K\times B)$. In the definition of weak solutions, we only required that $\psi \in C([0,\infty); L^1_w(K\times B))$ for any bounded subset K of Ω. Actually, using an argument similar to the one in Proposition 4.3 of [31] and Corollary B.7, we can prove that the solution we are constructing satisfies $\psi \in C([0,\infty); L^1(K\times B))$ (see [70] for more details).*

2) By $f \in L\log L(\Omega\times B, dx\rho(x)\psi_\infty dR)$ we mean that $\int_\Omega\int_B (f\log f - f + 1)\rho(x)\psi_\infty dRdx < \infty$. Notice that (7) does not really define a norm. One can of course define a norm using Orlicz spaces. However, we do not need to do that here.

3) If the domain Ω has finite measure (bounded domain or torus) then, the extra bound (8) reduces to $\int_\Omega \left[\int_B \psi_0 \log^2 \frac{\psi_0}{\rho_0\psi_\infty} dR\right]^{1/2} dx < \infty$. This extra bound (8) on the initial data allows us to prove the extra bound (57) on the solution. This is useful for two reasons. The first one is that it yields some sort of equi-integrability of the extra stress tensor (see the proof of Lemma 5.1). Actually, one can still prove Lemma 5.1 without the extra bound (57). Indeed, using de la Vallée-Poussin Theorem, we can deduce from (7) that there exists an increasing function Φ such that $\frac{\Phi(t)}{t}$ goes to infinity when t goes to infinity and such that $\Phi(\frac{\psi_0}{\rho_0\psi_\infty}\log\frac{\psi_0}{\rho_0\psi_\infty} - \frac{\psi_0}{\rho_0\psi_\infty} + 1)$ is in $L^1(\Omega\times B, dx\rho_0(x)\psi_\infty dR)$. One can then prove an estimate similar to (57) by replacing $\log^2$ by $\Phi(\log)$. We do not detail this here. The second reason is that the control on $\nabla_R g$ we get from (59) is crucial in the Gronwall argument in subsection 5.5 and it seems that this argument does not hold if we just replace $\log^2$ by $\Phi(\log)$.

Of course this is a very mild extra assumption. Actually, it is related to the fact that the Fokker-Planck equation (without the coupling) satisfies much stronger a priori estimates. Adding the term $\mathrm{div}_R(-\nabla u\, R\psi)$ destroys these estimates. However, one can still get a very mild estimate in the R variable that is crucial in the rest of the proof. Besides, even if one does not assume (8) initially, using the regularizing effect of the diffusion in the R variable we deduce that (8) holds for $t > 0$. Moreover, due to the local character of the weak compactness proof, the assumption (57) can be weakened by assuming the bound to hold locally in space, namely $\int_K \left[\int_B \psi_0\log^2\frac{\psi_0}{\rho_0\psi_\infty} dR\right]^{1/2} dx < \infty$ for any compact set K of Ω.

4) For simplicity of the presentation, the proof will be given in the case when $\rho_0(x)$ is constant, equal to 1, and Ω has finite measure. We will also indicate the necessary changes required in the general case.

The paper is organized as follows. In the next section, we give some preliminaries where in particular we prove some Hardy type inequalities. In Section 4, we derive some a priori estimates for the full model (1). In particular we recall the free energy estimate as well as a new "log^2" a priori estimate, which is very natural and comes from the initial bound (8). In Section 5, we prove the main theorem 2.1. As is classical when proving global existence of weak solutions, the main part of the proof is the proof of the weak compactness of a sequence of global solutions satisfying the a priori estimates and we will only detail this part of the proof. In Section 6, we present one way of approximating the system. In Section 7 we present some concluding remarks and open problems. In Appendix A, we recall few facts about DiPerna-Lions flows and in Appendix B, we prove few results about the linear operator $\mathcal{L}$.

3. Preliminaries

3.1. **Notations and conventions.** In this article ∇u will denote the matrix $(\nabla u)_{i,j} = \frac{\partial u_i}{\partial x_j}$. Many other authors in the non-Newtonian fluid mechanics literature use the alternative convention. The product of two matrices A and B will be denoted by AB and the identity matrix will be denoted by Id. For any two matrices A and B, $A : B$ will denote the trace of the matrix AB, namely $A : B = \mathrm{tr}(AB) = \sum_{i,j=1}^D A_{ij}B_{ij} = A_{ij}B_{ij}$. Here and below, we use the summation convention over repeated indices and hence $\sum_{i,j=1}^D$ will be omitted in many sums. For any two vectors R and Q, $R \otimes Q$ will denote the matrix whose entries are given by $(R\otimes Q)_{ij} = R_iQ_j$. In particular the definition of τ in (1) can be written as $\tau(\psi) = \int_B (R \otimes \nabla\mathcal{U})\psi(t, x, R)dR$. Also, when no confusion can occur we will use ∇ to denote the gradient of a function. In particular ∇u denotes the gradient of u with respect to the x variable and $\nabla\mathcal{U}$ denotes the gradient of $\mathcal{U}$ with respect to the R variable. For functions that depend on x and R, we will denote the gradient with respect to R by $\nabla_R\psi$. For any two distributions u_1 and $u_2 \in \mathcal{D}'$, defined on some domain ω, we will say that $u_1 \le u_2$ provided that $\langle u_2 - u_1, \phi\rangle_{\mathcal{D}'\times C_0^\infty} \ge 0$ for all nonnegative test functions $\phi \in C_0^\infty$ where here $\langle ., .\rangle_{\mathcal{D}'\times C_0^\infty}$ denotes the duality bracket between $\mathcal{D}'$ and C_0^∞. Also, we will denote by $\langle ., .\rangle_{\mathcal{M}\times C}$ the duality bracket between $\mathcal{M}$ and C, namely the space of Radon measures and the space of continuous functions. We will sometimes abuse the notation and write $\int u\phi$ instead of $\langle u, \phi\rangle_{\mathcal{D}'\times C_0^\infty}$. The notation $a \lesssim b$ means that there exists a universal positive constant C such that $a \le Cb$.

In a few formulas, we will omit the variable of integration if no confusion can occur, as in (53) for example. C will denote any positive constant that may change from one line to the other, C_0 will denote a constant that depends on the initial data and C_T a constant that depends on the initial data and the time T.

For $k \in \mathbb{R}$ and $p \ge 1$, we define the weighted Lebesgue and Sobolev spaces L^p_k and $W^{1,p}_k$ and their norms by

$$L^p_k = \{g \quad | \int_0^1 x^k\,|g|^p\,dx\, < \infty\} \quad \text{and} \quad \|g\|^p_{L^p_k} = \int_0^1 x^k\,|g|^p\,dx, \tag{11}$$

$$W^{1,p}_k = \{g \in L^p_k \quad | \int_0^1 x^k\,|g'|^p\,dx\, < \infty\} \quad \text{and} \quad \|g\|^p_{W^{1,p}_k} = \int_0^1 x^k\,[|g|^p + |g'|^p]\,dx. \tag{12}$$

We will mainly be working with the Hilbert spaces L^2_k and $H^1_k = W^{1,2}_k$. Also, we denote $\|g\|^2_{\dot H^1_k} = \int_0^1 x^k\,[g'^2]\,dx$. Finally, notice that $L^p_{k_1} \subset L^p_{k_2}$ if $k_1 \le k_2$.

We also define the higher dimensional versions of these spaces: for $k > 0$ and ψ_∞ defined in (6), namely behaving like $(1-|R|)^k$ when $|R|$ goes to 1,

$$\mathcal{L}^p_k = \{g \quad | \int_B \psi_\infty \, |g|^p \, dR < \infty\} \quad \text{and} \quad \|g\|^p_{\mathcal{L}^p_k} = \int_B \psi_\infty \, |g|^p \, dR, \tag{13}$$

$$\mathcal{W}^{1,p}_k = \{g \in \mathcal{L}^p_k \quad | \int_B \psi_\infty \, |\nabla g|^p \, dx < \infty\} \quad \text{and} \quad \|g\|^p_{\mathcal{W}^{1,p}_k} = \int_B \psi_\infty \, [|g|^p + |\nabla g|^p] \, dR. \tag{14}$$

Also, we denote $\|g\|^p_{\dot{\mathcal{W}}^{1,p}_k} = \int_B \psi_\infty \, |\nabla g|^p \, dR$ and $\mathcal{H}^1_k = \mathcal{W}^{1,2}_k$. We will also use the notation $\mathcal{L}^p_k = L^p(\psi_\infty dR)$.

3.2. **Hardy type inequalities.** The dissipation term in the free energy estimate (50) measures the distance between ψ and the equilibrium ψ_∞. We would like to use that bound to control the extra stress tensor in L^2. This will be done using an inequality of Hardy type [40]. First, we recall the classical weighted Hardy inequality [49, 41].

Lemma 3.1. *i) Assume that $g \in H^1_k$. Then the following hold:*
1) If $k > 1$, then $g \in L^2_{k-2}$ and $\|g\|_{L^2_{k-2}} \le C\|g\|_{H^1_k}$.
2) If $k = 1$, then we have

$$\int_0^1 \frac{g^2}{x(1+|\log x|)^2} dx \le C\|g\|^2_{H^1_k}. \tag{15}$$

3) If $k < 1$, then g has a trace at $x = 0$. Moreover, we have $\|g - g(0)\|_{L^2_{k-2}} \le C\|g\|_{H^1_k}$.
ii) We also have the following L^p version of 1), namely, if $k > p-1 > 0$ and $g \in W^{1,p}_k$ then $g \in L^p_{k-2}$ and $\|g\|_{L^p_{k-2}} \le C\|g\|_{W^{1,p}_k}$.

Proof. The proof of this lemma is classical and can be easily deduced from the change of variables we do in the proof of the next lemma. Let us only mention that in the case $k = 1$, by the change of variable $y = -\log(x)$ and $h(y) = g(x)$, (15) is equivalent to

$$\int_0^\infty \frac{h^2}{(1+y)^2} dy \le C \int_1^\infty [(h'(y)^2 + e^{-2y} h(y)^2] dy \tag{16}$$

which can be easily deduced from the classical Hardy inequality. Notice that there is no condition on the function h at $y = 0$ since we are dividing by $(1+y)^2$. ☐

Lemma 3.2. *i) For $k > 0$; take $\psi \in L^1(0,1)$ such that $\psi \ge 0$ and $\sqrt{\frac{\psi}{x^k}} \in H^1_k$, then $\psi(x)$ goes to 0 when x goes to 0.*
ii) If in addition $k > 1$, then we have

$$\int_0^1 \frac{\psi}{x^2} dx \le C \int_0^1 \left[x^k \left| \left(\sqrt{\frac{\psi}{x^k}} \right)' \right|^2 + \psi \right] dx. \tag{17}$$

iii) For $k > 0$, we have

$$\left(\int_0^1 \frac{\psi}{x} dx \right)^2 \le C \left(\int_0^1 \psi dx \right) \left(\int_0^1 \left[x^k | \left(\sqrt{\frac{\psi}{x^k}} \right)' |^2 + \psi \right] dx \right). \tag{18}$$

For $-1 \le \beta < k \le 1$, we have

$$\left(\int_0^1 \frac{\psi}{x^{1+\beta}} dx \right) \le C \left(\int_0^1 \psi dx \right)^{\frac{1-\beta}{2}} \left(\int_0^1 \left[x^k | \left(\sqrt{\frac{\psi}{x^k}} \right)' |^2 + \psi \right] dx \right)^{\frac{1+\beta}{2}} \tag{19}$$

and more generally for all $\gamma \geq 0$, if we assume in addition that $\int_0^1 \psi \log^{\frac{2\gamma}{1-\beta}} \left(C + \frac{\psi}{x^k}\right) dx < \infty$, then

$$\left(\int_0^1 \frac{\psi \log^\gamma \left(C + \frac{\psi}{x^k}\right)}{x^{1+\beta}} \, dx \right) \leq C \left(\int_0^1 \psi \log^{\frac{2\gamma}{1-\beta}} \left(C + \frac{\psi}{x^k}\right) dx \right)^{\frac{1-\beta}{2}} \left(\int_0^1 \left[x^k | \left(\sqrt{\frac{\psi}{x^k}} \right)' |^2 + \psi \right] dx \right)^{\frac{1+\beta}{2}}. \tag{20}$$

Remark 3.3. *Before giving the proof, let us mention that this lemma should be compared to the results of Section 3.2 of* [68]. *In particular Proposition 3.1 there was used to control the extra stress tensor. However, the main difference is that the results of Section 3.2 of* [68] *are derived in an L^2 framework since we were dealing with strong solutions there, whereas the results of Lemma 3.2 here are in an L^1 frame work since we only have control on the free energy and its dissipation.*

Proof. Inequality (17) for $k > 1$ is just Hardy's inequality. Notice that there is no requirement on the boundary data since $k > 1$. To prove it, we perform the change of variable $y = x^{1-k}$, namely $x^k = y^{-\alpha}$ and $\alpha = \frac{k}{k-1} > 1$. We also define $h(y) = g(x) = \sqrt{\frac{\psi(x)}{x^k}}$. Notice that the assumptions on ψ mean that $g \in H^1_k$. Hence, to prove (17), it is enough to prove that

$$\int_1^\infty \frac{h^2}{y^2} dy \leq C \int_1^\infty \left[h'(y)^2 + \frac{h^2}{y^{2\alpha}} \right] dy. \tag{21}$$

Let us explain how we get the term $\int_1^\infty h'(y)^2 dy$:

$$\frac{\partial}{\partial x} \left(\sqrt{\frac{\psi}{x^k}} \right) = \frac{\partial y}{\partial x} \frac{\partial h}{\partial y} = (1-k) x^{-k} \frac{\partial h}{\partial y}.$$

Also, $dy = (1-k)\frac{dx}{x^k}$ which yields $dx = \frac{x^k}{1-k} dy$. Hence,

$$\int_0^1 x^k \left| \left(\sqrt{\frac{\psi}{x^k}} \right)' \right|^2 dx = (k-1) \int_1^\infty h'(y)^2 dy. \tag{22}$$

To prove (21), we integrate by parts in

$$\int_1^A \frac{h\, h'}{y} dy = \int_1^A \frac{h^2}{2y^2} dy + \frac{h(A)^2}{2A} - \frac{h(1)^2}{2}$$

for each $A > 1$. The left-hand side is bounded by $C(\int_1^A \frac{h^2}{y^2} dy)^{1/2}$ $(\int_1^A h'(y)^2 dy)^{1/2}$. To bound, $h(1)^2$ by the right-hand side of (21), we use that $h(y) \leq C\sqrt{y}$ since $\int_1^\infty h'(y)^2 dy < \infty$. Hence, $\frac{h^2}{y^\alpha}$ goes to zero when y goes to infinity. This yields that

$$h^2(1) = -\int_1^\infty \left(\frac{h^2}{y^\alpha} \right)' dy = -\int_1^\infty \left[2\frac{h}{y^\alpha} h' - \alpha \frac{h^2}{y^{\alpha+1}} \right] dy$$

which is controlled by the right-hand side of (21) using the Cauchy-Schwarz inequality and the fact that $\alpha > 1$. Letting A go to infinity, we see that (21) holds. Also, the fact that $\frac{h^2}{y^\alpha}$ goes to 0 when y goes to infinity is equivalent to the fact that $\psi(x)$ goes to 0 when x goes to 0. Actually, one can even get a more precise statement regarding the behavior of ψ when x goes to zero. Indeed, writing $h(y) = h(A) + \int_A^y h'(z) dz$, we see that for each $\varepsilon > 0$, there exists a constant C_ε such that $h(y) \leq C_\varepsilon + \varepsilon\sqrt{y}$, hence, $g(x) \leq C_\varepsilon + \varepsilon x^{\frac{1-k}{2}}$ and

$\psi(x) = x^k h^2(y) \leq C_\varepsilon x^k + \varepsilon x$. Therefore $\frac{\psi(x)}{x}$ goes to 0 when x goes to 0 and *i)* follows in this case. The proof of (18) when $k > 1$ follows by the Cauchy-Schwarz inequality.

In the case $0 < k < 1$, (17) only holds if we add a vanishing boundary condition at $x = 0$ (see Lemma 3.1). Nevertheless, we can still prove that (18) holds without any extra condition. Indeed, making the change of variable $y = x^{1-k}$ and denoting $h(y) = \sqrt{\frac{\psi(x)}{x^k}}$, we see that (18) is equivalent to

$$\left(\int_0^1 y^{\alpha-1} h^2 dy\right)^2 \leq C \left(\int_0^1 y^{2\alpha} h^2\, dy\right) \left(\int_0^1 [h'(y)^2 + y^{2\alpha} h^2] dy\right) \tag{23}$$

where $\alpha = \frac{k}{1-k}$. Notice, here that (22) becomes

$$\int_0^1 x^k \left| \left(\sqrt{\frac{\psi}{x^k}} \right)' \right|^2 dx = (1-k) \int_0^1 h'(y)^2 dy. \tag{24}$$

In particular the fact that $\sqrt{\frac{\psi}{x^k}} \in H^1_k$ yields that $\int_0^1 h'(y)^2 dy$ is finite and hence h has a trace at 0. Hence, $\psi(x) = x^k h^2(y)$ goes to 0 when x goes to 0. More precisely, we see in this case that there exists C such $\psi(x) \leq C x^k$. To prove (23), we integrate by parts in

$$\int_0^1 y^\alpha h\, h' dy = -\frac{\alpha}{2} \int_0^1 y^{\alpha-1} h^2 dy + \frac{h^2(1)}{2}$$

and notice that the left-hand side is bounded by $\left(\int_0^1 y^{2\alpha} h^2 dy \int_0^1 h'(y)^2 dy\right)^{1/2}$ using the Cauchy-Schwarz inequality. Moreover, we have

$$\begin{aligned} h(1)^2 = \int_0^1 (y^{2\alpha+1} h^2)' dy &= \int_0^1 [y^{2\alpha+1} h\, h' + (2\alpha+1) y^{2\alpha} h^2] dy \\ &\leq C \left(\int_0^1 y^{2\alpha} h^2 dy \int_0^1 [h'(y)^2 + y^{2\alpha} h^2] dy \right)^{1/2}. \end{aligned} \tag{25}$$

Hence, (23) follows.

When $k = 1$, we make the change of variable $y = -\log x$ and hence (18) is equivalent to

$$\left(\int_0^\infty e^{-y} h^2 dy\right)^2 \leq C \left(\int_0^\infty e^{-2y} h^2\, dy\right) \left(\int_0^\infty [h'(y)^2 + e^{-2y} h^2] dy\right) \tag{26}$$

and the proof of (26) can be done in a similar way as that of (23). Also, arguing as in the case $k > 1$, for each $\varepsilon > 0$, there exists C_ε such that $h(y) \leq C_\varepsilon + \varepsilon\sqrt{y}$. Hence, $g(x) \leq C_\varepsilon + \varepsilon |\log x|$ and $\psi(x) \leq C_\varepsilon x + \varepsilon x |\log x|$.

To prove (19), we first notice that if $-1 \leq \beta \leq 0$, then the inequality can be easily deduced from (18) by interpolation. When $\beta > 0$, (19) is equivalent (in the case $k < 1$) to

$$\left(\int_0^1 y^{\alpha_\beta-1} h^2 dy\right) \leq C \left(\int_0^1 y^{2\alpha} h^2\, dy\right)^{\frac{1-\beta}{2}} \left(\int_0^1 [h'(y)^2 + y^{2\alpha} h^2] dy\right)^{\frac{1+\beta}{2}} \tag{27}$$

where $\alpha_\beta = \frac{k-\beta}{1-k}$ and $\alpha = \frac{k}{1-k}$. Applying (23) with α replaced by α_β, we get

$$\left(\int_0^1 y^{\alpha_\beta-1} h^2 dy\right) \leq C \left(\int_0^1 y^{2\alpha_\beta} h^2\, dy\right)^{1/2} \left(\int_0^1 [h'(y)^2 + y^{2\alpha} h^2] dy\right)^{1/2}. \tag{28}$$

Notice that we kept the exponent α in the last term instead of replacing it by α_β. Indeed, the last integral comes from the estimate of $h^2(1)$ and we can keep $\alpha = \frac{k}{1-k}$ in (25). Now, we can apply (28) replacing $\alpha_\beta - 1$ by $2\alpha_\beta$ and we get

$$\left(\int_0^1 y^{2\alpha_\beta} h^2 dy\right) \le C \left(\int_0^1 y^{2(2\alpha_\beta+1)} h^2\, dy\right)^{1/2} \left(\int_0^1 [h'(y)^2 + y^{2\alpha} h^2] dy\right)^{1/2}. \tag{29}$$

We can iterate this, replacing $\alpha_\beta - 1$ by $2\alpha_\beta$, $2(2\alpha_\beta+1)$, ... in (28) till we get an index greater than $2\alpha = 2\frac{k}{1-k}$. Interpolating with the last inequality, yields (19).

In the case $k = 1$, (19) is equivalent to

$$\left(\int_0^\infty e^{-(1-\beta)y} h^2 dy\right) \le C \left(\int_0^\infty e^{-2y} h^2\, dy\right)^{\frac{1-\beta}{2}} \left(\int_0^\infty [h'(y)^2 + e^{-2y} h^2]\, dy\right)^{\frac{1+\beta}{2}}. \tag{30}$$

The proof of (30) is similar and is left to the reader.

For the proof of (20), we use that it is equivalent (in the case $k < 1$) to

(31)

$$\int_0^1 y^{\alpha_\beta - 1} h^2 \log^\gamma(h^2) dy \le C \left(\int_0^1 y^{2\alpha} h^2 \log^{\frac{2\gamma}{1-\beta}}(h^2)\, dy\right)^{\frac{1-\beta}{2}} \left(\int_0^1 [h'(y)^2 + y^{2\alpha} h^2] dy\right)^{\frac{1+\beta}{2}}.$$

Again, one can prove (31) in the case $\beta = 0$ by an integration by parts similar to the one used in (23). The case where $-1 \le \beta \le 0$ can be deduced by interpolation from the case $\beta = 0$ and the case $0 < \beta < k$ can be deduced by a bootstrap argument similar to the one used in the proof of (19). ☐

Remark 3.4. *1) We can also state Lemma 3.2 in terms of the function $g = \sqrt{\frac{\psi}{\psi_\infty}}$. In particular,* (18) *becomes: if $g \in H^1_k$, then*

$$\left|\int_0^1 x^{k-1} g^2\, dx\right| \le \|g\|^{1/2}_{L^2_k} \|g\|^{1/2}_{H^1_k}. \tag{32}$$

This second formulation has the advantage that it does not require g to be nonnegative as can be easily seen from the proof of Lemma 3.2.

2) Notice that in terms of scaling and when $k = 1$, inequality (18) *and its version* (26) *written in the y variable (as well as inequality* (19) *and its y version* (30)*) are scaling invariant. However, this is not the case for inequality* (15) *and its y version* (16)*.*

Corollary 3.5. *Under the assumptions of Lemma 3.2, we have the following, more precise, bounds on ψ when x goes to zero*

$$\begin{cases} \psi(x) \le C\|\sqrt{\frac{\psi}{x^k}}\|^2_{H^1_k} x & \text{and} \quad \psi(x) \le C_\varepsilon x^k + \varepsilon x \quad \text{if} \quad k > 1, \\ \psi(x) \le C\|\sqrt{\frac{\psi}{x^k}}\|^2_{H^1_k} x|\log x| & \text{and} \quad \psi(x) \le C_\varepsilon x + \varepsilon x|\log x| \quad \text{if} \quad k = 1, \\ \psi(x) \le C\|\sqrt{\frac{\psi}{x^k}}\|^2_{H^1_k} x^k & \quad \text{if} \quad k < 1, \end{cases} \tag{33}$$

where C is a constant that only depends on k and C_ε is a constant that depends on $\varepsilon > 0$ and on the function $\sqrt{\psi}$.

In terms of $g = \sqrt{\frac{\psi}{x^k}}$, we have

$$\begin{cases} |g(x)| \le C\|g\|_{H^1_k} x^{\frac{1-k}{2}} & \text{and} \quad |g(x)| \le C_\varepsilon + \varepsilon x^{\frac{1-k}{2}} \quad \text{if} \quad k > 1, \\ |g(x)| \le C\|g\|_{H^1_k} |\log x|^{1/2} & \text{and} \quad |g(x)| \le C_\varepsilon + \varepsilon x|\log x|^{1/2} \quad \text{if} \quad k = 1, \\ |g(x)| \le C\|g\|_{H^1_k} & \quad \text{if} \quad k < 1. \end{cases} \tag{34}$$

3.3. **Control of the stress tensor.** We recall that

$$\psi_\infty(R) = \frac{e^{-\mathcal{U}(R)}}{\int_B e^{-\mathcal{U}(R')}dR'} = \frac{(1-|R|^2)^k}{\int_B (1-|R'|^2)^k \, dR'}.$$

Hence, $\psi_\infty(R)$ behaves like $(1-|R|)^k$ when $|R|$ goes to 1. In particular we will apply a variant of Lemma 3.2 with $x = 1-|R|$.

Corollary 3.6. *There exists a constant C such that for $\psi \geq 0$ and $\sqrt{\frac{\psi}{\psi_\infty}} \in \mathcal{H}^1_k$, we have the following bound*

$$|\tau(\psi)|^2 \leq C\left(\int_B \psi dR\right)\left[\int_B \left|\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 \psi_\infty dR\right]. \tag{35}$$

This Corollary can be seen as the L^1 version of Proposition 3.1 of [68]. It will allow us to control the extra stress tensor by the free energy dissipation.

To prove (35), we recall that since τ is given by the Kramers formula, we can write it as

$$\tau_{ij}(\psi) = \int_B \left(\frac{\psi(R)}{\psi_\infty} - a^2\right)\frac{(R_i\ R_j)}{1-R^2}\psi_\infty dR$$

for any constant a^2. Here, we take a such that $\int_B \psi_\infty(\sqrt{\frac{\psi(R)}{\psi_\infty}} - a)\, dR = 0$. Hence,

$$|\tau| \leq \int_B \left[|g|^2 + 2a\,|g|\right]\frac{1}{1-|R|}\psi_\infty dR \tag{36}$$

where $g = \sqrt{\frac{\psi(R)}{\psi_\infty}} - a$. Using (32) with $x = 1-|R|$, we deduce that the first term on the right-hand side of (36) is controlled by

$$\left[\int_B g^2\psi_\infty dR\right]^{1/2}\left[\int_B |\nabla_R g|^2\,\psi_\infty dR + \int_B g^2\psi_\infty dR\right]^{1/2}. \tag{37}$$

For the second term, we use the Cauchy-Schwarz inequality, namely

$$\int_0^1 x^{k-1}|g|\,dx \leq \left(\int_0^1 x^{k-1}\,dx\right)^{1/2}\left(\int_0^1 x^{k-1}|g|^2\,dx\right)^{1/2}$$

to reduce it to the first term. Hence, we deduce that

$$|\tau(\psi)|^2 \leq C\left(\int_B \psi_\infty g^2 dR\right)\left[\int_B |\nabla_R g|^2\,\psi_\infty dR + \int_B \psi_\infty g^2 dR\right]. \tag{38}$$

Using the weighted Poincare inequality since $\int_B \psi_\infty g\, dR = 0$, we deduce that

$$\int_B \psi_\infty g^2 dR \leq C\int_B |\nabla_R g|^2\,\psi_\infty dR$$

and hence (35) follows.

3.4. **Weighted Sobolev inequality.** In Subsection 5.1, we have to prove the equi-integrability of N_2^n. This will require the control of some higher L^p space norm of $\sqrt{\frac{\psi}{\psi_\infty}}$. We have

Proposition 3.7. *There exists $p > 2$ and a constant C such that for $\psi \geq 0$, $\sqrt{\frac{\psi}{\psi_\infty}} \in \mathcal{H}^1_k$, we have the following bound*

$$\left(\int_B \left|\sqrt{\frac{\psi}{\psi_\infty}}\right|^p \psi_\infty dR\right)^{1/p} \leq C\left[\int_B \left|\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 \psi_\infty dR + \int_B \psi dR\right]^{1/2}. \tag{39}$$

For the proof we first notice that the only difficulty comes from the weight and hence we can restrict ourselves to the region where $|R| > \frac{1}{2}$. Also, we use spherical polar coordinates, namely $R = (1-x)\omega$ where $\omega \in \mathbb{S}^{D-1}$ and $0 < x < \frac{1}{2}$. The square of the right-hand side of (39) can be written as the sum of a radial part and an angular part :

$$\int_{\mathbb{S}^{D-1}} \left(\int_0^{1/2} \left[\left| \partial_x \sqrt{\frac{\psi}{\psi_\infty}} \right|^2 + |\sqrt{\frac{\psi}{\psi_\infty}}|^2 \right] x^k \, dx \right) d\omega, \tag{40}$$

$$\int_0^{1/2} \left(\int_{\mathbb{S}^{D-1}} \left[\left| \partial_\omega \sqrt{\frac{\psi}{\psi_\infty}} \right|^2 + |\sqrt{\frac{\psi}{\psi_\infty}}|^2 \right] d\omega \right) x^k \, dx. \tag{41}$$

We recall the following one-dimensional weighted $L^p - L^q$ Hardy inequality (one can also call it weighted Sobolev inequality)

$$\left(\int_0^{1/2} |F(x)|^q x^k \, dx \right)^{1/q} \le C \left(\int_0^{1/2} |F'(x)|^2 x^k \, dx \right)^{1/2}. \tag{42}$$

This inequality can be easily deduced from Theorem 6 of [49], taking $u(x) = v(x) = x^k$ for any $q < \infty$ if $k \le 1$ and for $q \le \frac{2(k+1)}{k-1}$ if $k > 1$. Indeed, Theorem 6 of [49] states that (42) holds for any F, with $F(\frac{1}{2}) = 0$ if

$$\sup_{0<r<\frac{1}{2}} (\int_0^r x^k dx)^{1/q} (\int_r^{\frac{1}{2}} (x^k)^{-1} dx)^{1/2} < \infty.$$

If we do not assume that $F(\frac{1}{2}) = 0$, then the inequality (42) still holds if we replace the right-hand side by $\left(\int_0^{1/2} [|F'(x)|^2 + |F(x)|^2] x^k \, dx \right)^{1/2}$. Hence, we obtain control of $\sqrt{\frac{\psi}{\psi_\infty}}$ in the space $L^2(\mathbb{S}^{D-1}; L^q((0,\frac{1}{2}), x^k dx))$ using the radial part of the norm (40).

On the other hand we can use the classical Sobolev inequality in dimension $D-1$ to control $\sqrt{\frac{\psi}{\psi_\infty}}$ in the space $L^2_x((0,\frac{1}{2}); L^s(\mathbb{S}^{D-1}), x^k dx)$ where $s = \frac{2(D-1)}{(D-1)-2}$ if $D > 3$, $s < \infty$ if $D = 3$ and $s \le \infty$ if $D = 2$. Interpolating between the two spaces $L^2_\omega L^q_x$ and $L^2_x L^s_\omega$, we deduce the existence of some $p > 2$ such that (39) holds.

3.5. **Young measures and Chacon limit.**

We recall here two important weak convergence concepts used in this paper, namely the Young measure and Chacon's biting lemma. Actually, these two notions are closely related as was observed in Ball and Murat [5] (see also [73]).

Proposition 3.8. *(Young measures) If f^n is a sequence of functions bounded in $L^1(U;\mathbb{R}^m)$ where U is an open set of $\mathbb{R}^N$, then there exists a family $(\nu_x)_{x\in U}$ of probability measures on $\mathbb{R}^m$ (the Young measures), depending measurably on x and a subsequence also denoted by f^n such that if $g : \mathbb{R}^m \to \mathbb{R}$ is continuous, $A \subset U$ is measurable and*

$$g(f^n) \rightharpoonup z(x) \quad \textit{weakly in}\, L^1(A;\mathbb{R}),$$

then $g(.) \in L^1(\mathbb{R}^m;\nu_x)$ for a.e. $x \in A$ and

$$z(x) = \int_{\mathbb{R}^m} g(\lambda) d\nu_x(\lambda) \quad \textit{a.e.} \quad x \in A.$$

In the case where f^n is bounded in $L^p(U;\mathbb{R}^m)$ for some $p > 1$ (or when f^n is equi-integrable) and $|g(f)| \lesssim |f|$, we can always take $A = U$ and we have (extracting a subsequence)

$$g(f^n) \rightharpoonup \int_{\mathbb{R}^m} g(\lambda) d\nu_x(\lambda).$$

Proposition 3.9. *(Chacon limit) If f^n is a sequence of functions bounded in $L^1(U;\mathbb{R}^m)$ where U is an open set of $\mathbb{R}^N$, then there exists a function $f \in L^1(U;\mathbb{R}^m)$, a subsequence f^n and a non-increasing sequence of measurable sets E_k of U with $\lim_{k\to\infty} \mathcal{L}_N(E_k) = 0$ (where $\mathcal{L}_N$ is the Lebesgue measure on $\mathbb{R}^N$) such that for all $k \in \mathbb{N}$, $f^n \rightharpoonup f$ weakly in $L^1(U - E_k;\mathbb{R}^m)$ as n goes to infinity. The function f is called the Chacon limit of f^n. We denote it $f = ch(f^n)$.*

It is easy to see that if f^n is equi-integrable then the Chacon limit of f^n is equal to the weak limit of f^n in $L^1(U, \mathbb{R}^m)$.

If we consider continuous functions $g_k : \mathbb{R}^m \to \mathbb{R}^m$, $k \in \mathbb{N}$, satisfying the conditions :
(a) $g_k(\lambda) \to \lambda$ when $k \to \infty$, for each $\lambda \in \mathbb{R}^m$,
(b) $|g_k(\lambda)| \le C(1 + |\lambda|)$, for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}^m$,
(c) $\lim_{|\lambda|\to\infty} |\lambda|^{-1}|g_k(\lambda)| = 0$ for each k,
then, under the hypotheses of Proposition 3.8, for each fixed k, the sequence of functions $g_k(f^n)$ is equi-integrable and hence (extracting a subsequence) converges weakly in $L^1(U;\mathbb{R}^m)$, to some f_k. Applying a diagonal process, as k goes to infinity, the sequence f_k converges strongly to some f in $L^1(U;\mathbb{R}^m)$. The limit f is the Chacon's limit of the subsequence f^n and it is given by

$$f(x) = \int_{\mathbb{R}^m} \lambda d\nu_x(\lambda) \quad a.e. \quad x \in U.$$

This gives another possible definition of Chacon's limit, which is equivalent to the one given in Proposition 3.9. For the proof of these results we refer to the proposition on p.659 of [5].

Remark 3.10. *We end this subsection by the following important fact: Let f^n be a sequence of functions bounded in $L^2(U;\mathbb{R}^m)$. Extracting a subsequence, we can define f the weak limit of f^n, $\overline{|f^n|^2}$ the weak limit of $|f^n|^2$ in the sense of measures and $ch(|f^n|^2)$ the Chacon's limit of $|f^n|^2$. It is clear that $\overline{|f^n|^2} \ge \overline{|f^n|^2}^a \ge ch(|f^n|^2) \ge |f|^2$ where for any sequence of measures μ^n, we denote by $\overline{\mu^n}^a$ the part of the weak limit $\overline{\mu^n}$ which is absolutely continuous with respect to the Lebesgue measure. It is not difficult to give examples where the three inequalities are strict. In particular $\overline{|f^n|^2} - ch(|f^n|^2)$ measures the concentration in L^2 of the sequence f^n and $ch(|f^n|^2) - |f|^2$ measures the oscillations.*

Also, if f^n is bounded in $L^2(U;\mathbb{R}^m)$ such that f^n converges to f in all L^p, $1 \le p < 2$, then $ch(|f^n - f|^2) = 0$ and the defect of strong convergence in L^2 is only due to concentrations.

4. A priori estimates

In this section, we prove some a priori estimates. We will assume that we have a regular enough solution (u, ψ) of our system (1), which allows us to perform all the calculations.

4.1. **Mass conservation.** The second equation of (1) can be written as

$$\partial_t \psi + u.\nabla\psi = \mathrm{div}_R\Big[- \nabla u R\, \psi\Big] + div_R\Big[\psi_\infty \nabla_R \frac{\psi}{\psi_\infty}\Big] \tag{43}$$

and the boundary condition (2) can be written as

$$\psi_\infty \nabla_R \frac{\psi}{\psi_\infty}.\mathbf{n} = 0 \quad \text{on} \quad \partial B(0, R_0). \tag{44}$$

We define $\rho(t,x) = \int_B \psi dR$. Integrating (43) over R, we get the transport of ρ, namely $\partial_t\rho + u.\nabla\rho = 0$. In particular, if we assume initially that $\rho_0(x) \in L^1(\Omega) \cap L^\infty(\Omega)$; then, we can integrate in x and using that u vanishes at the boundary of Ω, we deduce that $\int_\Omega \rho(t,x)dx = \int_\Omega \rho_0(x)dx$, namely we deduce the global conservation of mass. Here, we will only assume that $\rho_0(x) \in L^\infty(\Omega)$. We would like to prove that if initially, $\psi_0(x,R) \geq 0$, then this property will be propagated in time. Let $\beta : \mathbb{R} \to \mathbb{R}$, be defined by

$$\begin{cases} \beta(s) = 0, & \text{for } s \leq 0 \\ \beta(s) = s, & \text{for } s \geq 0, \end{cases}$$

and let β_ε be a convex regularization of β that can be achieved by mollification.

Multiplying (43) by $\beta'_\varepsilon(\psi)$, we get

$$\begin{aligned} \partial_t \beta_\varepsilon(\psi) + u.\nabla \beta_\varepsilon(\psi) = \mathrm{div}_R\Big[-\nabla u R\, \beta_\varepsilon(\psi)\Big] \\ + \Delta_R \beta_\varepsilon(\psi) - \beta''_\varepsilon(\psi)|\nabla_R \psi|^2 + \nabla_R \mathcal{U}.\nabla_R \beta_\varepsilon(\psi) + \Delta_R \mathcal{U} \beta'_\varepsilon(\psi)\psi. \end{aligned} \tag{45}$$

Sending ε to zero in (45) and using that $\beta'_\varepsilon(s)s$ goes to $\beta(s)$ when ε goes to zero, we recover in the limit

$$\partial_t \beta(\psi) + u.\nabla \beta(\psi) \leq \mathrm{div}_R\Big[-\nabla u R \beta(\psi)\Big] + div_R\Big[\psi_\infty \nabla_R \frac{\beta(\psi)}{\psi_\infty}\Big], \tag{46}$$

where the inequality is understood in the sense of distributions. Integrating over R the difference between (43) and (46), we get

$$\partial_t \int_B [\psi - \beta(\psi)]\, dR + u.\nabla \int_B [\psi - \beta(\psi)]\, dR \geq 0.$$

Since, $\int_B [\psi - \beta(\psi)]dR = 0$ at $t = 0$ and $\int_B [\psi - \beta(\psi)]dR \leq 0$, we deduce that $\int_B [\psi - \beta(\psi)]dR = 0$ for all t and hence $\psi \geq 0$.

4.2. **Free energy.** Multiplying (43) by $\log \frac{\psi}{\rho\psi_\infty}$ and integrating over R and x, we get

$$\begin{aligned} &\partial_t \int_\Omega \int_B \Big[\psi \log(\frac{\psi}{\rho\psi_\infty}) - \psi + \rho\psi_\infty\Big]\, dRdx \\ &= \int_\Omega \int_B \nabla u : (R \otimes \nabla_R \mathcal{U})\psi\, dRdx - 4 \int_\Omega \int_B \psi_\infty \left|\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 dRdx \end{aligned} \tag{47}$$

where we have used that $\nabla\psi_\infty = -\psi_\infty \nabla \mathcal{U}$.

The first equation of (1) yields the classical energy estimate for the Navier-Stokes equation

$$\partial_t \int_\Omega \frac{|u|^2}{2} = - \int_\Omega \nabla u : \tau - \nu \int_\Omega |\nabla u|^2. \tag{48}$$

Adding (47) and (48) yields the following decay of the free-energy

$$\begin{aligned} &\partial_t \int_\Omega \Big[\int_B [\psi \log(\frac{\psi}{\rho\psi_\infty}) - \psi + \rho\psi_\infty]dR + \frac{|u|^2}{2}\Big]\, dx \\ &= -\nu \int_\Omega |\nabla u|^2 dx - 4 \int_\Omega \int_B \psi_\infty \left|\nabla_R \sqrt{\frac{\psi}{\psi_\infty}}\right|^2 dRdx. \end{aligned} \tag{49}$$

Integrating in time, we get the following uniform bound for all $t > 0$:

$$\begin{aligned}&\int_\Omega \left[\int_B [\psi \log(\frac{\psi}{\rho\psi_\infty}) - \psi + \rho\psi_\infty]dR + \frac{|u|^2}{2}\right](t)\,dx \\ &\quad + \nu\int_0^t\int_\Omega |\nabla u|^2 dxds + 4\int_0^t\int_\Omega\int_B \psi_\infty\left|\nabla_R\sqrt{\frac{\psi}{\psi_\infty}}\right|^2 dRdxds = C_0\end{aligned} \tag{50}$$

where C_0 is the initial free energy. To simplify the notations in the rest of this section, we will assume that $\rho_0(x) = 1$. The proof in the general case is identical and we will indicate the changes to be made at the end. The general idea is the following: When proving a priori estimates, one just has to replace ψ_∞ by $\rho(t,x)\psi_\infty$ and take advantage of the fact that ρ is merely transported by the flow. When proving weak compactness, one can use that ρ^n converges strongly to ρ in all $L^p((0,T)\times\Omega)$ spaces and use $\rho^n(t,x)\psi_\infty$ instead of ψ_∞. Due to the local character of the proof of weak compactness, a simpler way is just to use ψ_∞ and so the calculations given in Section 5 hold even when ρ_0 is not constant.

4.3. $\log^2$ **estimate.** The free energy only gives an $L\log L(\psi_\infty dR)$ bound on $\frac{\psi}{\psi_\infty}$. For some integrability reasons, we will need to control a slightly higher growth of ψ in the R variable.

We introduce $\tilde\psi = \psi + a\psi_\infty$ for some $a > 1$. This is done to insure that $\log\frac{\tilde\psi}{\psi_\infty}$ does not take negative values. It will also add a new term in the equation, which will not present any extra difficulties. Hence, $\tilde\psi$ solves

$$\partial_t\tilde\psi + u.\nabla\tilde\psi = \mathrm{div}_R\Big[-\nabla u\cdot R\tilde\psi\Big] + div_R\Big[\psi_\infty\nabla_R\frac{\tilde\psi}{\psi_\infty}\Big] - a\nabla_R\mathcal{U}.\nabla uR\psi_\infty. \tag{51}$$

We first derive this extra bound in the case the domain Ω is bounded and then discuss the modification of the argument in the whole space case.

4.3.1. *Case of a bounded domain.* Multiplying (51) by $\log^2\frac{\tilde\psi}{\psi_\infty}$ and integrating by parts in R, we get

$$\begin{aligned}&(\partial_t + u.\nabla_x)\int_B \tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]dR \\ &\quad = -2ak\nabla_i u_j\int_B \frac{R_iR_j}{1-|R|^2}\psi_\infty\log^2\frac{\tilde\psi}{\psi_\infty}dR \\ &\qquad + \int_B \tilde\psi\,2\log(\frac{\tilde\psi}{\psi_\infty})\frac{\psi_\infty}{\tilde\psi}\nabla uR\cdot\nabla_R\frac{\tilde\psi}{\psi_\infty}dR - 8\int_B\psi_\infty\left|\nabla_R\sqrt{\frac{\tilde\psi}{\psi_\infty}}\right|^2\log(\frac{\tilde\psi}{\psi_\infty})dR.\end{aligned} \tag{52}$$

Dividing (52) by $2\left(\int_B\tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2}$, we get

$$\begin{aligned}&(\partial_t + u.\nabla_x)\left(\int_B\tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2} = \frac{-ak\nabla_iu_j\int_B\frac{R_iR_j}{1-|R|^2}\psi_\infty\log^2\frac{\tilde\psi}{\psi_\infty}}{\left(\int_B\tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2}} \\ &\quad + \frac{\int_B\nabla u\cdot R\tilde\psi\,\log(\frac{\tilde\psi}{\psi_\infty})\frac{\psi_\infty}{\tilde\psi}\nabla_R\frac{\tilde\psi}{\psi_\infty}}{\left(\int_B\tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2}} - \frac{4\int_B\psi_\infty\left|\nabla_R\sqrt{\frac{\tilde\psi}{\psi_\infty}}\right|^2\log(\frac{\tilde\psi}{\psi_\infty})}{\left(\int_B\tilde\psi[\log^2(\frac{\tilde\psi}{\psi_\infty}) - 2\log(\frac{\tilde\psi}{\psi_\infty}) + 2]\right)^{1/2}} \\ &\quad = I_1 + I_2 + I_3.\end{aligned} \tag{53}$$

Let us introduce the notation

$$N_2 = \left(\int_B \tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}}{\psi_\infty}) + 2]dR \right)^{1/2}. \tag{54}$$

To bound I_1 we use that, $\psi_\infty \log^2 \frac{\tilde{\psi}}{\psi_\infty} \le C\tilde{\psi}$. Hence, the numerator of I_1 is bounded by $C|\nabla u| \int \frac{\tilde{\psi}}{1-|R|^2} dR$ which is clearly in $L^1((0,T)\times\Omega)$ using (18). Indeed,

$$\int \frac{\psi}{1-|R|^2} dR \le C \left(\int_B \left[\psi_\infty \left| \nabla_R \sqrt{\frac{\psi}{\psi_\infty}} \right|^2 + \psi \right] dR \right)^{1/2}.$$

To bound the second term on the right-hand side of (53), we use that the numerator can be bounded by

$$\begin{aligned}
&\left| \int_B \nabla u \cdot R\tilde{\psi} \log(\frac{\tilde{\psi}}{\psi_\infty}) \frac{\psi_\infty}{\tilde{\psi}} \nabla_R \frac{\tilde{\psi}}{\psi_\infty} \right| && (55)\\
&\le C|\nabla u| \left(\int_B \psi_\infty |\log(\frac{\tilde{\psi}}{\psi_\infty})| \left| \nabla_R \sqrt{\frac{\tilde{\psi}}{\psi_\infty}} \right|^2 \right)^{1/2} \left(\int_B \psi_\infty |\log(\frac{\tilde{\psi}}{\psi_\infty})| \frac{\tilde{\psi}}{\psi_\infty} \right)^{1/2} \\
&\le C|\nabla u|^2 \left(\int_B \tilde{\psi} |\log(\frac{\tilde{\psi}}{\psi_\infty})| \right) + \left(\int_B \psi_\infty |\log(\frac{\tilde{\psi}}{\psi_\infty})| \left| \nabla_R \sqrt{\frac{\tilde{\psi}}{\psi_\infty}} \right|^2 \right) \\
&\le C|\nabla u|^2 (1+a)^{1/2} \left(\int_B \tilde{\psi} \log^2(\frac{\tilde{\psi}}{\psi_\infty}) \right)^{1/2} + \left(\int_B \psi_\infty |\log(\frac{\tilde{\psi}}{\psi_\infty})| \left| \nabla_R \sqrt{\frac{\tilde{\psi}}{\psi_\infty}} \right|^2 \right). && (56)
\end{aligned}$$

Dividing by N_2 the inequality whose left-hand side is (55) and whose right-hand side in (56), we deduce that

$$I_2 \le C|\nabla u|^2 - \frac{1}{4} I_3.$$

Integrating (53) in time and space and using the fact that $-I_3$ is nonnegative, we deduce the following a priori bound

$$\begin{aligned}
&\int_\Omega \left(\int_B \tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}}{\psi_\infty}) + 2]dR \right)^{1/2} dx(t) \\
&+ \int_0^T \int_\Omega \frac{\int_B \psi_\infty \left| \nabla_R \sqrt{\frac{\tilde{\psi}}{\psi_\infty}} \right|^2 \log(\frac{\tilde{\psi}}{\psi_\infty}) dR}{\left(\int_B \tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}}{\psi_\infty}) + 2] \right)^{1/2} dR} dx ds \le C_T
\end{aligned} \tag{57}$$

for $0 \le t \le T$, if the initial condition satisfies $\int_\Omega \left(\int_B \tilde{\psi}_0 [\log^2(\frac{\tilde{\psi}_0}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}_0}{\psi_\infty}) + 2]dR \right)^{1/2} dx \le C_0$. Hence, we see that (53) can be written as

$$(\partial_t + u.\nabla)N_2 = F \tag{58}$$

where F is in $L^1((0,T)\times\Omega)$.

Later in the paper, when (57) will be stated for a sequence $\tilde{\psi}^n$ that approximates $\tilde{\psi}$, it will be unclear how one can pass to the limit and recover that the limiting function $\tilde{\psi}$ satisfies (57). Actually, one can find sequences of functions $\tilde{\psi}^n$ such that (57) holds and yet the weak limit $\tilde{\psi}$ does not satisfy (57). This is the reason why we will use the following bound, which is easily deduced from (57):

$$\begin{aligned} &\sup_{0\le t\le T}\int_\Omega \left(\int_B g^2\log(g^2)\psi_\infty dR\right)^{1/2} dx(t) \\ &+\int_0^T\int_\Omega\int_B \frac{\psi_\infty\left|\nabla_R g\right|^2\ dR}{\left(\int_B\tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty})-2\log(\frac{\tilde{\psi}}{\psi_\infty})+2]dR\right)^{1/2}}dxds\le C_T \end{aligned} \tag{59}$$

where g is given by $g=\sqrt{\frac{\tilde{\psi}}{\psi_\infty}}\log^{1/2}(\frac{\tilde{\psi}}{\psi_\infty})$. The advantage of (59) is that if g^n satisfies (59), then the weak limit g of g^n also satisfies (62).

4.3.2. *Case of an unbounded domain.* In the case $\Omega=\mathbb{R}^D$, we first take c_1 and c_2 to be two constants such that the function $\phi(x)=x[\log^2 x-2\log x+c_1]+c_2$ satisfies the equalities $\phi(1+a)=\phi'(1+a)=0$. This is achieved by taking $c_1=2-\log^2(1+a)$ and $c_2=2(1+a)[\log(1+a)-1]$. Notice also that the function $\phi(x)$ is nonnegative for $x\ge a$ since a is taken large enough. It is clear that the extra bound (8) implies that

$$\int_\Omega \frac{\int_B\phi(\frac{\tilde{\psi}_0}{\psi_\infty})dR}{1+\left[\int_B\phi(\frac{\tilde{\psi}_0}{\psi_\infty})dR\right]^{1/2}}dx\le C_0, \tag{60}$$

and hence, we can perform the same calculations as (52) and (53) with $\int_B\tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty})-2\log(\frac{\tilde{\psi}}{\psi_\infty})+2]dR$ replaced by $\int_B\phi(\frac{\tilde{\psi}}{\psi_\infty})dR$ and with the function $s\to\sqrt{s}$ used to go from (52) to (53) replaced by $s\to\frac{s}{1+\sqrt{s}}$ which behaves like $\phi_1(s)=\min(\sqrt{s},s)$. The rest of the proof is identical.

4.3.3. *Case when ρ is not constant.* In the case when ρ is not constant and we are in a bounded domain, we have to modify (52) slightly and multiply by $\log^2\frac{\tilde{\psi}}{\rho\psi_\infty}$. In the case we are also in an unbounded domain, we have to replace $\int_B\tilde{\psi}[\log^2(\frac{\tilde{\psi}}{\psi_\infty})-2\log(\frac{\tilde{\psi}}{\psi_\infty})+2]dR$ by $\int_B\phi(\frac{(1+a)\tilde{\psi}}{(\rho+a)\psi_\infty})dR$. The extra factor $\frac{1+a}{\rho+a}$ is used to insure that when ψ is at microscopic equilibrium, namely $\tilde{\psi}=(\rho+a)\psi_\infty$, the integrand reduces to $\phi(1+a)$. The rest of the proof is identical and yields at the end the following bound instead of (57):

$$\int_\Omega\phi_1\left(\int_B\phi(\frac{(1+a)\tilde{\psi}}{(\rho+a)\psi_\infty})\right)dx(t)+\int_0^T\int_\Omega\frac{\int_B\psi_\infty\left|\nabla_R\sqrt{\frac{\tilde{\psi}}{\psi_\infty}}\right|^2\log(\frac{(1+a)\tilde{\psi}}{(\rho+a)\psi_\infty})}{1+\left(\int_B\phi(\frac{(1+a)\tilde{\psi}}{(\rho+a)\psi_\infty})\right)^{1/2}}dxds\le C_T. \tag{61}$$

One can then deduce from (61) that (57) and (59) hold, with the integration set Ω replaced by any compact K of $\mathbb{R}^D$.

5. Weak compactness

As it is classical when proving global existence of weak solutions, it is enough to prove the weak compactness of a sequence of weak solutions satisfying the a priori estimates of the previous section. In the next section, we present one way of approximating the system.

We consider a sequence of weak solutions (u^n, ψ^n) to (1) in the sense of Subsection 2.1, satisfying, uniformly in n, the free energy bound (50) and the $\log^2$ bound (57). The sequence (u^n, ψ^n) has an initial data (u_0^n, ψ_0^n) such that (u_0^n, ψ_0^n) converge strongly to (u_0, ψ_0) in $L^2(\Omega)\times L^1_{loc}(\Omega; L^1(B))$ and $\psi_0^n \log \frac{\rho_0^n \psi_0^n}{\psi_\infty} - \psi_0^n + \rho_0^n \psi_\infty$ converges strongly to $\psi_0 \log \frac{\psi_0}{\rho_0 \psi_\infty} - \psi_0 + \rho_0 \psi_\infty$ in $L^1(\Omega \times B)$. We also assume that (u^n, ψ^n) has some extra regularity, with bounds that depend on n, such that we can perform all of the following calculations. In particular, we assume that $\frac{\psi^n}{\psi_\infty} \in C([0,T); \mathcal{L}^2_k) \cap L^2((0,T); \mathcal{H}^1_k)$. A sequence of approximate weak solutions that satisfy all of the above mentioned properties will be shown to exist in the next section.

We extract a subsequence such that u^n converges weakly to u in $L^p((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega))$ and ψ^n converges weakly to ψ in $L^p((0,T); L^1_{loc}(\Omega \times B))$ for each $p < \infty$. We would like to prove that (u, ψ) is still a solution of (1) in the sense of Subsection 2.1. The main difficulty is to pass to the limit in the nonlinear term $\nabla u^n R \psi^n$ appearing in the second equation of (1).

We introduce $g^n = \sqrt{\frac{\tilde\psi^n}{\psi_\infty}} \log^{1/2}(\frac{\tilde\psi^n}{\psi_\infty})$ and $f^n = \sqrt{\frac{\tilde\psi^n}{\psi_\infty}}$ where $\tilde\psi^n = \psi^n + a\psi_\infty$ and $a > 1$ is any constant. Notice that from the free energy bound (50), we deduce that g^n is uniformly bounded in $L^\infty((0,T)\times\Omega; \mathcal{L}^r_k)$ and in $L^2((0,T)\times\Omega; \mathcal{W}^{1,r}_k)$ for $r < 2$. We recall that $\mathcal{L}^r_k = L^r(\psi_\infty dR)$.

We also assume, extracting a subsequence if necessary, that g^n and f^n converge weakly to some g and f in $L^p((0,T); L^2(\Omega \times B,\, dx\psi_\infty dR))$ for each $p < \infty$. To prove that (u,ψ) is a solution of (1), it will be enough to prove that $(g^n)^2 = \frac{\tilde\psi^n}{\psi_\infty}\log(\frac{\tilde\psi^n}{\psi_\infty})$ converges weakly to $g^2 = \frac{\tilde\psi}{\psi_\infty}\log(\frac{\tilde\psi}{\psi_\infty})$ in the sense of distributions, which in turn will follow from showing that g^n converges strongly to g in $L^2((0,T); L^2(\Omega\times B,\, dx\psi_\infty dR))$.

First, it is clear that $u, \tilde\psi$ and g satisfy the same a priori estimates that the sequence $u^n, \tilde\psi^n$ and g^n satisfy since all those functionals have good convexity properties. In particular it is clear that u, ψ satisfy (50) with an inequality $\leq$. We just point out that to pass to the limit in the last term on the left-hand side of (50), we can use the fact that the function $\phi_2(x,y) = \frac{x^2}{y}$ is convex. To pass to the limit in (59), we also use the fact that $\phi_2(x,y)$ is convex. Hence, we deduce that

$$\sup_{0\le t\le T} \int_\Omega \left(\left(\int_B g^2 \log(g^2) \psi_\infty dR \right)^{1/2} + \overline{N_2^n} \right) dx(t) + \int_0^T \int_\Omega \int_B \frac{\psi_\infty |\nabla_R g|^2}{\overline{N_2^n}} \le C_T \tag{62}$$

where $\overline{N_2^n}$ is the weak limit of $\left(\int_B \tilde\psi^n [\log^2(\frac{\tilde\psi^n}{\psi_\infty}) - 2\log(\frac{\tilde\psi^n}{\psi_\infty}) + 2]\right)^{1/2}$.

5.1. **The renormalizing factor** N. Here, we construct N that will be used as a renormalizing factor in the next subsections. We first introduce the unique a.e. flow X^n in the sense of DiPerna and Lions [26, 3] of u^n (see Appendix A), i.e. the solution of

$$\partial_t X^n(t,x) = u^n(t, X^n(t,x)), \qquad X^n(t=0,x) = x. \tag{63}$$

We also denote by X the a.e. flow of u. Due to the fact that u^n and u are divergence free, we deduce that the a.e. mappings $x \to X^n(t,x)$ and $x \to X(t,x)$ are measure-preserving for all $t \in [0,T)$. We will also use the flow starting at time t_0, namely $X^n(t,t_0,x)$ such that

$$\partial_t X^n(t,t_0,x) = u^n(t, X^n(t,t_0,x)) \qquad X^n(t=t_0,t_0,x) = x.$$

In particular $X^n(t,0,x) = X^n(t,x)$.

Let Q^n be the solution of (58) with $F = F^n$ replaced by $|F^n|$ and $u = u^n$ and taking the same initial data as N_2^n at $t = 0$. Therefore, for a.e. $x \in \Omega$,

$$\frac{d}{dt}[Q^n(t, X^n(t,x))] = |F^n(t, X^n(t,x))| \tag{64}$$

and $Q^n(t=0,x) = N_2^n(t=0,x)$. Hence, if $\beta \in C^\infty(\mathbb{R})$, $\beta(s) = s$ for $|s| \le 1$, $\beta(s) = 2$ for $|s| \ge 4$ and $\beta_M(s) = M\beta(\frac{s}{M})$, we have

$$\frac{d}{dt}[\beta_M(Q^n)(t, X^n(t,x))] = \beta_M'(Q^n)|F^n(t, X^n(t,x))|. \tag{65}$$

To pass to the limit weakly in (65), we first write it in a weak form. Let $q^n(t,x) = Q^n(t, X^n(t,x))$ and $h^n(t,x) = |F^n(t, X^n(t,x))|$. Hence, we have

$$\begin{cases} \frac{\partial}{\partial t}[\beta_M(q^n)] = \beta_M'(q^n)h^n \\ \beta_M(q^n)(t=0,x) = \beta_M(N_2^n(t=0,x)). \end{cases} \tag{66}$$

For $\Phi(t,x) \in C_0^\infty([0,T) \times \Omega)$, we have

$$-\int_0^T \int_\Omega \beta_M(q^n)\partial_t\Phi \, dxdt - \int_\Omega [\beta_M(N_2^n)]\Phi(t=0,x)dx = \int_0^T \int_\Omega \beta_M'(q^n)h^n\Phi \, dxdt. \tag{67}$$

Recall that $\beta_M(q^n) \in L^\infty((0,T)\times\Omega)$ and that $\beta_M'(q^n)h^n$ is bounded in $L^1([0,T)\times\Omega)$. Passing to the limit weakly (extracting a subsequence if necessary) in (67), we deduce that

$$-\int_0^T \int_\Omega \overline{\beta_M(q^n)}\partial_t\Phi \, dxdt - \int_\Omega \overline{\beta_M(N_2^n(t=0,x))}\Phi(t=0,x)dx = \Big\langle \overline{\beta_M'(q^n)h^n}, \Phi \Big\rangle_{\mathcal{M}\times C} \tag{68}$$

where $\overline{\beta_M(q^n)}$ is the weak-star limit of $\beta_M(q^n)$ in $L^\infty((0,T)\times\Omega)$ and $\overline{\beta_M'(q^n)h^n} \in \mathcal{M}([0,T)\times\Omega)$ is the weak limit of $\beta_M'(q^n)g^n$ in the sense of measures. In the sequel, $\overline{a^n}$ will denote the weak limit of the sequence a^n in some appropriate space.

From the stability of the notion of a.e. flow with respect to the weak limit of u^n to u (see [21] and Proposition A.3), we get the following equality of weak limits

$$\overline{[\beta_M(q^n)(t,x)]} = \overline{[\beta_M(Q^n)(t, X^n(t,x))]} = \overline{\beta_M(Q^n)}(t, X(t,x)).$$

Hence, sending M to infinity in (68), we deduce that

$$-\int_0^T \int_\Omega Q(t, X(t,x))\partial_t\Phi \, dxdt - \int_\Omega N_{2,0}(x)\Phi(t=0,x)dx = \Big\langle F, \Phi \Big\rangle_{\mathcal{M}\times C} \tag{69}$$

where $Q = \lim_{M\to\infty} \overline{[\beta_M(Q^n)]}$ is the Chacon limit of Q^n, $N_{2,0}(x)$ is the Chacon limit of $N_2^n(t=0,x)$ (and hence depends only on the initial data) and $F = \lim_{M\to\infty} \overline{\beta_M'(q^n)g^n}$ is the limit of $\overline{\beta_M'(q^n)g^n}$ in the sense of measures, which exists since $\overline{\beta_M'(q^n)g^n}$ is increasing in M and is uniformly bounded in $\mathcal{M}([0,T)\times\Omega)$. Also, it is easy to see that $Q \in L^\infty(0,T; L^1(\Omega))$. One can deduce from (69) that

$$\frac{d}{dt}[Q(t, X(t,x))] = F \tag{70}$$

holds in $\mathcal{M}([0,T)\times\Omega)$. Notice that this does not imply necessarily that $Q(0, X(0,x)) = N_{2,0}(x)$ since F may have a Dirac part at $t=0$. Since, $F \ge 0$, we deduce that for a.e. $0 \le t < T_0 < T$, we have $Q(t, X(t,x)) \le Q(T_0, X(T_0,x))$. For $0 \le s \le T_0$, we define $N(s, X(s,x))$ by

$$\int_\Omega N(s, X(s,x))\phi(x)\, dx = \int_\Omega N_{2,0}(x)\phi(x)dx + \Big\langle F, h(t)\phi(x) \Big\rangle_{\mathcal{M}\times C} \tag{71}$$

where $h(t) = 1$ on $[0,T_0]$, $h'(t) \le 0$ and h has its support in $[0,T)$ and ϕ is any test function in $C_0^\infty(\Omega)$. Notice that for $0 \le s < T_0$, $N(s, X(s,x))$ only depends on x and that $Q(s, X(s,x)) \le N(s, X(s,x)) \le Q(T, X(T,x))$. Indeed, N is constant along the characteristics of u. Notice also that N is in $L^\infty(0,T_0; L^1(\Omega))$ and that $N(t, X(t,x))$ is in $L^1(\Omega; L^\infty(0,T_0))$. Hence,

$$\frac{d}{dt}\beta[N(t, X(t,x))] = 0 \quad \text{for } 0 < t < T_0 \text{ and for a.e. } x$$

for any bounded C^1 function on $\mathbb{R}_+$ and hence (see Appendix A)

$$\partial_t \beta(N) + u.\nabla \beta(N) = 0 \quad \text{holds in } \mathcal{D}'. \tag{72}$$

In the sequel, we will write $T = T_0$ and will not make the distinction between these two times. Since, N is bounded from below by 1, it also solves

$$(\partial_t + u.\nabla)\frac{1}{N} = 0. \tag{73}$$

Also, the following three inequalities hold

$$\overline{\beta_M(N_2^n)} \leq \overline{\beta_M(Q^n)} \leq Q \leq N. \tag{74}$$

The first one comes from the fact that $N_2^n \leq Q^n$ and hence for all $M > 0$, $\beta_M(N_2^n) \leq \beta_M(Q^n)$ and then we pass to the limit in n. The second inequality comes from the fact that Q is the limit of $\overline{\beta_M(Q^n)}$ when M goes to infinity and that $\overline{\beta_M(Q^n)}$ is increasing in M. Notice in particular that by monotone convergence $\overline{\beta_M(Q^n)}$ converges almost everywhere and in $L^1_{loc}((0,T)\times\Omega)$ to Q. From (74), we deduce that the weak limit of N_2^n, which is equal to the Chacon limit of N_2^n, is bounded by N, namely $\overline{N_2^n} = \lim_{M\to\infty} \overline{\beta_M(N_2^n)} \leq N$. The fact that the weak limit of N_2^n is equal to its Chacon limit comes from the fact that the sequence N_2^n is equiintegrable. This is a simple consequence of the dissipation of the free energy and the weighted Sobolev inequality (39). Indeed, on the one hand, from (39), we deduce that $\sqrt{\frac{\psi^n}{\psi_\infty}}$ is bounded in $L^2((0,T)\times\Omega; L^p(\psi_\infty dR))$ for some $p > 2$. On the other hand, from the conservation of mass, we know that $\sqrt{\frac{\psi^n}{\psi_\infty}}$ is bounded in $L^\infty((0,T)\times\Omega; L^2(\psi_\infty dR))$. Interpolating between these two bounds, we easily deduce that $\sqrt{\frac{\psi^n}{\psi_\infty}}$ is bounded in $L^r((0,T)\times \Omega\times B, dtdx\psi_\infty dR)$ for some $r > 2$ and hence N_2^n is equiintegrable. We also deduce that

$$f \text{ and } g \text{ are bounded in } L^r((0,T)\times\Omega\times B, dtdx\psi_\infty dR) \text{ for some } r > 2. \tag{75}$$

5.2. **Two ways of passing to the limit.** Replacing $(u, \tilde\psi)$ by $(u^n, \tilde\psi^n)$ in (51) and dividing by ψ_∞, we get

$$\begin{aligned}\partial_t \frac{\tilde\psi^n}{\psi_\infty} + u^n.\nabla \frac{\tilde\psi^n}{\psi_\infty} &= \operatorname{div}_R\Big[-\nabla u^n R \frac{\tilde\psi^n}{\psi_\infty}\Big] + \nabla\mathcal{U}.\nabla u^n R\frac{\tilde\psi^n}{\psi_\infty} \\ &+ \frac{1}{\psi_\infty}\operatorname{div}_R\Big[\psi_\infty \nabla_R \frac{\tilde\psi^n}{\psi_\infty}\Big] - a\nabla u^n R.\nabla_R\mathcal{U}.\end{aligned} \tag{76}$$

Hence, using Corollary B.7 and the fact that $\operatorname{div}_R(\nabla u^n R) = 0$, we deduce that, for any smooth function Θ from $(0,\infty)$ to $\mathbb{R}$ with Θ' and Θ'' bounded on $[a,\infty)$, we have

$$\begin{aligned}\partial_t\Theta(\frac{\tilde\psi^n}{\psi_\infty}) + u^n.\nabla\Theta(\frac{\tilde\psi^n}{\psi_\infty}) = &-\nabla u^n R\cdot\nabla_R\Theta(\frac{\tilde\psi^n}{\psi_\infty}) + \nabla_R\mathcal{U}.\nabla u R\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty}) \\ &+ \frac{1}{\psi_\infty}\operatorname{div}_R\Big[\psi_\infty\nabla_R\Theta(\frac{\tilde\psi^n}{\psi_\infty})\Big] - \Theta''(\frac{\tilde\psi^n}{\psi_\infty})|\nabla_R\frac{\tilde\psi^n}{\psi_\infty}|^2 \\ &- 2ak\nabla u^n : \frac{R\otimes R}{1-|R|^2}\Theta'(\frac{\tilde\psi^n}{\psi_\infty}),\end{aligned} \tag{77}$$

with the initial condition $\Theta(\frac{\tilde\psi^n}{\psi_\infty})(t=0) = \Theta(\frac{\tilde\psi_0}{\psi_\infty})$ and the boundary condition

$$\psi_\infty\nabla_R\Theta(\frac{\tilde\psi^n}{\psi_\infty}).\mathbf{n} = \Theta'(\frac{\tilde\psi^n}{\psi_\infty})\psi_\infty\nabla_R\frac{\tilde\psi^n}{\psi_\infty}.\mathbf{n} = 0 \quad \text{on} \quad \partial B(0,1). \tag{78}$$

In the case $k>1$, we can also justify (78) by the fact that for a.e. t and x, $\frac{\psi^n(t,x,.)}{\psi_\infty}\in D(\mathcal{L})$ and hence $\psi_\infty\nabla_R\frac{\psi^n}{\psi_\infty}.\mathbf{n}$ has a trace in $L^2(\partial B(0,1))$ (see also [70] for more details about the regularity of ψ in the variable R). In the sequel, we take $\Theta(t)=t^{1/2}\log^{1/2}(t)$ and recall that $g^n=\Theta(\frac{\tilde\psi^n}{\psi_\infty})$. We introduce the following defect measures γ_{ij},γ'_{ij} and β_{ij} such that

$$\begin{aligned}\nabla u^n g^n\to\nabla u g+\gamma,\qquad \nabla u^n\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\to\nabla u\overline{\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})}+\gamma',\\ \nabla u^n\tilde\psi^n\to\nabla u\tilde\psi+\beta\end{aligned}\tag{79}$$

where $\gamma,\gamma'\in L^2((0,T)\times\Omega;L^r(\psi_\infty dR))\cap L^{3/2}((0,T)\times\Omega;\mathcal{L}^p_{k-1})$ for all $r<2$ and $p<\min(2,1+k)$ and $\beta\in L^1((0,T)\times\Omega;L^1(\frac{dR}{1-|R|}))$ are matrix valued. We recall that $\overline{F^n}$ denotes the weak limit of F^n (modulo a sequence extraction). Indeed, from the bounds on ∇u^n and g^n, we deduce that $\nabla u^n g^n$ is bounded in $L^2((0,T)\times\Omega;L^r(\psi_\infty dR))$ and in $L^1((0,T)\times\Omega;\mathcal{W}^{1,r}_k)$ for $r<2$. By Hardy's inequality (see Lemma 3.1 ii)), we deduce that $\nabla u^n g^n$ is bounded in $L^1((0,T)\times\Omega;\mathcal{L}^p_{k-2})$ for $p-1<k$ and $p<2$. Interpolating with the previous bound, we deduce that $\nabla u^n g^n$ is bounded in $L^{3/2}((0,T)\times\Omega;\mathcal{L}^p_{k-1})$. The proof that $\beta\in L^1((0,T)\times\Omega;L^1(\frac{dR}{1-|R|}))$ is given in Lemma 5.2.

On one hand, we pass to the limit in (77) with $\Theta(t)=t^{1/2}\log^{1/2}(t)$. For this, we first have to pass to the limit in its weak formulation. For all $h\in C^1([0,T]\times\overline{\Omega}\times\overline{B})$, $h(T)=0$, we have

$$\begin{aligned}&-\int_0^T\iint_{\Omega\times B}g^n(\partial_t h+u^n.\nabla h)\psi_\infty\,dtdxdR-\iint_{\Omega\times B}g^n(t=0)h(t=0)\psi_\infty\,dxdR\\ &=\int_0^T\iint_{\Omega\times B}\left[g^n\nabla u^nR\cdot\nabla_R(h\psi_\infty)+\nabla_R\mathcal{U}.\nabla u^nR\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})h\psi_\infty\right]dtdxdR\\ &-\int_0^T\iint_{\Omega\times B}\psi_\infty\left[\nabla_Rg^n.\nabla_Rh+h\Theta''(\frac{\tilde\psi^n}{\psi_\infty})|\nabla_R\frac{\tilde\psi^n}{\psi_\infty}|^2\right]dtdxdR\\ &-\int_0^T\iint_{\Omega\times B}2ak\nabla u^n:\frac{R\otimes R}{1-|R|^2}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})h\psi_\infty\,dtdxdR.\end{aligned}\tag{80}$$

In view of the density of $C^1(\overline{B})$ in $\mathcal{H}^1_k$ (see Proposition B.2), we see that (80) still holds if $h\in C^1([0,T]\times\overline{\Omega};\mathcal{H}^1_k)\cap C([0,T]\times\overline{\Omega}\times\overline{B})$ and $h(T)=0$. A simple calculation gives

$$\begin{cases}\Theta'(s)=\frac12 s^{-1/2}(\log^{1/2}(s)+\log^{-1/2}(s)),\\ \Theta''(s)=-\frac14 s^{-3/2}(\log^{1/2}(s)+\log^{-3/2}(s)).\end{cases}$$

Now, using that $-\psi_\infty\Theta''(\frac{\tilde\psi^n}{\psi_\infty})|\nabla_R\frac{\tilde\psi^n}{\psi_\infty}|^2=\psi_\infty|\nabla_Rf^n|^2\frac{(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})}{f^n}$, we deduce that this term is bounded in $L^1([0,T]\times\overline{\Omega}\times\overline{B})$. We denote by $\overline{\psi_\infty|\nabla_Rf^n|^2\frac{(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})}{f^n}}$ its weak limit in the sense of measures in $\mathcal{M}([0,T)\times\overline{\Omega}\times\overline{B})$. We recall that for any sequence of measures μ^n, we denote by $\overline{\mu^n}^a$ the part of the weak limit which is absolutely continuous with respect to the Lebesgue measure. Passing to the limit in (80), we deduce that g solves the following inequality in the sense of Proposition B.6 (the definition there was given without the x dependence):

$$\begin{aligned}\partial_t g + u.\nabla g \geq \operatorname{div}_R\Big[-\nabla_i u_j R_j g - \gamma_{ij} R_j\Big] + \nabla_R \mathcal{U}.\nabla u R \overline{\frac{\tilde{\psi}^n}{\psi_\infty}\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})} + \nabla_R \mathcal{U} R : \gamma' \\ + \frac{1}{\psi_\infty}\operatorname{div}_R\Big[\psi_\infty \nabla_R g\Big] + \overline{\frac{|\nabla_R f^n|^2(\log^{1/2} + \log^{-3/2})(\frac{\tilde{\psi}^n}{\psi_\infty})}{f^n}}^a - ak\overline{\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})\nabla u^n} : \frac{2R\otimes R}{1-|R|^2},\end{aligned} \tag{81}$$

with the initial condition $g(t=0,x,R) \geq \Theta(\frac{\tilde{\psi}_0}{\psi_\infty})$ and the boundary condition $\psi_\infty \nabla_R g.\mathbf{n} \geq 0$. We only point out that to pass to the limit in the transport term $u^n.\nabla g^n$, we write it as $\operatorname{div}(g^n u^n)$ and then use Lemma 5.1 of [61]. In the sequel all the transport terms $u.\nabla b$ that we are going to write can be understood as $\operatorname{div}(bu)$.

Notice that we only get an inequality in (81), as well as in the corresponding initial and boundary conditions because the measure $\psi_\infty \overline{|\nabla_R f^n|^2 \frac{(\log^{1/2} + \log^{-3/2})(\frac{\tilde{\psi}^n}{\psi_\infty})}{f^n}}$ may have a singular part, which can only be nonnegative.

Also, notice that (62) does not yield that $g \in L^2((0,T)\times\Omega;\dot{\mathcal{H}}^1_k)$. One has to divide it by $\sqrt{\overline{N^n_2}}$. In Subsection 5.1, we have constructed a renormalizing factor N that satisfies the inequality $\overline{N^n_2} \leq N$ and hence $\tilde{g} = \frac{g}{N^2} \in L^2((0,T)\times\Omega;\dot{\mathcal{H}}^1_k)$ (the reason that we divide by N^2 and not only $\sqrt{N}$ comes from Subsection 5.5). Also, it is easy to see that $\frac{g}{N^2} \in L^\infty((0,T)\times\Omega;\mathcal{L}^2_k)$ since $\int_B g^2 \leq N^2$. Moreover, since N satisfies $(\partial_t + u.\nabla)\frac{1}{N} = 0$, this will allow us to divide (81) by N^2 and deduce the following inequality for $\tilde{g}$:

$$\begin{aligned}\partial_t \tilde{g} + u.\nabla \tilde{g} \geq \operatorname{div}_R\Big[-\nabla_i u_j R_j \tilde{g} - \frac{\gamma_{ij}}{N^2} R_j\Big] \\ + \nabla_R \mathcal{U}.\nabla u R \frac{1}{N^2}\overline{\frac{\tilde{\psi}^n}{\psi_\infty}\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})} + \nabla_R \mathcal{U}\otimes R : \frac{\gamma'}{N^2} \\ - \mathcal{L}(\tilde{g}) + \frac{1}{N^2}\overline{\frac{|\nabla_R f^n|^2(\log^{1/2} + \log^{-3/2})(\frac{\tilde{\psi}^n}{\psi_\infty})}{f^n}}^a \\ - \frac{ak}{N^2}\overline{\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})\nabla u^n} : \frac{2R\otimes R}{1-|R|^2},\end{aligned} \tag{82}$$

with the initial condition $\tilde{g}(t=0,x,R) \geq \frac{1}{N^2}\Theta(\frac{\tilde{\psi}_0}{\psi_\infty})$ and the boundary condition $\psi_\infty \nabla_R \tilde{g}.\mathbf{n} \geq 0$. Actually, to rigorously deduce this inequality from (81) (as well as to perform the next multiplication by $\chi'_\varepsilon(g)$), we have first to mollify (81) and (73) in the x variable and use the regularization Lemma 2.3 of [60] and then pass to the limit. We do not include the details here. It is also important to note that $\frac{\gamma_{ij}}{N^2}, \frac{\gamma'_{ij}}{N^2} \in L^2((0,T)\times\Omega;L^2(\psi_\infty dR))$. Indeed, $|\gamma|, |\gamma'| \leq C(\overline{|\nabla u^n|^2}^a)^{1/2}(\overline{|g^n|^2})^{1/2}$ and $\frac{(\overline{|g^n|^2})^{1/2}}{N} \in L^\infty((0,T)\times\Omega;L^2(\psi_\infty dR))$. This bound on $\frac{\gamma}{N^2}$ is crucial so as to be able to apply Proposition B.6 below. Let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = t^2$ for $0 \leq t \leq \frac{1}{2}$, $\chi(t) = t$ for $t \geq 1$, $\chi'(t) \geq 0$ and $\chi_\varepsilon(t) = \frac{1}{\varepsilon^2}\chi(\varepsilon t)$. By Proposition B.6, we deduce that

$$
\begin{aligned}
\partial_t \chi_\varepsilon(\tilde g) + u.\nabla \chi_\varepsilon(\tilde g) \geq \operatorname{div}_R\Big[- \nabla u R \chi_\varepsilon(\tilde g)\Big] + \nabla u R \cdot \nabla_R \mathcal{U}\overline{\Big(\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\Big)}\frac{\chi'_\varepsilon(\tilde g)}{N^2} \\
- \frac{1}{\psi_\infty}\operatorname{div}_R(\psi_\infty \chi'_\varepsilon(\tilde g)\frac{\gamma R}{N^2}) + \nabla_R \mathcal{U}\otimes R : \frac{\gamma' - \gamma}{N^2}\chi'_\varepsilon(\tilde g) + \frac{\gamma R}{N^2}.\nabla_R \chi'_\varepsilon(\tilde g) \\
- \mathcal{L}(\chi_\varepsilon(\tilde g)) - \chi''_\varepsilon(\tilde g)|\nabla_R \tilde g|^2 + \overline{\frac{|\nabla_R f^n|^2(\log^{1/2} + \log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})|^2}{f^n}}^a \frac{\chi'_\varepsilon(\tilde g)}{N^2} \\
- ak\, \overline{\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\nabla u^n} : \frac{2R\otimes R}{1-|R|^2}\frac{\chi'_\varepsilon(\tilde g)}{N^2},
\end{aligned}
\tag{83}
$$

with the initial condition $\chi_\varepsilon(\tilde g(t=0,x,R)) \geq \chi_\varepsilon(\frac{1}{N^2}\Theta(\frac{\tilde\psi_0}{\psi_\infty}))$ and the boundary condition $\psi_\infty \nabla_R \chi_\varepsilon(\tilde g).\mathbf{n} \geq 0$. The inequality (83) holds in the sense of distributions (see Proposition B.6). Multiplying (83) by ψ_∞ and integrating in R yields (this is equivalent to taking $\phi = 1$ in the weak formulation of (83), see also (153))

$$
\begin{aligned}
(\partial_t + u.\nabla)\int_B \psi_\infty \chi_\varepsilon(\tilde g)\, dR \geq -\nabla u : \tau\left(\psi_\infty\Big(\chi_\varepsilon(\tilde g) - \frac{\chi'_\varepsilon(\tilde g)}{N^2}\overline{\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})}\Big)\right) \\
+ \int_B \Big[\psi_\infty \nabla_R \mathcal{U}\otimes R : \frac{(\gamma'-\gamma)}{N^2}\chi'_\varepsilon(\tilde g) + \psi_\infty \frac{\gamma R}{N^2}.\nabla_R \chi'_\varepsilon(\tilde g)\Big]\, dR \\
+ \int_B \left[\psi_\infty \overline{\frac{|\nabla_R f^n|^2(\log^{1/2} + \log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})|^2}{f^n}}^a \frac{\chi'_\varepsilon(\tilde g)}{N^2} - \psi_\infty \chi''_\varepsilon(\tilde g)|\nabla_R \tilde g|^2\right] dR \\
- \int_B \left[\psi_\infty ak\, \overline{\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\nabla u^n} : \frac{2R\otimes R}{1-|R|^2}\frac{\chi'_\varepsilon(\tilde g)}{N^2}\right] dR,
\end{aligned}
\tag{84}
$$

with the initial data $\int_B \psi_\infty \chi_\varepsilon(\tilde g(t=0))dR \geq \int_B \psi_\infty \chi_\varepsilon(\frac{1}{N^2}\Theta(\frac{\tilde\psi_0}{\psi_\infty}))dR$. We recall that $\tau_{ij}(\psi) = 2k\int_B \psi \frac{R_i R_j}{1-|R|^2}dR$.

On the other hand, passing to the limit in the equation (51) with $(u, \tilde\psi)$ replaced by $(u^n, \tilde\psi^n)$, we get that the following holds in the weak sense (see (10)):

$$
\partial_t \tilde\psi + u.\nabla \tilde\psi = \operatorname{div}_R\Big[- \nabla u \cdot R\tilde\psi - \beta_{ij} R_j\Big] + div_R\Big[\psi_\infty \nabla_R \frac{\tilde\psi}{\psi_\infty}\Big] - 2ak\psi_\infty \nabla u : \frac{R\otimes R}{1-|R|^2}.
\tag{85}
$$

To deduce that $\tilde\psi(t=0) = \tilde\psi_0$ and that $\psi_\infty \nabla_R \frac{\tilde\psi}{\psi_\infty}.\mathbf{n} = 0$ on ∂B, we have to pass to the limit in the weak formulation (10) of (51) with $(u, \tilde\psi)$ replaced by $(u^n, \tilde\psi^n)$. Notice also that since $\partial_t \tilde\psi^n$ is bounded in $L^2(0,T; W^{-1,1}(\Omega\times B))$, we deduce that $\tilde\psi^n$ is equi-continuous in time with values in $W^{-1,1}(\Omega\times B)$. Besides, $\tilde\psi^n \log(\frac{\tilde\psi^n}{\psi_\infty})$ satisfies

$$
\begin{aligned}
(\partial_t + u^n.\nabla)\left[\int_B \tilde\psi^n \log(\frac{\tilde\psi^n}{\psi_\infty})dR\right] = \nabla u^n : \tau(\tilde\psi^n) \\
- 4\int_B \psi_\infty \left|\nabla_R \sqrt{\frac{\tilde\psi^n}{\psi_\infty}}\right|^2 dR - 2ak\int_B \nabla u^n : \frac{R\otimes R}{1-|R|^2}\psi_\infty \log(\frac{\tilde\psi^n}{\psi_\infty})dR.
\end{aligned}
\tag{86}
$$

Here we have used that $\int_B \nabla u^n : \frac{R\otimes R}{1-|R|^2}\psi_\infty dR = C\nabla u^n : Id = 0$ since u^n is divergence-free. We would like to pass to the limit weakly in (86) and deduce that

$$
\begin{aligned}
(\partial_t + u.\nabla)\int_B \overline{\tilde{\psi}^n \log(\frac{\tilde{\psi}^n}{\psi_\infty})}dR &= \nabla u : \tau(\tilde{\psi}) + \int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}dR \\
- 4\int_B \psi_\infty \overline{\left|\nabla_R \sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\right|^2} dR &- 2ak\int_B \psi_\infty \overline{\log(\frac{\tilde{\psi}^n}{\psi_\infty})\nabla u^n} : \frac{R\otimes R}{1-|R|^2}dR.
\end{aligned} \tag{87}
$$

However, we can not use (79) to pass to the limit in $\nabla u^n : \tau(\tilde{\psi}^n) = \int_B \nabla u^n \frac{R_iR_j}{1-|R|^2}\tilde{\psi}^n dR$ and deduce that

$$
\nabla u^n : \tau(\tilde{\psi}^n) \rightharpoonup \nabla u : \tau(\tilde{\psi}) + \int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}dR \tag{88}
$$

since $\nabla_i u_j^n \frac{R_iR_j}{1-|R|^2}\tilde{\psi}^n$ is only bounded in $L^1(dt\, dx\, dR)$. Besides, we can not pass to the limit in the transport term even if we write it in divergence form.

To overcome these difficulties, we will divide (86) by $1+\delta N_2^n$ where N_2^n solves (58) before passing to the limit. Then, we will send δ to zero. To be able to deal with the limit δ to zero, we need to renormalize (86) too. We denote $N_1^n = \int_B \tilde{\psi}^n \log(\frac{\tilde{\psi}^n}{\psi_\infty})dR$ and $\theta_\kappa(s) = \frac{s}{1+\kappa s}$ and recall that $N_2^n = \left(\int_B \tilde{\psi}^n[\log^2(\frac{\tilde{\psi}^n}{\psi_\infty}) - 2\log(\frac{\tilde{\psi}^n}{\psi_\infty}) + 2]dR\right)^{1/2}$. We first multiply (86) by $\theta'_\kappa(N_1^n)$ and get an equation for $\theta_\kappa(N_1^n)$. Dividing the resulting equation by $1+\delta N_2^n$, using (58) and writing the weak formulation, we get, for all $\phi \in C_0^\infty([0,T)\times\Omega)$,

$$
\begin{aligned}
&-\int_\Omega \frac{\theta_\kappa(N_{1,0}^n)}{1+\delta N_{2,0}^n}\phi(t=0)dx - \int_0^T\int_\Omega \frac{\theta_\kappa(N_1^n)}{1+\delta N_2^n}(\partial_t + u.\nabla)\phi\, dxdt \\
&= \int_0^T\int_\Omega \left[\frac{\nabla u^n : \tau(\tilde{\psi}^n)}{(1+\delta N_2^n)(1+\kappa N_1^n)^2} - \frac{4}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}\int_B \psi_\infty \left|\nabla_R\sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\right|^2 dR\right]\phi dxdt \\
&- \int_0^T\int_\Omega \left[\frac{2ak}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}\int_B \nabla u^n : \frac{R\otimes R}{1-|R|^2}\psi_\infty \log(\frac{\tilde{\psi}^n}{\psi_\infty})dR\right]\phi dxdt \\
&- \int_0^T\int_\Omega \left[\frac{\delta F^n}{(1+\delta N_2^n)^2}\theta_\kappa(N_1^n)\right]\phi dxdt.
\end{aligned}
$$

Taking the weak limit when n goes to infinity (extracting a subsequence if necessary), we get, for $\kappa,\delta > 0$,

$$
\begin{aligned}
(\partial_t + u.\nabla)\overline{\frac{\theta_\kappa(N_1^n)}{1+\delta N_2^n}} =& \overline{\frac{\nabla u^n : \tau(\tilde{\psi}^n)}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}} - \overline{\frac{4}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}\int_B \psi_\infty\left|\nabla_R\sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\right|^2 dR} \\
&- \overline{\frac{2ak}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}\int_B \nabla u^n : \frac{R\otimes R}{1-|R|^2}\psi_\infty \log(\frac{\tilde{\psi}^n}{\psi_\infty})dR} \\
&- \overline{\frac{\delta F^n}{(1+\delta N_2^n)^2}}\theta_\kappa(N_1^n).
\end{aligned} \tag{89}
$$

The last term on the right-hand side of (89) is bounded in $\mathcal{M}([0,T)\times\Omega)$ by $\frac{\delta}{\kappa}\overline{|F^n|}$. Hence, if we denote by F_0 the Dirac part of this measure at time $t=0$, we deduce that the initial condition satisfies $\overline{\frac{\theta_\kappa(N_1^n)}{1+\delta N_2^n}}(t=0) \le \frac{\theta_\kappa(N_{1,0})}{1+\delta N_{2,0}} + \frac{\delta}{\kappa}F_0$. The fact that we get only an inequality comes from the fact that the sum of the other terms on the right-hand side of (89) is a measure in $\mathcal{M}([0,T)\times\Omega)$, which has a non-positive singular part. Indeed, as we will see from Corollary 5.3, the first term on the right-hand of (89) is in $L^1([0,T)\times\Omega)$. Moreover, it is easy to see that the third one is in $L^p_{loc}([0,T)\times\Omega)$ for all $p<2$.

Before sending δ and κ to zero, we divide the inequality we get after keeping only the regular part with respect to the Lebesgue measure in (89) by N^4. For $\phi \in C_0^\infty([0,T)\times\Omega)$, $\phi \ge 0$, we have

$$
\begin{aligned}
&-\int_\Omega \overline{\frac{\theta_\kappa(N_{1,0}^n)}{1+\delta N_{2,0}^n}}\frac{\phi(t=0)}{N^4}dx - \int_0^T\int_\Omega \frac{1}{N^4}\overline{\frac{\theta_\kappa(N_1^n)}{1+\delta N_2^n}}(\partial_t + u.\nabla)\phi\, dxdt \\
&\le \int_0^T\int_\Omega \frac{1}{N^4}\left[\overline{\frac{\nabla u^n : \tau(\tilde\psi^n)}{N^{\delta,\kappa}}} - \overline{\frac{4}{N^{\delta,\kappa}}\int_B \psi_\infty\left|\nabla_R\sqrt{\frac{\tilde\psi^n}{\psi_\infty}}\right|^2 dR}^{a}\right]\phi dxdt \\
&- \int_0^T\int_\Omega \frac{1}{N^4}\left[\overline{\frac{2ak}{N^{\delta,\kappa}}\int_B \nabla u^n : \frac{R\otimes R}{1-|R|^2}\psi_\infty \log(\frac{\tilde\psi^n}{\psi_\infty})dR}\right]\phi dxdt + C\frac{\delta}{\kappa}\sup\phi,
\end{aligned} \tag{90}
$$

where we have denoted $N^{\delta,\kappa} = (1+\delta N_2^n)(1+\kappa N_1^n)^2$. Now, we can send δ to zero. Notice that due to the fact that $\theta_\kappa(N_1^n)$ is bounded by $1/\kappa$ and that F^n is bounded in L^1, we deduce that the last term goes to zero when δ goes to zero. Then, we send κ to zero and recover in the limit

$$
\begin{aligned}
(\partial_t + u.\nabla)\frac{\theta}{N^4} \le& \frac{1}{N^4}\overline{\nabla u^n : \tau(\tilde\psi^n)}^{\delta,\kappa} - \overline{\frac{4}{N^4}\int_B \psi_\infty\left|\nabla_R\sqrt{\frac{\tilde\psi^n}{\psi_\infty}}\right|^2 dR}^{\delta,\kappa,a} \\
&- \frac{2ak}{N^4}\int_B \overline{\nabla u^n : \frac{R\otimes R}{1-|R|^2}\psi_\infty \log(\frac{\tilde\psi^n}{\psi_\infty})}^{\delta,\kappa} dR,
\end{aligned} \tag{91}
$$

where $\theta = \lim_{\kappa\to 0}\lim_{\delta\to 0}\overline{\frac{\theta_\kappa(N_1^n)}{1+\delta N_2^n}} = \lim_{\kappa\to 0}\overline{\theta_\kappa(N_1^n)}$ is the Chacon limit of N_1^n and

$$
\overline{F^n}^{\delta,\kappa} = \lim_{\kappa\to 0}\lim_{\delta\to 0}\overline{\frac{F^n}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}}
$$

for any sequence F^n bounded in L^1 and $\overline{F^n}^{\delta,\kappa,a}$ is the part of that measure which is absolutely continuous with respect to the Lebesgue measure. It is worth pointing out (see also Remark 3.10) that for any sequence F^n bounded in L^1, we have $\overline{F^n} \ge \overline{F^n}^{\delta,\kappa} \ge \overline{F^n}^{\delta,\kappa,a} \ge \mathrm{ch}(F^n)$. Also, we deduce from the limit of (90) that $\theta(t=0) \le N_{1,0} = \int_B \tilde\psi_0 \log(\frac{\tilde\psi_0}{\psi_\infty})dR$. Since $\sqrt{\frac{\psi^n}{\psi_\infty}}$ is bounded in $L^r((0,T)\times\Omega\times B, dtdx\psi_\infty dR)$ for some $r>2$, we deduce that N_1^n is equi-integrable and hence θ, which is the Chacon limit of N_1^n, is equal to the weak limit of N_1^n. Indeed, it is easy to see that if F^n is equi-integrable then $\overline{F^n}^{\delta,\kappa} = \overline{F^n}$. Also, we can invert the integral sign and the weak limit and get that $\theta = \overline{N_1^n} = \int_B \overline{\tilde\psi^n \log(\frac{\tilde\psi^n}{\psi_\infty})}dR$ since the sequence $\tilde\psi^n \log(\frac{\tilde\psi^n}{\psi_\infty})$ is equi-integrable.

5.3. **The term $\overline{\nabla u^n : \tau(\tilde\psi^n)}^{\delta,\kappa}$.** In this subsection, we will prove that $\overline{\nabla u^n : \tau(\tilde\psi^n)}^{\delta,\kappa} = \nabla u : \tau + \int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}$. This will follow from the following two lemmas

Lemma 5.1.

$$\overline{\frac{\nabla u^n : \tau(\tilde{\psi}^n)}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}} = \int_B z^{\delta,\kappa} : \frac{R\otimes R}{1-|R^2|}\psi_\infty dR, \tag{92}$$

where $z^{\delta,\kappa} = \overline{\frac{\tilde{\psi}^n\,\nabla u^n}{\psi_\infty(1+\delta N_2^n)(1+\kappa N_1^n)^2}}$

Lemma 5.2. $z^{\delta,\kappa}$ *converges strongly to* $\overline{\nabla u^n \tilde{\psi}^n} = \nabla u\tilde{\psi}+\beta$ *in* $L^1((0,T)\times\Omega\times B; dt\,dx\,\psi_\infty \frac{dR}{1-|R|})$ *when* δ *goes to zero and then* κ *goes to zero.*

Denoting $\tau^{n,\delta,\kappa} = \frac{\tau(\tilde{\psi}^n)}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}$, we get that

Corollary 5.3.

$$\overline{\nabla u^n : \tau(\tilde{\psi}^n)}^{\delta,\kappa} = \lim_{\kappa\to 0}\lim_{\delta\to 0}\overline{\nabla u^n : \tau^{n,\delta,\kappa}} = \nabla u : \tau(\psi) + \int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}dR. \tag{93}$$

Proof of Lemma 5.1. The proof of (92) follows from the fact that $z^{n,\delta,\kappa} = \frac{\nabla u^n \tilde{\psi}^n}{\psi_\infty(1+\delta N_2^n)(1+\kappa N_1^n)^2}$ is equi-integrable in $L^1((0,T)\times\Omega\times B; dt\,dx\,\frac{\psi_\infty dR}{1-|R|})$ for δ,κ fixed. Indeed, consider the real valued function $\Phi(x) = x\log(1+x)+1$. It is enough to prove that $\Phi(|z^{n,\delta,\kappa}|) = \Phi\left(\frac{|\nabla u^n\tilde{\psi}^n|}{\psi_\infty(1+\delta N_2^n)(1+\kappa N_1^n)^2}\right)$ is bounded in $X = L^1((0,T)\times\Omega\times B; dt\,dx\,\frac{\psi_\infty dR}{1-|R|})$. To simplify the notation, we denote $N^n = (1+\delta N_2^n)(1+\kappa N_1^n)^2$. Hence, it is enough to bound

$$\frac{|\nabla u^n|}{N^n}\left[\frac{\tilde{\psi}^n}{\psi_\infty}\log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right) + \frac{\tilde{\psi}^n}{\psi_\infty}\log\left(3+\frac{|\nabla u^n|}{N^n}\right)\right] \tag{94}$$

in X (see definition above).

To bound the first term appearing in (94) we use the Hardy type inequality (18) to get that

$$\begin{aligned}
&\frac{|\nabla u^n|}{N^n}\int_B \tilde{\psi}^n \log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)\frac{1}{1-|R|}\,dR\\
&\lesssim \frac{|\nabla u^n|}{N^n}\left[\int_B \tilde{\psi}^n\log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)dR\right]^{1/2}\left[\int_B \psi_\infty\left|\nabla\Big(\sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\log^{1/2}\frac{\tilde{\psi}^n}{\psi_\infty}\Big)dR\right|^2\right]^{1/2}\\
&\lesssim |\nabla u^n|^2 + \frac{1}{(N^n)^2}\left[\int_B \tilde{\psi}^n\log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)dR\right]\left[\int_B \psi_\infty\left|\nabla\Big(\sqrt{\frac{\tilde{\psi}^n}{\psi_\infty}}\log^{1/2}\frac{\tilde{\psi}^n}{\psi_\infty}\Big)dR\right|^2\right],
\end{aligned} \tag{95}$$

and using the a priori bound (57), we see that the last term is uniformly bounded in $L^1((0,T)\times\Omega)$. To bound the second term in (94), we first use the inequality $x\,y \le C(x^2\log^2(x) + \frac{y^2}{\log^2 y})$ for $x,y\ge 2$ with $x = \int_B \frac{\tilde{\psi}^n}{\psi_\infty}\frac{\psi_\infty}{1-|R|}\,dR$ and $y = \left(3+\frac{|\nabla u^n|}{N^n}\right)\log\left(3+\frac{|\nabla u^n|}{N^n}\right)$ and then apply Jensen's inequality. Hence,

$$\begin{aligned}
&\frac{|\nabla u^n|}{N^n}\log\left(3+\frac{|\nabla u^n|}{N^n}\right)\int_B \frac{\tilde{\psi}^n}{\psi_\infty}\frac{\psi_\infty}{1-|R|}\,dR \\
&\lesssim 1+|\nabla u^n|^2+\frac{1}{(N^n)^2}\left[\int_B \frac{\tilde{\psi}^n}{\psi_\infty}\frac{\psi_\infty}{1-|R|}\,dR\right]^2\log^2\left[\int_B \frac{\tilde{\psi}^n}{\psi_\infty}\frac{\psi_\infty}{1-|R|}\,dR\right] \\
&\lesssim 1+|\nabla u^n|^2+\frac{1}{(N^n)^2}\left[\int_B \frac{\tilde{\psi}^n}{\psi_\infty}\log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)\frac{\psi_\infty}{1-|R|}\,dR\right]^2,
\end{aligned} \tag{96}$$

and the last term can be bounded as in (95). We notice here that the last inequality implies in particular that $|\tau^{n,\delta,\kappa}|^2$ is equi-integrable in L^1 for fixed δ and κ. This is actually a very important fact that will be used again later.

Proof of Lemma 5.2. To prove this lemma, we use the dominated convergence and monotone convergence theorems. Indeed, $|z^{n,\delta,\kappa}|$ is decreasing in δ,κ, namely for $0<\delta\le\delta'$ and $0<\kappa\le\kappa'$, we have $|z^{n,\delta',\kappa'}|\le|z^{n,\delta,\kappa}|\le|\nabla u^n\tilde{\psi}^n|$. Passing to the limit weakly in n, we deduce that

$$\overline{|z^{n,\delta',\kappa'}|}\le\overline{|z^{n,\delta,\kappa}|}\le\overline{|\nabla u^n\tilde{\psi}^n|}$$

and by monotone convergence, we deduce that $G=\overline{\overline{|z^{n,\delta,\kappa}|}}^{\delta,\kappa}\in X$ and that for all $0<\delta$ and $0<\kappa$, we have $|z^{\delta,\kappa}|\le G$. Moreover, we have, for $0<\delta\le\delta'$ and $0<\kappa\le\kappa'$,

$$|z^{\delta,\kappa}-z^{\delta',\kappa'}|\le\left|\overline{|z^{n,\delta,\kappa}|}-\overline{|z^{n,\delta',\kappa'}|}\right|. \tag{97}$$

Indeed, recall that

$$|z^{\delta,\kappa}-z^{\delta',\kappa'}|=\left|\overline{z^{n,\delta,\kappa}-z^{n,\delta',\kappa'}}\right|\le\overline{|z^{n,\delta,\kappa}-z^{n,\delta',\kappa'}|}$$

where we have used that $x\to|x|$ is convex. Now, notice that for each t, x, R, $0<\delta\le\delta'$, $0<\kappa\le\kappa'$ and n, there exists $0<\lambda\le1$ such that $z^{n,\delta',\kappa'}=\lambda z^{n,\delta,\kappa}$. Hence, $|z^{n,\delta,\kappa}-z^{n,\delta',\kappa'}|=|z^{n,\delta,\kappa}|-|z^{n,\delta',\kappa'}|$ and (97) follows.

Hence, there exists $g\in X$ such that $z^{\delta,\kappa}$ converges strongly to g in X. Now, we would like to prove that the limit g is equal to $\overline{\nabla u^n\tilde{\psi}^n}$. This follows from the fact that $\nabla u^n\tilde{\psi}^n$ is equi-integrable in $L^1((0,T)\times\Omega\times B;dt\,dx\,dR)$ (without the weight). Indeed, denoting $\Phi(x)=|x|\log^{1/2}(1+|x|)$, we have

$$\begin{aligned}
\Phi(|\nabla u^n\tilde{\psi}^n|)&\lesssim|\nabla u^n\tilde{\psi}^n|\Big[\log(1+\tilde{\psi}^n)+\log(1+|\nabla u^n|)\Big] \\
&\lesssim\tilde{\psi}^n\Big[|\nabla u^n|^2+\log(1+\tilde{\psi}^n)\Big],
\end{aligned}$$

which is clearly a bounded sequence in $L^1(dt\,dx\,dR)$. Indeed, for the first term, we use that $\tilde{\psi}^n$ is a bounded sequence in $L^\infty(dt\,dx;L^1(dR))$ and that $|\nabla u^n|^2$ is a bounded sequence in $L^1(dt\,dx)$. For the second term, we use that $\tilde{\psi}^n\log(1+\tilde{\psi}^n)\lesssim\tilde{\psi}^n+\tilde{\psi}^n\log\left(\frac{\tilde{\psi}^n}{\psi_\infty}\right)$.

5.4. Identification of $\int_B\beta_{ij}\frac{R_iR_j}{1-|R^2|}dR$.

In this subsection, we give a relation between β and some defect measure related to the lack of strong convergence of ∇u^n in L^2. First, we give we consider the case of a smooth bounded domain Ω. To state the main proposition of this subsection, we introduce a few notations. Let $u^n=v^n+w^n$, where v^n and w^n solve

$$\begin{cases}\partial_t v^n-\Delta v^n+\nabla p_1^n=\nabla.\tau^n & \text{in}\quad\Omega\\ v^n(t=0)=0,\qquad v^n=0 & \text{on}\quad\partial\Omega,\end{cases} \tag{98}$$

(99) $$\begin{cases} \partial_t w^n - \Delta w^n + \nabla p_2^n = -u^n.\nabla u^n & \text{in} \quad \Omega \\ w^n(t=0) = u^n(t=0), \qquad w^n = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We further split w^n into $w_1^n + w_2^n$ where w_1^n is the solution with zero initial data and w_2^n is the solution with zero right-hand side.

In the rest of this subsection we will use $\boldsymbol{\delta}$ to denote the pair δ, κ. We define $v^{n,\boldsymbol{\delta}} = v^{n,\delta,\kappa}$ the solution of

(100) $$\begin{cases} \partial_t v^{n,\boldsymbol{\delta}} - \Delta v^{n,\boldsymbol{\delta}} + \nabla p_1^{n,\boldsymbol{\delta}} = \nabla.\tau^{n,\boldsymbol{\delta}} & \text{in} \quad \Omega \\ v^{n,\boldsymbol{\delta}}(t=0) = 0, \qquad v^{n,\boldsymbol{\delta}} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Extracting a subsequence, we assume that $(\tau^{n,\boldsymbol{\delta}}, \nabla v^{n,\boldsymbol{\delta}}, \nabla v^n, \nabla w^n)$ converges weakly in L^2 to some $(\tau^{\boldsymbol{\delta}}, \nabla v^{\boldsymbol{\delta}}, \nabla v, \nabla w)$ and that

$$\overline{|\nabla v^{n,\boldsymbol{\delta}}|^2} = |\nabla v^{\boldsymbol{\delta}}|^2 + \mu^{\boldsymbol{\delta}}$$

for some defect measure $\mu^{\boldsymbol{\delta}} \in \mathcal{M}((0,T)\times\Omega)$. We also denote by μ the limit of $\mu^{\boldsymbol{\delta}}$, when δ and then κ go to zero in the sense of measures (extracting a subsequence if necessary), namely

$$\mu = \lim_{\kappa\to 0}\lim_{\delta\to 0}\mu^{\boldsymbol{\delta}} = \lim_{\boldsymbol{\delta}\to 0}\mu^{\boldsymbol{\delta}}.$$

The fact that this limit exists comes from the fact that $\mu^{\boldsymbol{\delta}}$ is uniformly bounded in $\mathcal{M}((0,T)\times\Omega)$, since its total mass is controlled by the initial free energy.

Proposition 5.4. *We have*

$$\mu = -\int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}dR.$$

Proof. We introduce the following weak limits

$$\overline{\tau^{n,\boldsymbol{\delta}} : \nabla v^{n,\boldsymbol{\delta}}} = W^{\boldsymbol{\delta}\boldsymbol{\delta}},$$
$$\overline{\tau^{n,\boldsymbol{\delta}} : \nabla v^{n}} = W^{\boldsymbol{\delta}}.$$

Step 1: First, we would like to prove that $W^{\boldsymbol{\delta}\boldsymbol{\delta}}$ and $W^{\boldsymbol{\delta}}$ have the same limit W when $\boldsymbol{\delta}$ goes to zero and that this limit is in $L^1((0,T)\times\Omega)$. To prove this, we introduce, for $M>0$, the following weak limits:

$$\overline{\tau^{n,\boldsymbol{\delta}}1_{|\tau^{n,\boldsymbol{\delta}}|\le M} : \nabla v^{n,\boldsymbol{\delta}}} = W_M^{\boldsymbol{\delta}\boldsymbol{\delta}},$$
$$\overline{\tau^{n,\boldsymbol{\delta}}1_{|\tau^{n,\boldsymbol{\delta}}|> M} : \nabla v^{n,\boldsymbol{\delta}}} = W^{\boldsymbol{\delta}\boldsymbol{\delta}} - W_M^{\boldsymbol{\delta}\boldsymbol{\delta}},$$
$$\overline{\tau^{n,\boldsymbol{\delta}}1_{|\tau^{n,\boldsymbol{\delta}}|\le M} : \nabla v^{n}} = W_M^{\boldsymbol{\delta}},$$
$$\overline{|\tau^{n,\boldsymbol{\delta}}1_{|\tau^{n,\boldsymbol{\delta}}|\le M}|^2} = G_M^{\boldsymbol{\delta}} \qquad \text{and} \qquad \overline{|\tau^{n,\boldsymbol{\delta}}|^2} = G^{\boldsymbol{\delta}}.$$

Since for a fixed $\boldsymbol{\delta}$, $|\tau^{n,\boldsymbol{\delta}}|^2$ is equi-integrable, we deduce that $G_M^{\boldsymbol{\delta}}$ converges to $G^{\boldsymbol{\delta}}$ in $L^1((0,T)\times\Omega)$ when M goes to infinity and is monotone in M. Also, by monotone convergence, we deduce that there exists $G\in L^1$ such that $G^{\boldsymbol{\delta}}$ converges to G in $L^1((0,T)\times\Omega)$ when $\boldsymbol{\delta}$ goes to zero. Actually, G is the weak limit of $|\tau^n|^2$ in the sense of Chacon.

Let us fix $\varepsilon>0$. We choose $\boldsymbol{\delta}_0$ and M_0 such that for $\boldsymbol{\delta}<\boldsymbol{\delta}_0$ and $M>M_0$, we have $\|G-G^{\boldsymbol{\delta}}\|_{L^1} + \|G-G_M^{\boldsymbol{\delta}}\|_{L^1}\le\varepsilon$. We have

$$\begin{aligned}\overline{|\tau^{n,\boldsymbol{\delta}}|^2} &= \overline{|\tau^{n,\boldsymbol{\delta}}1_{|\tau^{n,\boldsymbol{\delta}}|\le M}|^2} + \overline{|\tau^{n,\boldsymbol{\delta}}1_{|\tau^{n,\boldsymbol{\delta}}|> M}|^2} \\ &= G_M^{\boldsymbol{\delta}} + \quad (G^{\boldsymbol{\delta}} - G_M^{\boldsymbol{\delta}}).\end{aligned}$$

Hence, we deduce that for $\boldsymbol{\delta}<\boldsymbol{\delta}_0$ and $M>M_0$, we have for all n, $\|\tau^{n,\boldsymbol{\delta}}1_{|\tau^{n,\boldsymbol{\delta}}|>M}\|_{L^2}^2\le\varepsilon$ and hence, by Cauchy-Schwarz we deduce that $\|W^{\boldsymbol{\delta}\boldsymbol{\delta}} - W_M^{\boldsymbol{\delta}\boldsymbol{\delta}}\|_{L^1}\le C\sqrt{\varepsilon}$ and that $\|W^{\boldsymbol{\delta}} -$

$W^{\boldsymbol\delta}_M\|_{L^1} \le C\sqrt{\varepsilon}$. Hence to prove that $\lim_{\boldsymbol\delta} W^{\boldsymbol\delta\boldsymbol\delta} = \lim_{\boldsymbol\delta} W^{\boldsymbol\delta}$, it is enough to prove it for the M approximation, namely that

$$\lim_{\boldsymbol\delta} W^{\boldsymbol\delta\boldsymbol\delta}_M = \lim_{\boldsymbol\delta} W^{\boldsymbol\delta}_M. \tag{101}$$

To prove (101), we first notice that since Ω is a bounded domain, we have that $\tau^{n,\boldsymbol\delta} - \tau^n$ goes to zero in $L^p((0,T)\times\Omega)$ for $p<2$ when $\boldsymbol\delta$ goes to zero uniformly in n. Then, by parabolic regularity of the Stokes system (see (8.5) in [2] as well as the earlier works [36, 80]), we deduce that $\|\nabla v^{n,\boldsymbol\delta} - \nabla v^n\|_{L^p((0,T)\times\Omega)}$ goes to zero when $\boldsymbol\delta$ goes to zero uniformly in n for $p<2$. Hence, (101) holds.

Step 2: In this second step, we will compare the local energy identity of the weak limit of (100) with the weak limit of the local energy identity of (100).

On one hand, passing to the limit in (100) and multiplying by $v^{\boldsymbol\delta}$, we deduce that

$$\partial_t \frac{|v^{\boldsymbol\delta}|^2}{2} - \Delta\frac{|v^{\boldsymbol\delta}|^2}{2} + |\nabla v^{\boldsymbol\delta}|^2 + \mathrm{div}(p_1^{\boldsymbol\delta} v^{\boldsymbol\delta}) = \mathrm{div}(v^{\boldsymbol\delta}.\tau^{\boldsymbol\delta}) - \nabla v^{\boldsymbol\delta} : \tau^{\boldsymbol\delta} \tag{102}$$

On the other hand, reversing the order, we get

$$\partial_t \frac{|v^{\boldsymbol\delta}|^2}{2} - \Delta\frac{|v^{\boldsymbol\delta}|^2}{2} + |\nabla v^{\boldsymbol\delta}|^2 + \mu_{\boldsymbol\delta} + \mathrm{div}(p_1^{\boldsymbol\delta} v^{\boldsymbol\delta}) = \mathrm{div}(v^{\boldsymbol\delta}.\tau^{\boldsymbol\delta}) - W^{\boldsymbol\delta\boldsymbol\delta}. \tag{103}$$

For a justification of these two calculations, we refer to [63]. Comparing (102) and (103), we deduce that $W^{\boldsymbol\delta\boldsymbol\delta} = \nabla v^{\boldsymbol\delta} : \tau^{\boldsymbol\delta} - \mu_{\boldsymbol\delta}$. We would like now to send $\boldsymbol\delta$ to zero.

First, it is clear that $\tau^{\boldsymbol\delta}$ converges strongly to τ in $L^2((0,T)\times\Omega)$ when $\boldsymbol\delta$ goes to zero. Hence, $\nabla v^{\boldsymbol\delta}$ also converges to ∇v in $L^2((0,T)\times\Omega)$. Besides, from the energy estimate, we recall that u^n is bounded in $L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);\dot H^1(\Omega))$ and hence by Sobolev embeddings that u^n is bounded in $L^{\frac{2(D+2)}{D}}((0,T)\times\Omega)$ and that $u^n\nabla u^n$ is bounded in $L^{\frac{D+2}{D+1}}((0,T)\times\Omega)$. By parabolic regularity of the Stokes operator applied to (99) with zero initial data, we deduce that ∇w_1^n is bounded in $L^{\frac{D+2}{D+1}}((0,T);W^{1,\frac{D+2}{D+1}}\Omega)$ and that $\partial_t w_1^n$ is bounded in $L^{\frac{D+2}{D+1}}((0,T)\times\Omega)$. Since τ^n is bounded in $L^2((0,T)\times\Omega)$, we deduce from (98) that ∇v^n is also bounded in $L^2((0,T)\times\Omega)$ and hence ∇w^n is also bounded in $L^2((0,T)\times\Omega)$. Moreover, it is clear that ∇w_2^n is compact in $L^2((0,T)\times\Omega)$ and hence ∇w_1^n is also bounded in $L^2((0,T)\times\Omega)$ and from the previous bounds on ∇w_1^n, we deduce that ∇w_1^n is compact in $L^p((0,T)\times\Omega)$ for $p<2$. Hence, we deduce that $\overline{\nabla w^n : \tau(\tilde\psi^n)}^{\delta,\kappa} = \nabla w : \tau(\psi)$ (where we have used that $\tau^{n,\boldsymbol\delta}$ is equi-integrable for each fixed $\boldsymbol\delta$) and from Corollary 5.3 that $\lim_{\boldsymbol\delta} W^{\boldsymbol\delta\boldsymbol\delta} = \lim_{\boldsymbol\delta} W^{\boldsymbol\delta} = \overline{\nabla v^n : \tau(\tilde\psi^n)}^{\delta,\kappa} = \nabla v : \tau(\psi) + \int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}dR$. Finally, we deduce that $\mu = \lim_{\boldsymbol\delta\to0}\mu^{\boldsymbol\delta} = -\int_B \beta_{ij}\frac{R_iR_j}{1-|R^2|}dR$.

Remark 5.5. *For later application, it is worth noting that if h^n is a sequence bounded in $L^2((0,T)\times\Omega\times B)$ and such that $|h^n|^2$ is equi-integrable, then $|\overline{\nabla u^n h^n} - \nabla u h| = |\overline{\nabla v^n h^n} - \nabla v h| \le C\mu\overline{(h^n-h)^2}^{1/2}$. In particular one can deduce (see Remark 3.10), that $ch(|\nabla u^n - \nabla u|^2) = ch(|\nabla v^n - \nabla v|^2) \le \mu$.*

Let us now indicate the changes to be made in the whole space case. The only problem is that we do not necessarily have that τ^n is in $L^p((0,T)\times\Omega)$ for some $p<2$ and hence the argument after (101) does not hold. One has just to localize the last argument at the end of Step 1. This can be done in several different ways and we leave this to the reader. One way is to use the parabolic regularity of the Stokes operator in L^p_{loc}. □

5.5. **Gronwall along the characteristics.** We recall that $g^n = \Theta(\frac{\tilde\psi^n}{\psi_\infty})$ where $\Theta(t) = t^{1/2}\log^{1/2}(t)$ and that g is the weak limit of g^n in $L^2((0,T);L^2(\Omega\times B, dx\psi_\infty dR))$. We also denote $\eta = \overline{N_1^n} - \int_B g^2\psi_\infty = \int_B[\overline{(g^n)^2} - g^2]\psi_\infty dR$. Hence $\frac{\eta}{N^4} = \lim_{\varepsilon\to0}\eta_\varepsilon$ weakly in

$L^2_{loc}((0,T)\times\Omega)$ where $\eta_\varepsilon = \frac{\theta}{N^4} - \int_B \psi_\infty \chi_\varepsilon(\tilde g)dR$. In particular η measures the lack of strong convergence of g^n to g in $L^2((0,T)\times\Omega;\mathcal{L}^2_k)$. Notice also that by the choice of the renormalizing factor N, the defect measure $\frac{\eta}{N}$ is in $L^\infty((0,T)\times\Omega)$. The following Proposition holds.

Proposition 5.6. *For a.e. $x\in\Omega$, the following holds in $\mathcal{D}'(0,T)$:*

$$\frac{d}{dt}\Big(\frac{\eta}{N^4}(t,X(t,x))\Big) \lesssim \left[1+\int_B \psi_\infty \frac{|\nabla_R g|^2}{N} + \frac{|\nabla u|^2}{N^3}\right]\frac{\eta}{N}(1+|\log(\frac{\eta}{N})|)(t,X(t,x)) \tag{104}$$

and $\frac{\eta}{N^4}(0,X(0,x))=0$ for a.e. $x\in\Omega$. Besides, the factor $(1+|\log(\frac{\eta}{N})|)$ is not needed if $k>1$.

Proof. Taking the difference between (91) and (84), we get that the inequality

(105)
$$\begin{aligned}(\partial_t + u.\nabla)\eta_\varepsilon \le\ & \frac{1}{N^4}\overline{\nabla u^n : \tau(\tilde\psi^n)}^{\delta,\kappa} + \nabla u : \tau\left(\psi_\infty\Big(\chi_\varepsilon(\tilde g) - \frac{\chi'_\varepsilon(\tilde g)}{N^2}\overline{\frac{\tilde\psi^n}{\psi_\infty}\Theta'(\frac{\tilde\psi^n}{\psi_\infty})}\Big)\right)\\
&- \frac{4}{N^4}\overline{\int_B \psi_\infty|\nabla_R f^n|^2}^{\delta,\kappa,a} - \int_B\psi_\infty\left[\overline{\frac{|\nabla_R f^n|^2(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})}{f^n}}^{a}\frac{\chi'_\varepsilon(\tilde g)}{N^2} - \chi''_\varepsilon(\tilde g)|\nabla_R\tilde g|^2\right]dR\\
&- 2ak\int_B\psi_\infty\left[\left(\frac{1}{N^4}\overline{\Big(\log(\frac{\tilde\psi^n}{\psi_\infty})+1\Big)\nabla u^n}^{\delta,\kappa} - \frac{\chi'_\varepsilon(\tilde g)}{N^2}\overline{\Theta'(\frac{\tilde\psi^n}{\psi_\infty})\nabla u^n}\right) : \frac{R\otimes R}{1-|R|^2}\right]dR\\
&- \int_B\psi_\infty\left[\nabla_R\mathcal{U}\otimes R : \frac{(\gamma'-\gamma)}{N^2}\chi'_\varepsilon(\tilde g) + \frac{\gamma R}{N^2}.\nabla_R\chi'_\varepsilon(\tilde g)\right]dR\end{aligned}$$

holds in $\mathcal{D}'((0,T)\times\Omega)$ where we recall that $\eta_\varepsilon = \frac{\theta}{N^4} - \int_B\psi_\infty\chi_\varepsilon(\tilde g)dR$ and $\eta_\varepsilon(t=0)\le \frac{N_{1,0}}{N^4} - \int\psi_\infty\chi_\varepsilon(\frac{1}{N^2}\Theta(\frac{\tilde\psi_0}{\psi_\infty}))\,dR$. We denote by $-A_i^\varepsilon$, $1\le i\le 4$, the term appearing on the i-th line of the right-hand side of (105). We would like to send ε to zero and deduce that (106) holds. It is not difficult to prove (see also the proof of Lemma 5.7 below) that all the terms appearing on the right-hand side of (105) are bounded in $L^1((0,T)\times K)$ for any compact set K of Ω (and actually dominated by a fixed L^1_{loc} function $G(t,x)$), except the term $\overline{\frac{|\nabla_R f^n|^2(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})|^2}{f^n}}^{a}\frac{\chi'_\varepsilon(\tilde g)}{N^2}$ which is nonnegative. Testing (105) with a nonnegative test function, we deduce that for $t<T$ we have $\int_0^t\int_K \overline{\frac{|\nabla_R f^n|^2(\log^{1/2}+\log^{-3/2})(\frac{\tilde\psi^n}{\psi_\infty})|^2}{f^n}}^{a}\frac{\chi'_\varepsilon(\tilde g)}{N^2}\le C_{t,K}$, independent of ε. Hence, we can use Fatou's Lemma to pass to the limit in that term and use dominated convergence to pass to the limit in the remaining terms and deduce that

$$
\begin{aligned}
(\partial_t + u.\nabla)\frac{\eta}{N^4} &\le -\sum_{i=1}^{4} A_i(t,x) \\
&= \frac{1}{N^4}\left[\overline{\nabla u^n : \tau(\tilde{\psi}^n)}^{\delta,\kappa} + \nabla u : \tau\left(\psi_\infty\Big(g^2 - 2g\overline{\frac{\tilde{\psi}^n}{\psi_\infty}\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})}\Big)\right)\right] \\
&- \frac{4}{N^4}\overline{\int_B \psi_\infty |\nabla_R f^n|^2}^{\delta,\kappa,a} + \frac{1}{N^4}\int_B \psi_\infty\left[2|\nabla_R g|^2 - \frac{\overline{|\nabla_R f^n|^2(\log^{1/2}+\log^{-3/2})(\frac{\tilde{\psi}^n}{\psi_\infty})}^{a}}{f^n}\,2g\right]dR \qquad (106)\\
&- \frac{2ak}{N^4}\int_B \psi_\infty\left(\overline{\Big(\log(\frac{\tilde{\psi}^n}{\psi_\infty})+1\Big)\nabla u^n}^{\delta,\kappa} - \overline{\Big(2\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})\nabla u^n\Big)}g\right) : \frac{R\otimes R}{1-|R|^2}\,dR \\
&+ \frac{2}{N^4}\int \psi_\infty \left|\gamma : R\otimes\nabla g - \nabla_R\mathcal{U}\otimes R : (\gamma-\gamma')g\right| dR
\end{aligned}
$$

and $\frac{\eta}{N^4}(0, X(0,x)) \le 0$ for a.e. $x \in \Omega$. Of course, one has to pass to the limit in the weak formulation of (105) with a test function $\phi \in C^\infty([0,T)\times\Omega)$ to also deduce the bound on the initial data. We skip the details. Here, we denote the terms appearing on the $(i{+}1)$st line of (106) by $-A_i, 1 \le i \le 4$. Notice that, from Proposition A.5, one can understand (106) in different ways. In particular for a.e. $x \in \Omega$, $\frac{\eta}{N^4}(t, X(t,x)) \in BV(0,T)$.

In the next Lemma, we will estimate the right-hand side of (106). We denote by ϖ, the following integral $\varpi = \overline{\int_B \psi_\infty |\nabla_R(f^n - f)|^2\, dR}^{\delta,\kappa,a}$. Notice that

$$
\varpi = \overline{\int_B \psi_\infty |\nabla_R(f^n - f)|^2\, dR}^{\delta,\kappa,a} \ge \int_B \psi_\infty \overline{|\nabla_R(f^n - f)|^2}^{\delta,\kappa,a} dR.
$$

Lemma 5.7. *We have the following bounds:*

$$
\begin{aligned}
A_2 &\ge \frac{1}{N^4}\overline{\int_B \psi_\infty |\nabla_R(f^n - f)|^2\, dR}^{\delta,\kappa,a} = \frac{1}{N^4}\varpi, \\
A_1 &\ge \frac{\mu}{N^4} - C(1+|\nabla u|^2)\frac{\eta}{N^4} - \frac{1}{10N^4}\varpi, \\
|A_3| &\le \frac{1}{10N^4}(\mu+\varpi) + \frac{C}{N^4}|\nabla u|^2\eta, \\
|A_4| &\le \frac{1}{10N^4}(\mu+\varpi) + C\left[1 + \int_B \psi_\infty \frac{|\nabla_R g|^2}{N}\right]\frac{\eta}{N}(1+|\log(\frac{\eta}{N})|).
\end{aligned}
$$

For the proof of the first estimate, we rewrite $|\nabla_R g|^2$ as

$$
\begin{aligned}
|\nabla_R g|^2 &= \left|\overline{\nabla_R f^n(\log^{1/2}(f^n)^2 + \log^{-1/2}(f^n)^2)}\right|^2 \\
&= \left|\overline{\nabla_R f^n(\log^{1/2}(f^n)^2)}\right|^2 + \left|\overline{\nabla_R f^n(\log^{-1/2}(f^n)^2)}\right|^2 \qquad (107)\\
&\quad + 2\overline{\nabla_R f^n(\log^{1/2}(f^n)^2)}\cdot\overline{\nabla_R f^n(\log^{-1/2}(f^n)^2)}.
\end{aligned}
$$

Hence, we deduce that

$$
A_2 = \frac{1}{N^4}\left[\int_B \psi_\infty(\boldsymbol{\alpha}+\boldsymbol{\beta}+\boldsymbol{\gamma}_c+\boldsymbol{\gamma}_o) + 4\overline{\int_B \psi_\infty|\nabla_R f^n|^2 dR}^{\delta,\kappa,a} - 4\int_B \psi_\infty\overline{|\nabla_R f^n|^2}^{\delta,\kappa,a} dR\right],
$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$, $\boldsymbol{\gamma}_c$ and $\boldsymbol{\gamma}_o$ are given by

$$\frac{\boldsymbol{\alpha}}{2} = \overline{\frac{|\nabla_R f^n|^2(\log^{1/2}(\frac{\tilde{\psi}^n}{\psi_\infty}))}{f^n}}^{a} \overline{f^n \log^{1/2}(\frac{\tilde{\psi}^n}{\psi_\infty})} - \left|\overline{(\nabla_R f^n)\log^{1/2}(\frac{\tilde{\psi}^n}{\psi_\infty})}\right|^2,$$

$$\frac{\boldsymbol{\beta}}{2} = \overline{\frac{|\nabla_R f^n|^2(\log^{-3/2}(\frac{\tilde{\psi}^n}{\psi_\infty}))}{f^n}}^{a} \overline{f^n \log^{1/2}(\frac{\tilde{\psi}^n}{\psi_\infty})} - \left|\overline{(\nabla_R f^n)\log^{-1/2}(\frac{\tilde{\psi}^n}{\psi_\infty})}\right|^2,$$

$$\frac{\boldsymbol{\gamma}_c}{2} = 2\overline{|\nabla_R f^n|^2}^{\delta,\kappa,a} - 2ch(|\nabla_R f^n|^2),$$

$$\frac{\boldsymbol{\gamma}_o}{2} = 2ch(|\nabla_R f^n|^2) - 2\overline{(\nabla_R f^n)\log^{1/2}(\frac{\tilde{\psi}^n}{\psi_\infty})} \,.\, \overline{(\nabla_R f^n)\log^{-1/2}(\frac{\tilde{\psi}^n}{\psi_\infty})}.$$

Notice that $\boldsymbol{\gamma}_c$ is some measure of concentration of the sequence $|\nabla_R f^n|^2$ whereas $\boldsymbol{\gamma}_o$ is some measure of oscillations. We introduce the Young measure $\nu_{t,x,R}(\Lambda,\lambda)$ associated with the sequence $(\nabla_R f^n, f^n)$ where $\Lambda \in \mathbb{R}^D$ and $\lambda \in \mathbb{R}$. Hence, the defect measure $\overline{|\nabla_R(f^n - f)|^2}^{\delta,\kappa,a}$ satisfies (see Remark 3.10) :

$$\begin{aligned}
\overline{|\nabla_R(f^n - f)|^2} \geq \overline{|\nabla_R(f^n - f)|^2}^{\delta,\kappa,a} &= \frac{\boldsymbol{\gamma}_c}{4} + \int |\Lambda - \int \Lambda' \nu_{t,x,R}(\Lambda',\lambda')|^2 \nu_{t,x,R}(\Lambda,\lambda) \\
&= \frac{\boldsymbol{\gamma}_c}{4} + \frac{1}{2}\int\int |\Lambda - \Lambda'|^2 \nu_{t,x,R}(\Lambda',\lambda')\nu_{t,x,R}(\Lambda,\lambda).
\end{aligned}$$

Indeed, it is easy to see that $\overline{|\nabla_R(f^n - f)|^2}^{\delta,\kappa,a}$ is bounded from above by the weak limit and that it is given by the sum of $\frac{\boldsymbol{\gamma}_c}{4}$ and the Chacon limit of $|\nabla_R(f^n - f)|^2$. In the sequel, we will drop the t, x and R dependence of ν and will denote $\nu' = \nu(\Lambda',\lambda')$ and $\nu = \nu(\Lambda,\lambda)$. Besides, $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}_o$ satisfy

$$\boldsymbol{\alpha} \geq \int\int A(\Lambda,\lambda,\Lambda',\lambda')\nu(\Lambda',\lambda')\nu(\Lambda,\lambda)$$

and the same is true for $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}_o$, with A replaced by B or C where A, B and C are given by

$$\begin{aligned}
A &= \frac{|\Lambda|^2\log^{1/2}(\lambda^2)}{\lambda}\lambda'\log^{1/2}(\lambda')^2 + \frac{|\Lambda'|^2\log^{1/2}(\lambda')^2}{\lambda'}\lambda\log^{1/2}(\lambda^2) - 2\Lambda.\Lambda'\log^{1/2}(\lambda^2)\log^{1/2}(\lambda')^2, \\
B &= \frac{|\Lambda|^2\log^{-3/2}(\lambda^2)}{\lambda}\lambda'\log^{1/2}(\lambda')^2 + \frac{|\Lambda'|^2\log^{-3/2}(\lambda')^2}{\lambda'}\lambda\log^{1/2}(\lambda^2) - 2\Lambda.\Lambda'\log^{-1/2}(\lambda^2)\log^{-1/2}(\lambda')^2, \\
C &= 2|\Lambda|^2 + 2|\Lambda'|^2 - 2\Lambda.\Lambda'\Big(\log^{1/2}(\lambda^2)\log^{-1/2}(\lambda'^2) + \log^{-1/2}(\lambda^2)\log^{1/2}(\lambda')^2\Big).
\end{aligned}$$

To prove the first estimate of Lemma 5.7, it is enough to show that $A + B + C \geq \frac{c}{2}|\Lambda - \Lambda'|^2$. First, we rewrite $A + B + C$ as

$$\begin{aligned}
A + B + C &= |\Lambda|^2 B_1 + |\Lambda'|^2 B_2 - 2\Lambda.\Lambda' B_3 \\
&= |\Lambda - \Lambda'|^2 + |\Lambda|^2(B_1 - 1) + |\Lambda'|^2(B_2 - 1) - 2\Lambda.\Lambda'(B_3 - 1),
\end{aligned}$$

where B_1, B_2 and B_3 are given by

$$B_1 = \frac{\log^{1/2}(\lambda^2)}{\lambda}\lambda' \log^{1/2}(\lambda'^2) + \frac{\lambda' \log^{1/2}(\lambda'^2)}{\lambda \log^{3/2}(\lambda^2)} + 2,$$
$$B_2 = \frac{\log^{1/2}(\lambda'^2)}{\lambda'}\lambda \log^{1/2}(\lambda^2) + \frac{\lambda \log^{1/2}(\lambda^2)}{\lambda' \log^{3/2}(\lambda'^2)} + 2,$$
$$B_3 = \log^{1/2}(\lambda^2)\log^{1/2}(\lambda'^2) + \frac{1}{\log^{1/2}(\lambda^2)\log^{1/2}(\lambda'^2)} + \frac{\log^{1/2}(\lambda^2)}{\log^{1/2}(\lambda'^2)} + \frac{\log^{1/2}(\lambda'^2)}{\log^{1/2}(\lambda^2)}.$$

Actually, we will prove that if a is chosen large enough then $(B_1-1)(B_2-1) \geq (B_3-1)^2$ from which we deduce that $A + B + C \geq |\Lambda - \Lambda'|^2$ and the lemma will follow. Indeed, after simple calculations, we get

$$\begin{aligned}
&(B_1-1)(B_2-1) - (B_3-1)^2 = \\
&\quad \log^{1/2}(\lambda^2)\log^{1/2}(\lambda'^2)\left[\frac{\lambda}{\lambda'} + \frac{\lambda'}{\lambda} + 2 - 2\frac{\log^{1/2}(\lambda^2)}{\log^{1/2}(\lambda'^2)} - 2\frac{\log^{1/2}(\lambda'^2)}{\log^{1/2}(\lambda^2)}\right] \\
&\quad + 2\left[\frac{\log^{1/2}(\lambda^2)}{\log^{1/2}(\lambda'^2)} + \frac{\log^{1/2}(\lambda'^2)}{\log^{1/2}(\lambda^2)} - 2\right] \\
&\quad + \frac{1}{\log^{1/2}(\lambda^2)\log^{1/2}(\lambda'^2)}\left[\frac{\lambda\log(\lambda^2)}{\lambda'\log(\lambda'^2)} + \frac{\lambda'\log(\lambda'^2)}{\lambda\log(\lambda^2)} + 2 - 2\frac{\log^{1/2}(\lambda^2)}{\log^{1/2}(\lambda'^2)} - 2\frac{\log^{1/2}(\lambda'^2)}{\log^{1/2}(\lambda^2)}\right].
\end{aligned}$$

We will prove that the three terms appearing inside the brackets are nonnegative. This is obvious for the second one since it is of the form $x + \frac{1}{x} - 2$ for some $x > 0$. We recall that since $(f^n)^2 \geq a$, we get that $\lambda \geq \sqrt{a}$ on the support of ν. For the first bracket, we assume that $\lambda' \geq \lambda$ and write $\lambda' = \lambda(1+\varepsilon)$. Hence, the term in the first bracket is given by

$$1 + \varepsilon + \frac{1}{1+\varepsilon} + 2 - 2\sqrt{1 + \frac{\log(1+\varepsilon)}{\log\lambda}} - 2\frac{1}{\sqrt{1 + \frac{\log(1+\varepsilon)}{\log\lambda}}} \tag{108}$$

and one can check easily that if $\lambda \geq \sqrt{a}$ is big enough then (108) is nonnegative. The same argument can be used for the third bracket. This completes the proof of the estimate of A_2.

To bound A_1, we first observe that

$$g^2 - 2g\overline{\frac{\tilde{\psi}^n}{\psi_\infty}\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})} = -\overline{f^n \log^{1/2}(f^n)^2}\;\overline{f^n \log^{-1/2}(f^n)^2}$$

and hence,

$$\begin{aligned}
A_1 &= -\frac{1}{N^4}\left[\overline{\nabla u^n : \tau(\tilde{\psi}^n)}^{\delta,\kappa} + \nabla u : \tau\left(\psi_\infty\Big(g^2 - 2g\overline{\frac{\tilde{\psi}^n}{\psi_\infty}\Theta'(\frac{\tilde{\psi}^n}{\psi_\infty})}\Big)\right)\right] \\
&= -\frac{1}{N^4}\left[\overline{\nabla u^n : \tau(\tilde{\psi}^n - \psi)}^{\delta,\kappa} + \nabla u : \tau(\psi - \psi_\infty(\overline{f^n \log^{1/2}(f^n)^2}\;\overline{f^n \log^{-1/2}(f^n)^2}))\right] \\
&= \frac{1}{N^4}\left[\mu - \nabla u : \tau(\psi - \psi_\infty(\overline{f^n \log^{1/2}(f^n)^2}\;\overline{f^n \log^{-1/2}(f^n)^2}))\right].
\end{aligned}$$

By convexity, it is clear that $\overline{(f^n-f)^2} = \overline{(f^n)^2} - f^2 \geq \overline{(f^n)^2} - \overline{f^n \log^{1/2}(f^n)^2}\ \overline{f^n \log^{-1/2}(f^n)^2}$ and hence, using (37), we get

$$|\tau(\psi - \psi_\infty(\overline{f^n \log^{1/2}(f^n)^2}\ \overline{f^n \log^{-1/2}(f^n)^2}))|$$
$$\leq \left(\int_B \psi_\infty \overline{(f^n-f)^2} dR\right)^{1/2} \left(\overline{\int_B \psi_\infty |\nabla(f^n-f)|^2 dR}^{\delta,\kappa,a} + \int_B \psi_\infty \overline{(f^n-f)^2} dR\right)^{1/2}.$$

Hence,

$$-A_1 \leq -\frac{\mu}{N^4} + \frac{|\nabla u|}{N^4}\eta + C\frac{|\nabla u|}{N^4}\left(\int_B \psi_\infty \overline{(f^n-f)^2} dR \overline{\int_B \psi_\infty |\nabla(f^n-f)|^2 dR}^{\delta,\kappa,a}\right)^{1/2}$$
$$\leq -\frac{\mu}{N^4} + C(|\nabla u| + |\nabla u|^2)\frac{\eta}{N^4} + \frac{1}{10N^4}\overline{\int_B \psi_\infty |\nabla(f^n-f)|^2 dR}^{\delta,\kappa,a}.$$

For A_3, we use that the term between parentheses in the definition of A_3 can be written as

$$\overline{\frac{\nabla u^n}{f^n}\Big(\log^{1/2}(f^n)^2 + \log^{-1/2}(f^n)^2\Big)\Big[f^n \log^{1/2}(f^n)^2 - \overline{f^n \log^{1/2}(f^n)^2}\Big]}.$$

If we denote by $\nu_{t,x,R}(\Pi,\lambda)$ the Young measure associated to the sequence $(\nabla_x u^n, f^n)$, then we see easily that A_3 is given by

$$A_3 = -\frac{2ak}{N^4}\int_B \int\int \left(\frac{\Pi}{\lambda}(\log^{1/2}\lambda^2 + \log^{-1/2}\lambda^2) - \frac{\Pi'}{\lambda'}(\log^{1/2}\lambda'^2 + \log^{-1/2}\lambda'^2)\right)$$
$$(\lambda \log^{1/2}\lambda^2 - \lambda' \log^{1/2}\lambda'^2) : \frac{R\otimes R}{1-|R|^2}\psi_\infty \, d\nu\, d\nu'\, dR.$$

The absolute value of the two factors inside the integral can be bounded respectively by

$$|\Pi - \Pi'|(\frac{\log^{1/2}\lambda^2}{\lambda} + \frac{\log^{1/2}\lambda'^2}{\lambda'}) + (|\Pi| + |\Pi'|)(\frac{\log^{1/2}\lambda^2}{\lambda} - \frac{\log^{1/2}\lambda'^2}{\lambda'})$$

and

$$|\lambda - \lambda'|(\log^{1/2}\lambda^2 + \log^{1/2}\lambda'^2).$$

Hence

$$|A_3| \leq \frac{1}{10N^4}\int_B \int\int |\Pi - \Pi'|^2(\frac{\log\lambda^2}{\lambda} + \frac{\log\lambda'^2}{\lambda'})^2 \frac{1}{1-|R|^2}\psi_\infty \, d\nu\, d\nu'\, dR$$
$$+ \frac{C}{N^4}\int_B \int\int (|\Pi| + |\Pi'|)|\lambda - \lambda'|^2(\frac{\log\lambda^2}{\lambda^2} + \frac{\log\lambda'^2}{\lambda'^2})^2 \frac{1}{1-|R|^2}\psi_\infty \, d\nu\, d\nu'\, dR$$
$$+ \frac{C}{N^4}\int_B \int\int (1 + |\Pi| + |\Pi'|)|\lambda - \lambda'|^2 \frac{1}{1-|R|^2}\psi_\infty \, d\nu\, d\nu'\, dR$$
$$\leq \frac{1}{10N^4}\mu + \frac{1}{10N^4}\varpi + \frac{C}{N^4}|\nabla u|^2\eta,$$

where we have used that $\int\int |\lambda - \lambda'|^2 d\nu\, d\nu' = \mathrm{ch}(|\nabla u^n - \nabla u|^2) \leq \mu$ (see Remark 5.5).

Finally, to bound $-A_4$, we split it into two terms. For A_4^1, we use Cauchy-Schwarz to deduce that

$$
\begin{aligned}
|A_4^1| &\le \frac{2}{N^4}\int_B \psi_\infty |\gamma_{ij}||\nabla g|\ dR \\
&\le \frac{1}{10N^4}\overline{|\nabla v^n - \nabla v|^2}^{\delta,\kappa} + \frac{C}{N^4}\overline{\left(\int_B \psi_\infty (g^n-g)|\nabla_R g|\, dR\right)^2} \\
&\le \frac{1}{10N^4}\mu + \frac{C}{N^4}\left(\int_B \psi_\infty |\nabla_R g|^2 dR\right)\overline{\int_B \psi_\infty (g^n-g)^2 dR}.
\end{aligned}
$$

To bound A_4^2, we first consider the case $k>1$ where the term can be treated using (17):

$$
\begin{aligned}
|A_4^2| &\le \frac{2}{N^4}\int_B \psi_\infty (|\gamma_{ij}|+|\gamma_{ij}'|)\frac{g}{1-|R|}\ dR \\
&\le \frac{1}{10N^4}\overline{|\nabla v^n - \nabla v|^2}^{\delta,\kappa} + \frac{C}{N^4}\overline{\left(\int_B \psi_\infty \frac{|g^n-g|\, g}{(1-|R|^2)}\ dR\right)^2} \\
&\le \frac{1}{10N^4}\mu + \frac{C}{N^4}\overline{\int_B \psi_\infty (g^n-g)^2 dR}\int_B \psi_\infty \frac{|g|^2}{(1-|R|)^2}\ dR \qquad (109)\\
&\le \frac{1}{10N^4}\mu + \frac{C}{N^4}\left(\int_B \psi_\infty |\nabla_R g|^2\, dR\right)\eta.
\end{aligned}
$$

In the case $k \le 1$, we have to use (20) instead of (17) to control the second term on the second line of (109). We define f_g by $g = f_q \log^{1/2}(f_g^2)$. We have

$$
|g^n - g| \lesssim (|f^n - f| + |f - f_g|)\Big(\log^{1/2}(g^2) + \log^{1/2}(C + (f^n-f)^2)\Big) \qquad (110)
$$

where we have used that $\log^{1/2}(f_g^2) + \log^{1/2}((f^n)^2) \lesssim \log^{1/2}(g^2) + \log^{1/2}(C+(f^n-f)^2)$. To control the four terms appearing on the right-hand side of (110), it is enough to control the two terms coming from $|f^n - f|$ since $|f-f_g|^2 \lesssim \overline{|f^n - f|^2}$.

The term involving $\log^{1/2}(C+(f^n-f)^2)$ can be treated as follows:

$$
\begin{aligned}
&\left(\int_B \psi_\infty \frac{|f^n-f|\log^{1/2}(C+(f^n-f)^2)g}{(1-|R|^2)}\ dR\right)^2 \qquad (111)\\
&\le \int_B \psi_\infty \frac{(f^n-f)^2\log^{1/2}(C+(f^n-f)^2)}{(1-|R|^2)} dR \int_B \psi_\infty \frac{g^2\log^{1/2}(C+(f^n-f)^2)}{(1-|R|^2)} dR.
\end{aligned}
$$

To bound the second term on the right-hand side of (111), we use the following Young's inequality for $a, b \ge 1$, $ab \le a\log^\gamma a + e^{(b^{\frac{1}{\gamma}})}$ with $\gamma = 1/2$. We denote $d = 1-|R|^2$ and hence

$$
\int_B \psi_\infty \frac{g^2\log^{1/2}(C+(f^n-f)^2)}{(1-|R|^2)}\ dR \le \int_B \psi_\infty \left[\frac{g^2}{d}\log^{1/2}\frac{g^2}{d} + C + |f^n-f|^2\right] dR.
$$

On the set $\{g^2 \ge \frac{1}{d^\varepsilon}\}$ where $\varepsilon = \frac{k}{2}$, we have $\log^{1/2}\frac{g^2}{d} \le C\log^{1/2} g^2$. Using (20), we have,

$$
\int_B \psi_\infty \frac{g^2\log^{1/2} g^2}{d} dR\ \le \left(\int_B \psi_\infty g^2 \log g^2 dR\right)^{\frac{1}{2}}\left(\int_B \psi_\infty (|\nabla_R g|^2 + g^2) dR\right)^{\frac{1}{2}}. \qquad (112)
$$

On the set $\{g^2 \le \frac{1}{d^\varepsilon}\}$, we have

$$
\frac{g^2}{d}\log^{1/2}\frac{g^2}{d} \le \frac{C}{d^{1+\varepsilon}}\log^{1/2}(\frac{1}{d}), \qquad (113)
$$

which is integrable in the ball B with the measure $\psi_\infty dR$.

To bound the first term on the right-hand side of (111), we use (20):

$$\begin{aligned}(114)\quad &\Big|\int_B \psi_\infty \frac{(f^n-f)^2\log^{1/2}(C+(f^n-f)^2)}{(1-|R|^2)}\,dR\Big| \\ &\le \left(\int_B \psi_\infty (f^n-f)^2\log(C+(f^n-f)^2)dR\right)^{\frac12}\left(\int_B \psi_\infty|\nabla_R(f^n-f)|^2dR\right)^{\frac12}\\ &\le \frac{C}{\lambda^2}\left(\int_B \psi_\infty (f^n-f)^2\log(C+(f^n-f)^2)dR\right)+\lambda^2\left(\int_B \psi_\infty|\nabla_R(f^n-f)|^2dR\right),\end{aligned}$$

for each $\lambda>0$. Passing to the limit weakly (more precisely, applying $\overline{F_n}^{\delta,\kappa,a}$) to both sides of (111) and optimizing in λ, we deduce that,

$$\frac{1}{N^4}\overline{\left(\int_B \psi_\infty \frac{|f^n-f|\log^{\frac12}(C+(f^n-f)^2)g}{(1-|R|^2)}\,dR\right)^2} \le \frac{C}{N^4}\left(\int_B \psi_\infty g^2\log g^2\int_B\psi_\infty(|\nabla_R g|^2+g^2)\right)^{\frac12}\eta^{\frac12}\varpi^{\frac12}.$$

The term involving $\log^{1/2}(g^2)$ in (110) can be treated as follows: Let $\varepsilon>0$ be such that $3\varepsilon<r-2$ where r is as in (75). On the set where $\{g^\varepsilon\ge\frac1d\}$, we have

$$\begin{aligned}(115)\quad \left(\int_{B\cap\{g^\varepsilon\ge\frac1d\}}\psi_\infty\frac{|f^n-f|\log^{1/2}(g^2)g}{(1-|R|^2)}\,dR\right)^2 &\lesssim \left(\int_B\psi_\infty|f^n-f|\log^{1/2}(g^2)g^{1+\varepsilon}\,dR\right)^2\\ &\lesssim \int_B\psi_\infty(f^n-f)^2dR\int_B\psi_\infty g^{2+3\varepsilon}dR.\end{aligned}$$

On the set where $\{g^\varepsilon\le\frac1d\}$, we have $\log^{1/2}(g^2)\lesssim\log^{1/2}(\frac1d)$, hence

$$(116)\quad \left(\int_{B\cap\{g^\varepsilon\le\frac1d\}}\psi_\infty\frac{|f^n-f|\log^{\frac12}(g^2)g}{(1-|R|^2)}\,dR\right)^2\lesssim\int_B\psi_\infty\frac{|f^n-f|^2\log^{\frac12}(\frac1d)}{d}dR\int_B\psi_\infty\frac{g^2\log^{\frac12}(g^2)}{d}dR.$$

The second term on the right-hand side of (116) can be estimated as in (112). For the first term, we first look at the set $\{\frac{\eta}{N}\frac{1}{d^\varepsilon}\le C+|f^n-f|^2\}$ where $\varepsilon=k/2$:

$$(117)\quad \int_{B\cap\{\frac{\eta}{N}\frac{1}{d^\varepsilon}\le C+|f^n-f|^2\}}\frac{\psi_\infty|f^n-f|^2\log^{\frac12}(\frac1d)}{d}dR\lesssim\int_B\psi_\infty\frac{|f^n-f|^2\Big(\log^{\frac12}(C+|f^n-f|^2)+|\log(\frac{\eta}{N})|^{\frac12}\Big)}{d}dR$$

The first term on the right-hand side of (117), involving $\log^{1/2}(C+|f^n-f|^2)$, is treated as in (114). For the second term, we use that

$$(118)\quad \int_B\psi_\infty\frac{|f^n-f|^2|\log(\frac{\eta}{N})|^{1/2}}{d}dR\;\lesssim\;\eta^{\frac12}\varpi^{\frac12}|\log(\frac{\eta}{N})|^{1/2}.$$

On the set $\{\frac{\eta}{N}\frac{1}{d^\varepsilon}\ge C+|f^n-f|^2\}$, we have

$$(119)\quad \int_{B\cap\{\frac{\eta}{N}\frac{1}{d^\varepsilon}\ge C+|f^n-f|^2\}}\psi_\infty\frac{|f^n-f|^2\log^{\frac12}(\frac1d)}{d}dR\lesssim\int_B\psi_\infty\frac{\eta\log^{\frac12}(\frac1d)}{N\,d\,d^\varepsilon}dR\lesssim\frac{\eta}{N}.$$

Next, we claim that $\int_B \psi_\infty g^2 \log g^2 dR \leq CN^2$. Indeed, if we introduce $h^n = g^n \log^{1/2} g^n$, we see that $N_2^n \geq \left(\int_B \psi_\infty (h^n)^2\right)^{1/2}$ and then it is easy to see using that $(x,y) \to \frac{x^2}{y}$ is convex that

$$\overline{\left(\int_B \psi_\infty (h^n)^2\right)^{1/2}} \geq \left(\int_B \psi_\infty h^2\right)^{1/2},$$

from which we deduce the claim. Hence, we get

$$\frac{1}{N^4}\overline{\left(\int_B \psi_\infty \frac{|g^n - g|\, g}{(1-|R|^2)}\, dR\right)^2} \leq \frac{\varpi}{10N^4} + C\left[1 + \int_B \psi_\infty \frac{|\nabla_R g|^2}{N}\right] \frac{\eta}{N}(1 + |\log(\frac{\eta}{N})|)(t, X(t,x)).$$

This ends the proof of Lemma 5.7. Putting the estimates of the terms A_i, $1 \leq i \leq 4$, together, and writing the outcome along the characteristics of u using Proposition A.5, we see that (106) implies (104). The fact that we also get that the initial condition $\frac{\eta}{N^4}(0, X(0,x)) = 0$ for a.e. $x \in \Omega$ comes from the fact that $\frac{\eta}{N^4}(0, X(0,x)) \leq 0$ and that $\eta \geq 0$. This completes the proof of Proposition 5.6. □

Using (104), we can now conclude the proof of the weak compactness argument. Indeed, notice that the right-hand side of (104) is in $L^1((0,T) \times K)$ for any bounded measurable subset K of Ω. To prove this, we can observe that $\frac{\eta}{N}$ is bounded and that using (62), the term between the brackets in (104) is in $L^1((0,T) \times K))$.

Now, since the term between the brackets in (104) is in $L^1((0,T) \times K)$, we deduce that for almost all x, $\int_0^T \left[1 + \int_B \psi_\infty \frac{|\nabla_R g|^2}{N}\right](t, X(t,x))$ is finite. Besides, for almost all x, $N(t, X(t,x))$ (which is constant in t) is bounded. Hence, we deduce that for almost all x,

$$\int_0^T N^3 \left[1 + \int_B \psi_\infty \frac{|\nabla_R g|^2}{N}\right] + |\nabla u|^2(t, X(t,x))\, dt \quad \text{is finite.}$$

If $k > 1$, then we can apply Gronwall's lemma and deduce that for a.e. x, we have for all $t < T$, $\frac{\eta(t,x)}{N^4} \leq \frac{\eta(0,x)}{N^4} e^{C_T(x)}$ and since $\eta(0,x) = 0$ due to the initial strong convergence, we deduce that $\frac{\eta(t,x)}{N^4} = 0$ and hence $\eta = 0$ for a.e. $x \in \Omega$. If $k < 1$, we have to replace Gronwall's lemma by Osgood's Lemma (see Lemme 5.2.1 in [13]) to infer that $\eta = 0$ for all $t < T$, for a.e. $x \in \Omega$. Hence, we deduce the strong convergence of g^n to g. This yields that (u, ψ) is a weak solution of (1) with the initial data (u_0, ψ_0).

Remark 5.8. *Notice that one also gets that $\mu = \varpi = 0$ and that equality should hold in* (91)*,* (81)*,... In particular this means that* (77) *holds with $(\tilde{\psi^n}, u^n)$ replaced by $(\tilde{\psi}, u)$ and so the defect measure in* (81) *has to vanish. It seems unclear if one can deduce this fact just from the limit equation* (85)*. Let us compare this on the one hand to other works (related to the Boltzmann equation) where a defect measure due to renormalization or absence of equi-integrability is present in the final formulation even if it formally has to vanish (see for instance* [1] *and* [64]*). On the other hand, we can compare the fact that we also obtain a renormalized solution of* (85) *to the works* [61, 31] *about the compressible Navier-Stokes system where the notion of renormalized solution to the continuity equation is very important.*

6. Approximate system

In the previous section, we proved the weak compactness of a sequence of solutions to the system (1). Of course one has to construct a sequence of (approximate) weak solutions to which we can apply the strategy of the previous sections. The only thing we have to make sure is that the calculations performed in the previous section can be carried out on the approximate system. We consider a sequence of global smooth solutions (u^n, ψ^n) to the following regularized system

$$(120)\qquad \begin{cases} \partial_t u^n + (\tilde{u}^n \cdot \nabla) u^n - \nu \Delta u^n + \nabla p^n = \operatorname{div}(\tau^n \star \omega_n), \quad \operatorname{div} u = 0, \\ \\ \partial_t \psi^n + \tilde{u}^n . \nabla \psi^n = \operatorname{div}_R \Big[- \nabla u^n \star \omega_n \, R \psi^n + \beta \nabla \psi^n + \nabla \mathcal{U} \psi^n \Big] \\ \\ \tau^n = \int_B (R \otimes \nabla \mathcal{U}) \psi^n(t,x,R) dR \qquad (\nabla \mathcal{U} \psi^n + \beta \nabla \psi^n).\mathbf{n} = 0 \text{ on } \partial B(0, R_0). \end{cases}$$

where $\omega_n(x) = n^D \omega(nx)$, $\omega \in C_0^\infty(\mathbb{R}^D)$, $\int_{\mathbb{R}^D} \omega = 1$, $\operatorname{Supp}(\omega) \in B(0,1)$ and $\tau^n \star \omega_n$ denotes the convolution in the x variable. Here, when performing convolutions in the case of a bounded domain Ω, all functions are extended by 0 in Ω^c. Besides, $\tilde{u}^n$ is a regularized version of u^n that has the same boundary condition. In particular, if $\Omega = \mathbb{R}^D$ or $\mathbb{T}^D$, we can take $\tilde{u}^n = u^n \star \omega_n$. If Ω is a smooth bounded domain, we take $\tilde{u}^n = v^n \star \omega_n$ where v^n solves $-\Delta v^n + v^n + \nabla p^n = -\Delta u^n + u^n$ in $\Omega_{\frac{1}{n}}$, $\operatorname{div} v^n = 0$ in $\Omega_{\frac{1}{n}}$ and $\Omega_{\frac{1}{n}} = \{x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \frac{1}{n}\}$. System (120) is complemented with smooth initial data (u_0^n, ψ_0^n) such that (u_0^n, ψ_0^n) converges strongly to (u_0, ψ_0) in $L^2(\Omega) \times L^1(\Omega \times B)$ and $\psi_0^n \log \frac{\psi_0^n}{\rho_0^n \psi_\infty} - \psi_0^n + \rho_0^n \psi_\infty$ converges strongly to $\psi_0 \log \frac{\psi_0}{\rho_0 \psi_\infty} - \psi_0 + \rho_0 \psi_\infty$ in $L^1(\Omega \times B)$. We also assume that (8) holds uniformly in n. In the case Ω is a bounded domain of $\mathbb{R}^D$, we also impose the Dirichlet boundary condition $u^n = 0$ at the boundary $\partial\Omega$. We do not detail the proof of existence for the system (120). We only mention that we have to combine classical results about strong solutions to the Navier-Stokes system with the study of the linear Fokker-Planck equation (see [68]). Following the proof of existence given in [68], we can prove the following result.

Proposition 6.1. *Take $u_0^n \in H^s(\Omega)$ and $\psi_0^n \geq 0$ such that $\psi_0^n - \rho_0^n \psi_\infty \in H^s(\Omega; \psi_\infty L^2(\frac{dR}{\psi_\infty}))$ with $\rho_0^n = \int \psi_0^n dR \in L^\infty(\Omega)$. Then, there exists a global unique solution (u^n, ψ^n) to (120) such that $(u^n, \psi^n - \rho^n \psi_\infty)$ is in $C([0,T); H^s) \times C([0,T); H^s(\mathbb{R}^N; L^2(\frac{dR}{\psi_\infty})))$ for all $T > 0$. Moreover, $u^n \in L^2([0,T); H^{s+1})$ and $\psi^n - \rho^n \psi_\infty \in L^2([0,T); H^s(\mathbb{R}^N; \psi_\infty \mathcal{H}_k^1))$.*

Remark 6.2. *The proof is exactly the same as the proof of Theorem 2.1 in* [68] *with a few differences:*

- *In* [68]*, we only had local existence whereas here, we have global existence since we have a regularized system.*
- *Theorem 2.1 of* [68] *was stated in the whole space. Of course in the case of a bounded domain, we have to use energy bounds for Navier-Stokes written in a bounded domain.*
- *In Theorem 2.1 of* [68] *we assumed that $\int \psi_0 dR = 1$. The result can be easily extended to this more general case. We also point out that there is a small mistake in the statement of the Theorem 2.1 of* [68]*. Indeed, one has to read $\psi_0 - \psi_\infty \in H^s(\Omega; L^2(\frac{dR}{\psi_\infty}))$ instead of $\psi_0 \in H^s(\Omega; L^2(\frac{dR}{\psi_\infty}))$ when the problem is in the whole space.*

It is clear that the solutions constructed in Proposition 6.1 satisfy the free-energy bound (50) and the extra bound (57) (with Ω replaced by K in the whole space case).

Once we have our sequence of regular approximate solutions, we have to check that all the computations performed in the previous section can be done on this sequence (u^n, ψ^n). The only point to be checked is that Proposition 5.4 still holds since the rest of the proof only involves the transport equation. Now, v^n and w^n solve

$$(121)\qquad \begin{cases} \partial_t v^n - \Delta v^n + \nabla p_1^n = \nabla.(\tau^n \star \omega_n) & \text{in} \quad \Omega \\ v^n(t=0) = 0, \qquad v^n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

$$(122) \qquad \begin{cases} \partial_t w^n - \Delta w^n + \nabla p_2^n = -\tilde{u}^n.\nabla u^n & \text{in} \quad \Omega \\ w^n(t=0) = u^n(t=0), \qquad w^n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and we define $v^{n,\boldsymbol{\delta}}$ to be the solution of

$$(123) \qquad \begin{cases} \partial_t v^{n,\boldsymbol{\delta}} - \Delta v^{n,\boldsymbol{\delta}} + \nabla p_1^{n,\boldsymbol{\delta}} = \nabla.\tau^{n,\boldsymbol{\delta}} & \text{in} \quad \Omega \\ v^{n,\boldsymbol{\delta}}(t=0) = 0, \qquad v^{n,\boldsymbol{\delta}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\tau^{n,\boldsymbol{\delta}}$ is now given by $\tau^{n,\boldsymbol{\delta}} = \frac{\tau^n \star \omega_n}{(1+\delta N_2^n)(1+\kappa N_1^n)^2}$. Step 1 of the proof of Proposition 5.4 is the same with the only difference that we replace τ^n by $\tau^n \star \omega_n$. Hence, we deduce that $\|\nabla v^{n,\boldsymbol{\delta}} - \nabla v^n\|_{L^p((0,T)\times\Omega)}$ goes to zero when $\boldsymbol{\delta}$ goes to zero uniformly in n for $p < 2$. The second step is then identical to the one in the proof of Proposition 5.4.

7. Conclusion

In this paper we gave a proof of existence of weak solutions to the system (1), using the fact that a sequence of regular solutions to the approximate system (120) converges weakly to a weak solution of (1). We would like here to mention few important open problems (with increasing level of difficulty, at least in the author's opinion):

- *The zero diffusion limit in x.* If we add a center-of-mass diffusion term $\varepsilon\Delta_x\psi$ in the Fokker-Planck equation of (1) with a boundary condition for ψ when $x \in \partial\Omega$ in the case Ω is a smooth bounded domain, then one can prove the global existence of weak solutions to the model (see [9] for an existence result for the FENE model with center-of-mass diffusion). A natural question is whether we recover a weak solution of the unregularized system (1) when ε goes to zero. This is the object of a forthcoming paper [71]. The difficulty comes from the fact that the calculation of section 5 used in a critical way the fact that we had a transport equation in the x variable.
- *Relaxing the assumption* (57). This extra bound was only used to give some extra control for the stress tensor and to justify the calculation in section 5. Can we prove the same existence result without it?
- *Other models.* A natural question is whether we can extend the result of this paper to the Hookean model (where the system can be reduced to a macroscopic model). We were not able to perform this. The main difficulty is that we do not know whether the extra stress tensor τ is in L^2. Nevertheless, we know how to use the strategy of this paper to prove global existence for the FENE-P model [69]. We should also mention the new paper [10] where global existence to the Hookean-type bead-spring chain model is proved when center-of-mass diffusion is taken into account and the potential $U(R)$ grows faster than $|R|^2$ when R goes to infinity. Moreover, in [20], global existence is proved for the Hooke model (which is equivalent to the macroscopic Oldroyd-B model) when the data is small in L^∞.
- *Regularity in 2D.* Many works on polymeric flows are motivated by similar known results for the Navier-Stokes system. In particular a natural question is whether one can prove global existence of smooth solutions to (1) in 2D. We point out that this is known for the co-rotational model [59, 68]. This seems to be a difficult problem since, we only have an L^2 bound on τ and that an L^∞ bound on τ was necessary in the previously mentioned works. In particular a similar result is not known for the co-rotational Oldroyd-B model where one can prove L^p bounds on τ for each $p > 1$.
- *Is system* (1) *better behaved than Navier-Stokes?* One does not expect to prove results on (1) which are not known for Navier-Stokes since (1) is more complicated than Navier-Stokes. However, one can speculate that due to the polymer molecules and the extra stress tensor, system (1) may behave better than Navier-Stokes and that one can prove global existence of smooth solutions to (1) even if such result is not proved or disproved for the Navier-Stokes system.

8. Acknowledgments

The work of N. M. is partially supported by NSF-DMS grant 0703145. The author would like to thank the referees for their very careful reading of the paper which improved the presentation tremendously. He also would like to thank P.-L. Lions and Ping Zhang for many discussions about this model. Finally, he thanks the IMA where part of this work was done.

Appendix A. About the DiPerna-Lions flows

Most papers dealing with the theory of DiPerna-Lions flows for Sobolev vector fields u are stated in the whole space. Extending them to the case of a bounded domain, when u vanishes at the boundary, is easy. One has just to extend u by 0 in Ω^c (see also [24]). In addition, in that case the flow keeps Ω invariant. In the sequel, we will consider the problem in $\mathbb{R}^D$.

Proposition A.1. *(Existence of DiPerna-Lions flow) If $u \in L^2(0,T;H^1(\mathbb{R}^D))$ and $div(u) = 0$. Then there exists a unique flow $X(t,t_0,x)$ such that for all $t_0 \in (0,T)$ and for a.e. $x \in \mathbb{R}^D$, $t \to X(t,t_0,x)$ is absolutely continuous and satisfies*

$$\begin{cases} \frac{\partial X}{dt}(t,t_0,x) = u(t,X(t,t_0,x)), \quad t \in (0,T) \\ X(t=t_0,t_0,x) = x, \end{cases} \tag{124}$$

and for $t, t_0 \in (0,T)$, the map $x \to X(t,t_0,x)$ is measure-preserving.

We will also denote $X(t,x) = X(t,0,x)$. One of the main ingredients in proving Proposition A.1 is the following stability result.

Proposition A.2. *(Stability of the flow) Assume that u^n is bounded in $L^2(0,T;H^1(\mathbb{R}^D))$, $div(u^n) = 0$ and u^n converges to u in $L^2_{loc}((0,T)\times\mathbb{R}^D)$. Let $X^n(t,t_0,x)$ and $X(t,t_0,x)$ be the DiPerna-Lions flows associated respectively with u^n and u. Then the flows $X^n(t,t_0,x)$ and $X(t,t_0,x)$ satisfy the following stability result. For all $t_0 \in (0,T)$, we have $X^n(t,t_0,x)$ converges to $X(t,t_0,x)$ locally in measure in $\mathbb{R}^D$, uniformly with respect to $t \in (0,T)$.*

In particular this means that, for all $t_0 \in (0,T)$,

$$\lim_{n\to\infty} |\{x \in B(0,R) \,|\, |X^n(t,t_0,x) - X(t,t_0,x)| > \delta\}| = 0, \quad \text{for every } R > 0 \text{ and } \delta > 0, \tag{125}$$

uniformly in $t \in (0,T)$. This is also equivalent to the fact that

$$\min(1, |X^n(t,t_0,x) - X(t,t_0,x)|) \to 0, \text{ in } L^1_{loc}(\mathbb{R}^D) \text{ uniformly in } t \in (0,T). \tag{126}$$

Proposition A.3. *(Weak limit along the flow) Assume that u^n is bounded in $L^2(0,T;H^1(\mathbb{R}^D))$, $div(u^n) = 0$ and u^n converges to u in $L^2_{loc}((0,T)\times\mathbb{R}^D)$. Let f^n be a sequence bounded in $L^\infty((0,T)\times\mathbb{R}^D)$ such that f^n converges weakly to f in $L^p_{loc}((0,T)\times\mathbb{R}^D)$, $1 < p < \infty$. Let $X^n(t,t_0,x)$ and $X(t,t_0,x)$ be the DiPerna-Lions flows associated respectively with u^n and u. Then, $f^n(t,X^n(t,x))$ converges weakly to $f(t,X(t,x))$ in $L^p_{loc}((0,T)\times\mathbb{R}^D)$, $1 < p < \infty$.*

Let us denote $q^n(t,x) = f^n(t,X^n(t,x))$. Hence, for any test function $\phi(t,x) \in C^\infty_0((0,T)\times\Omega)$, we have

$$\int_0^T \int_{\mathbb{R}^D} q^n(t,x)\phi(t,x)\,dxdt = \int_0^T \int_{\mathbb{R}^D} f^n(t,y)\phi(t,(X^n)^{-1}(t,y))dydt, \tag{127}$$

where for each time t, we made the change of variable $y = X^n(t,x)$. Hence, $x = (X^n)^{-1}(t,y) = X^n(0,t,y)$. Recall that this change of variable is measure-preserving. The left-hand side of (127) converges to $\int_0^T \int_{\mathbb{R}^D} \overline{q^n}\phi(t,x)\,dxdt$. To pass to the limit in the right-hand side, we notice that f^n converges to f weak* in $L^\infty((0,T)\times\mathbb{R}^D)$ and that for each $t_0 \in (0,T)$, we have

$$\begin{aligned}
&\|\phi(t_0,(X^n)^{-1}(t_0,y)) - \phi(t_0,(X)^{-1}(t_0,y))\|_{L^1(\mathbb{R}^D)} \\
&\qquad \le 2\|\phi\|_{W^{1,\infty}} \int_{X(t_0,K)\cup X^n(t_0,K)} \min(1,|(X^n)^{-1}(t_0,y) - (X)^{-1}(t_0,y)|)dy \\
&\qquad = 2\|\phi\|_{W^{1,\infty}} \int_{X(t_0,K)\cup X^n(t_0,K)} \min(1,|X^n(0,t_0,y) - X(0,t_0,y)|)dy
\end{aligned} \tag{128}$$

where K is a compact set that contains the support of ϕ in the x variable, namely $\mathrm{Supp}(\phi) \subset (0,T)\times K$. In the sequel, we assume that K is included in a ball $B(0,r)$ for some $r > 0$. From Proposition A.2, we know that

$$\min(1,|X^n(0,t_0,x) - X(0,t_0,x)|) \to 0, \text{ in } L^1_{loc}(\mathbb{R}^D). \tag{129}$$

However, the set on which we are integrating in (128), namely $X(t_0,K)\cup X^n(t_0,K)$ is not necessarily included in a compact set since it is not necessarily bounded. To overcome this problem, we have to adapt the proof of Proposition A.2 to include our case. Arguing as in [21], we introduce, for each $\lambda > 0$, the set

$$G_\lambda = \{x \in \mathbb{R}^D \mid \quad |X(t,x)| \le \lambda, \quad \forall t \in [0,T]\} \tag{130}$$

and the set G^n_λ where $X(t,x)$ is replaced by $X^n(t,x)$. Notice that if we knew that u is bounded in $L^\infty((0,T)\times\mathbb{R}^D)$ then the set G_λ would be equal to the whole ball $B(0,\lambda)$. From Proposition 3.2 of [21], we know that

$$|B(0,R)\setminus G_\lambda| + |B(0,R)\setminus G^n_\lambda| \le \varepsilon(R,\lambda), \tag{131}$$

where $\varepsilon(R,\lambda)$ only depends on the norm of u and u^n in $L^2((0,T)\times\mathbb{R}^D)$ and for each R fixed, $\varepsilon(R,\lambda)$ goes to zero when λ goes to infinity. Arguing as in the proof of Theorem 3.8 of [21], we introduce, for each $\lambda > 0$, the function

$$g^n(t) = \int_{X(t_0,G_\lambda)\cap X(t_0,G^n_\lambda)} \log\left(\frac{|X(t,t_0,x) - X^n(t,t_0,x)|}{\delta} + 1\right) dx, \tag{132}$$

where $\delta = \delta^n(\lambda) = \|u - u^n\|_{L^1((0,T)\times B(0,\lambda))}$. A calculation similar to the one given in the proof of Theorem 3.8 of [21] yields that $|(g^n)'(t)| \le C_\lambda$, where C_λ depends on λ, but does not depend on n or $t_0 \in (0,T)$. Hence $|g(0)| \le C_\lambda T$ since $g(t_0) = 0$. Fix $\varepsilon > 0$, let $R = 2r$ and recall that $K \subset B(0,r)$. Using (131), we deduce that there exists λ_0 such that if $\lambda > \lambda_0$, then $|B(0,R)\setminus(G_\lambda\cap G^n_\lambda)| \le \varepsilon$. In particular this implies that $|K\setminus(G_\lambda\cap G^n_\lambda)| \le \varepsilon$ and since $X(t_0,.)$ and $X^n(t_0,.)$ are measure-preserving, we deduce that $|X(t_0,K)\setminus X(t_0,(G_\lambda\cap G^n_\lambda))| \le \varepsilon$ and $|X^n(t_0,K)\setminus X^n(t_0,(G_\lambda\cap G^n_\lambda))| \le \varepsilon$. Using that $|g^n(0)| \le C_\lambda T$, we can find a measurable set $P^n \subset X(t_0,(G_\lambda\cap G^n_\lambda))$ such that $|X(t_0,(G_\lambda\cap G^n_\lambda))\setminus P^n| \le \varepsilon$ and

$$\log\left(\frac{|X(0,t_0,x) - X^n(0,t_0,x)|}{\delta} + 1\right) \le \frac{C_\lambda T}{\varepsilon} \quad \text{for all } x \in P^n. \tag{133}$$

Hence, we deduce that

$$\begin{aligned}
&\int_{X(t_0,K)\cup X^n(t_0,K)} \min(1,|X^n(0,t_0,x) - X(0,t_0,x)|)dx \\
&\quad \le |X(t_0,K)\setminus X(t_0,(G_\lambda\cap G^n_\lambda))| + |X^n(t_0,K)\setminus X^n(t_0,(G_\lambda\cap G^n_\lambda))| \\
&\qquad + |X(t_0,(G_\lambda\cap G^n_\lambda))\setminus P^n| + \int_{P^n} |X^n(0,t_0,x) - X(0,t_0,x)| \\
&\quad \le 3\varepsilon + C\delta e^{\frac{C_\lambda T}{\varepsilon}} \le 3\varepsilon + Ce^{\frac{C_\lambda T}{\varepsilon}} \|u - u^n\|_{L^1((0,T)\times B(0,\lambda))}.
\end{aligned} \tag{134}$$

Now, we choose N large enough, so that $Ce^{\frac{C_\lambda T}{\varepsilon}}\|u-u^n\|_{L^1((0,T)\times B(0,\lambda))} \leq \varepsilon$ for all $n \geq N$. Hence, we deduce that the left-hand side of (134) goes to zero when n goes to infinity, uniformly in $t_0 \in (0,T)$, and we infer from (128) that $\phi(t_0,(X^n)^{-1}(t_0,y))$ converges strongly to $\phi(t_0,(X)^{-1}(t_0,y))$ in $L^1(\mathbb{R}^D)$ when n goes to infinity uniformly in $t_0 \in (0,T)$. Since, this holds for all $t_0 \in (0,T)$, we deduce that $\phi(t,(X^n)^{-1}(t,y))$ converges strongly to $\phi(t,(X)^{-1}(t,y))$ in $L^1((0,T)\times\mathbb{R}^D)$ when n goes to infinity. Therefore, using that f^n converges to f weak* in $L^\infty((0,T)\times\mathbb{R}^D)$, we deduce that the right-hand side of (127) goes to $\int_0^T\int_{\mathbb{R}^D} f(t,y)\phi(t,(X)^{-1}(t,y))dydt$ when n goes to infinity. Hence, we get that $\overline{f^n(t,X^n(t,x))} = \overline{q^n(t,x)} = f(t,X(t,x))$.

Remark A.4. *One can extend Proposition A.3 to the case where f^n converges weakly to f in $L^1_{loc}((0,T)\times\mathbb{R}^D)$. Indeed, this implies that locally, f^n is equiintegrable and hence we can approximate f^n by* $\min(f^n,M)$ *uniformly in n and apply Proposition A.3 to* $\min(f^n,M)$ *and conclude by sending M to infinity. In our paper, we only use the case where f^n is bounded.*

Finally, we prove a Proposition similar to Theorem A on p.361 of [25].

Proposition A.5. *(mild formulation) Assume that $u \in L^2(0,T;H^1_0(\Omega))$ and that $X(t,x)$ is its DiPerna-Lions flow. Let $f \in L^\infty((0,T)\times\Omega)$, $f_0 \in L^\infty(\Omega)$ and $h \in L^1((0,T)\times\Omega)$. The following three systems are equivalent :*

$$\begin{cases} \partial_t f + u.\nabla f \geq h \quad in \quad \mathcal{D}'((0,T)\times\Omega) \\ f(t=0,x) \geq f_0(x), \end{cases} \tag{135}$$

$$\begin{cases} \frac{d}{dt}[f(t,X(t,x))] \geq h(t,X(t,x)) \quad in \quad \mathcal{D}'((0,T)\times\Omega) \\ f(t=0,x) \geq f_0(x), \end{cases} \tag{136}$$

$$\begin{cases} \frac{d}{dt}[f(t,X(t,x))] \geq h(t,X(t,x)) \quad in \quad \mathcal{D}'((0,T)) \quad for\ a.e.\ x \in \Omega, \\ f(t=0,x) \geq f_0(x). \end{cases} \tag{137}$$

In this case, we also have that $f(t,X(t,x)) \in BV(0,T;\mathcal{M}(\Omega))$ and that for a.e. $x \in \Omega$, the function $f(t,X(t,x)) \in BV(0,T)$ and $h(t,X(t,x)) \in L^1(0,T)$.

Recall that f solves (135) in the weak sense means that for $\phi \in C_0^\infty([0,T)\times\Omega)$, we have

$$\int_0^T\int_\Omega f(-\partial_t\phi - u.\nabla\phi)dxdt - \int_\Omega f_0\phi(t=0)dx \geq \int_0^T\int_\Omega h\phi dxdt. \tag{138}$$

We will only give a sketch of the proof. We first use Lemma 2.3 of [60] to regularize (135) in the x variable. Let $\omega_\varepsilon(x) = \varepsilon^{-D}\omega(\frac{x}{\varepsilon})$, $\omega \in C_0^\infty(\mathbb{R}^D)$, $\int_{\mathbb{R}^D}\omega = 1$ and $\mathrm{Supp}(\omega) \in B(0,1)$. Hence, (135) yields

$$\begin{cases} \partial_t f_\varepsilon + u.\nabla f_\varepsilon \geq h_\varepsilon + r_\varepsilon \quad \text{in} \quad \mathcal{M}((0,T)\times\Omega) \\ f_\varepsilon(t=0,x) \geq f_{0,\varepsilon}(x), \end{cases} \tag{139}$$

where $f_\varepsilon = f\star\omega_\varepsilon$, $h_\varepsilon = h\star\omega_\varepsilon$, $f_{0,\varepsilon} = f_0\star\omega_\varepsilon$ and r_ε goes to zero in $L^1((0,T)\times\Omega)$. Hence, we deduce by making the change of variable $x \to X(t,x)$ that $\frac{d}{dt}[f_\varepsilon(t,X(t,x))] \geq (h_\varepsilon + r_\varepsilon)(t,X(t,x))$ and then we send ε to zero to deduce (136). Taking a test function of the form $\phi_1(t)\phi_2(x)$ in (136), we deduce that (137) holds. Finally, to prove that (137) yields (135), we take $\phi(t,y) \in C_0^\infty([0,T)\times\Omega)$ and denote $\psi(t,x) = \phi(t,X(t,x))$ and hence, $\partial_t\psi = \partial_t\phi + u.\nabla_y\phi \in L^2((0,T)\times\Omega)$. Hence, for a.e. $x \in \Omega$ we can use $\psi(t,x)$ as a test function in (137) and deduce that for a.e. $x \in \Omega$, we have

$$-\int_0^T f(t,X(t,x))\partial_t\psi(t,x)dt - f_0(x)\psi(t=0,x) \geq \int_0^T h(t,X(t,x))\psi(t,x)dt.$$

Since, both sides are in $L^1(\Omega)$, we can integrate in Ω and making the change of variable $y = X(t,x)$, we infer that (138) holds for $\phi(t,y)$.

Appendix B. The linear operator $\mathcal{L}$

Here, we study the linear operator $\mathcal{L}$ in the R variable given by

$$\mathcal{L}h = -\frac{1}{\psi_\infty} div(\psi_\infty \nabla h) \tag{140}$$

on the space $\mathcal{H}^0 = \mathcal{L}^2_k = L^2(\psi_\infty dR)$ and with the domain

$$D(\mathcal{L}) = \{h \in \mathcal{H}^0 | \nabla h \in \mathcal{H}^0, \quad \frac{1}{\psi_\infty} div(\psi_\infty \nabla h) \in \mathcal{H}^0 \quad \text{and}\, \psi_\infty \mathbf{n}.\nabla h|_{\partial B} = 0\}. \tag{141}$$

For a more detailed version of this appendix, we refer to [70]. We also refer to Subsection 3.3 of [68] where the operator $L\phi = -\mathrm{div}(\psi_\infty \nabla \frac{\psi}{\psi_\infty})$ is studied. It is easy to see that $\mathcal{L}h = \frac{1}{\psi_\infty} L(\psi_\infty h)$ and hence properties of $\mathcal{L}$ can be deduced from properties of L. We also recall that $\mathcal{H}^1_k$ was defined in (14) and we define $\mathcal{H}^2_k$ by

$$\mathcal{H}^2_k = \{g \in \mathcal{H}^1_k \quad | \int_B \psi_\infty \, |\nabla^2 g|^2 \, dx \, < \infty\} \quad \text{and} \quad \|g\|^2_{\mathcal{H}^2_k} = \int_B \psi_\infty \, [g^2 + |\nabla g|^2 + |\nabla^2 g|^2] \, dR.$$

It is not difficult to see (using elliptic regularity) that $D(\mathcal{L}) = \left\{h \in \mathcal{H}^2_k | \; \psi_\infty \mathbf{n}.\nabla h_{|\partial B} = 0\right\}$. One can give a sense to the boundary condition in the sense of traces or in a weak sense, namely $h \in D(\mathcal{L})$ if and only if $h \in \mathcal{H}^2_k$ and for any $g \in \mathcal{H}^1_k$, we have

$$\int_B \psi_\infty (\mathcal{L}h) g dR = \int_B \psi_\infty \nabla h.\nabla g \; dR. \tag{142}$$

We have the following proposition. For a proof, we refer to Proposition 3.6 of [68].

Proposition B.1. *$\mathcal{L}$ is self-adjoint and positive. Moreover, it has a discrete spectrum formed by a sequence (ℓ_n) such that $\ell_n \to \infty$ when $n \to \infty$.*

Let us define $\mathcal{H}^1_{k,0} = \overline{C^\infty_0(B)}^{\mathcal{H}^1_k}$. Hence, we have the following result.

Proposition B.2. *If $k > 0$, then $\overline{C^1(\overline{B})}^{\mathcal{H}^1_k} = \mathcal{H}^1_k$ and if $k \geq 1$ then*

$$\overline{C^\infty_0}^{\mathcal{H}^1_k} = \mathcal{H}^1_{k,0} = \mathcal{H}^1_k. \tag{143}$$

Proof. We will only give the proof of the second statement when $k = 1$ (see also Remark 3.7 of [68]). From (15), we have

$$\int_B \frac{|h|^2}{x(1 + \log(x)^2)} dR \leq C |h|^2_{\mathcal{H}^1_k}. \tag{144}$$

We define the function $\chi \in C^\infty(\mathbb{R})$ by $\chi(t) = 1$ for $0 \leq t \leq 1$, $|\chi'(t)| \leq 2$ for $1 \leq t \leq 2$ and $\chi(t) = 0$ for $t \geq 2$. For $h \in \mathcal{H}^1_k$, we take

$$h_n(R) = h(R) \chi(\frac{-\log(1 - |R|)}{n}).$$

It is clear that $h_n \in \mathcal{H}^1_k$. Moreover,

$$\|h - h_n\|^2_{H^1_k} \leq C \int_{1 - |R| \leq e^{-n}} \frac{|h|^2}{x(1 + \log(x)^2)} + \psi_\infty |\nabla \psi|^2 \quad dR$$

which goes to 0 when n goes to infinity. Now, it is easy to see (using mollifiers) that h_n can be approximated in $\mathcal{H}^1_k$ by a sequence in $C^\infty_0(B)$. This completes the proof of (143). □

It is clear that (143) does not hold when $k < 1$. Indeed, when $k < 1$, functions in $\mathcal{H}^1_k$ have a trace on ∂B, namely for $h \in \mathcal{H}^1_k$, $\gamma(h) = h_{|\partial B} \in H^{\frac{1-k}{2}}(\partial B)$ and γ is surjective from $\mathcal{H}^1_k$ onto $H^{\frac{1-k}{2}}(\partial B)$.

Remark B.3. *We point out that if $k \geq 1$, then the boundary condition $\psi_\infty \mathbf{n}.\nabla h|_{\partial B} = 0$ is a consequence of the fact that $h \in \mathcal{H}^2_k$ and hence $D(\mathcal{L}) = \mathcal{H}^2_k$. For the proof, we use the fact that, for all $h \in \mathcal{H}^2$, (142) holds for all $g \in C_0^\infty$. Then, we use that any $g \in \mathcal{H}^1$ can be approximated in $\mathcal{H}^1$ by a sequence $g_n \in C_0^\infty$. Hence, by passing to the limit we deduce that (142) holds for all $g \in \mathcal{H}^1$. This implies that $\psi_\infty \mathbf{n}.\nabla h|_{\partial B} = 0$ and hence, $D(\mathcal{L}) = \mathcal{H}^2_k$ (see Remark 3.8 of* [68] *and* [44] *for a probabilistic interpretation).*

The previous remark shows that when $k \geq 1$, $\mathcal{L}$ is similar to a Dirichlet boundary value problem. Whereas when $k < 1$, it is similar to a Neumann boundary value problem.

We can also define $\mathcal{H}^{-1}_k = (\mathcal{H}^1_{k,0})'$. Since C_0^∞ is dense in $\mathcal{H}^1_{k,0}$,we deduce that $\mathcal{H}^{-1}_k \subset \mathcal{D}'(B)$. It is not difficult to see that

$$\mathcal{H}^{-1}_k = \{\phi \in \mathcal{D}'(B) \,|\, \exists\, f_i \in \mathcal{L}^2_k \,|\, \phi = \frac{1}{\psi_\infty} \partial_i(\psi_\infty f_i)\} \quad \text{and}$$

that the operator $T(g) = -\frac{1}{\psi_\infty} div(\psi_\infty \nabla g)$ is an isomorphism from $\mathcal{H}^1_{k,0}$ into $\mathcal{H}^{-1}_k$ (see [70]).

B.1. **More about the boundary condition.**

Definition B.4. *i) For a vector $v \in L^1(B)$, we say that $v.\mathbf{n} = 0$ on ∂B if for all $\phi \in C^\infty(\overline{B})$, $\phi \geq 0$, there exists a sequence $\chi_n \in C_0^\infty(B)$ such that $0 \leq \chi_n \leq 1$ and χ_n converges to 1 in all $L^p(B)$, $1 \leq p < \infty$, and*

$$\int_B \operatorname{div}(v) \phi \chi_n \, dR + \int_B v.\nabla \phi \, dR \quad \to \quad 0 \tag{145}$$

when n goes to infinity. Here, we abused the notation since $\operatorname{div}(v)$ *is only a distribution and hence the left-hand side should be understood as* $\langle \operatorname{div}(v), \phi \chi_n \rangle_{\mathcal{D}' \times C_0^\infty}$.

ii) Under the same hypotheses as in i), we say that $v.\mathbf{n} \geq 0$ on ∂B if (145) *is replaced by*

$$\liminf_{n \to \infty} \int_B \operatorname{div}(v) \phi \chi_n \, dR + \int_B v.\nabla \phi \, dR \quad \geq \quad 0. \tag{146}$$

In the sequel, we will mostly use (145) and (146) when we also have a bound on $\operatorname{div}(u)$ in $L^1(B)$. If we also know that $\operatorname{div}(v) \in L^1(B)$, then (145) is equivalent to the fact that

$$\int_B \operatorname{div}(v) \phi \, dR = \int_B -u.\nabla \phi \, dR \tag{147}$$

and (146) is equivalent to $\int_B \operatorname{div}(v) \phi \, dR + \int_B u.\nabla \phi \, dR \geq 0$. One can then easily extend the Definition B.4 to the case where v also depends on other variables such as t and x. In particular if $v \in L^1((0,T) \times B)$ and $\operatorname{div}(v) \in W^{-s,p}((0,T); L^1(B))$, then $v.\mathbf{n} = 0$ on ∂B means that for $\phi \in C^\infty(\overline{B})$ and $\phi_1 \in C_0^\infty(0,T)$, we have

$$\left\langle \int_B \operatorname{div}(v) \phi \, dR, \phi_1 \right\rangle_{\mathcal{D}' \times C_0^\infty} = \left\langle \int_B -u.\nabla \phi \, dR, \phi_1 \right\rangle_{\mathcal{D}' \times C_0^\infty}. \tag{148}$$

It is easy to see that (148) and (3) give equivalent interpretations of (2).

Proposition B.5. *If $v \in L^1(B)$ and $\frac{v.R}{d(1+|\log d|)} \in L^1(B)$ then $v.\mathbf{n} = 0$ on ∂B, where we recall that $d = 1 - |R|$.*

Proof. Let χ be defined as in the proof of Proposition B.2 and $\chi_n(R) = \chi(\frac{-\log(1-|R|)}{n})$. Hence, for $\phi \in C^\infty(\overline{B})$, we have

$$\int_B \operatorname{div}(v) \phi \chi_n \, dR = -\int_B v.\nabla \phi \chi_n + \phi\, v \cdot \frac{R}{|R|} \chi'(\frac{-\log(d)}{n}) \frac{1}{d\, n} \, dR. \tag{149}$$

Now, notice that $\chi'(\frac{-\log(d)}{n})$ vanishes unless $n \leq -\log(d) \leq 2n$ and hence the second term on the right-hand side is bounded by

$$\int_{d\leq e^{-n}} |v.\mathbf{n}| \frac{1}{d\,(1-\log(d))}\,dR \tag{150}$$

which goes to zero when n goes to infinity. □

B.2. **Renormalization of $\mathcal{L}$.** We would like to understand how we can renormalize the singular parabolic inequality

$$\begin{cases} \partial_t h + \mathcal{L}h \quad \geq \quad \mathrm{div}_R(A(t,R)) + B(t,R), \\ \psi_\infty \nabla h.\mathbf{n} \geq 0, \qquad\qquad h(t=0,R) \geq h_0(R), \end{cases} \tag{151}$$

with the force term $F = \mathrm{div}_R(A(t,R)) + B(t,R)$, where $A \in L^1((0,T);\mathcal{L}^1_{k-1}) \cap L^2((0,T);\mathcal{L}^2_k)$ and $B \in L^1((0,T);L^1(\psi_\infty dR))$. We have the following Proposition, which is used in justifying the passage from (82) to (83).

Proposition B.6. *Assume that $h \in L^\infty((0,T);\mathcal{L}^2_k) \cap L^2((0,T);\mathcal{H}^1_k)$ solves* (151) *in the weak sense, namely for all $\phi \in C^\infty([0,\infty)\times \overline{B};\mathbb{R})$ compactly supported in $[0,\infty)\times\overline{B}$, $\phi \geq 0$,*

$$\int_0^\infty \int_B \psi_\infty[\nabla h.\nabla\phi - h\partial_t\phi]dRdt + \int_B \phi(t=0)h_0 dRdx \geq \int_0^\infty \int_B [\psi_\infty \phi B - A.\nabla(\psi_\infty\phi)]dRdt. \tag{152}$$

Hence, for $\beta \in C^\infty(\mathbb{R})$, β convex and β' bounded and nonnegative, $\beta(h) \in L^\infty((0,T);\mathcal{L}^2_k) \cap L^2((0,T);\mathcal{H}^1_k)$ solves

$$\begin{cases} \partial_t\beta(h) + \mathcal{L}\beta(h) + \beta''(h)|\nabla h|^2 \quad \geq \quad \mathrm{div}_R(\beta'(h)A(t,R)) - \beta''(h)\nabla h.A + \beta'(h)B(t,R), \\ \psi_\infty\nabla\beta(h).\mathbf{n} \geq 0, \qquad\qquad \beta(h)(t=0,R) \geq \beta(h_0)(R). \end{cases} \tag{153}$$

Proof. From the fact that $\partial_t h = -\mathcal{L}h + \mathrm{div}_R(A(t,R)) + B(t,R) + \mu$ for some measure $\mu \geq 0$ such that $\psi_\infty\mu \in \mathcal{M}([0,T)\times\overline{B})$, we deduce as in the Appendix C of [60] (see also Lemma 6.3 of [31] and [62]), that $h \in BV([0,T);\mathcal{L}^2_k - w)$ where $\mathcal{L}^2_k - w$ denotes the space $\mathcal{L}^2_k$ endowed with its weak topology (we refer to Chapter 5 of [30] for more about BV functions). Hence, the initial condition can be understood as $\mathrm{ess}\liminf_{t\to 0^+} h(t) \geq h_0$ where the limit is understood in the weak topology of $\mathcal{L}^2_k$. In other words, for all $\phi \in \mathcal{L}^2_k$, $\phi \geq 0$, we have $\mathrm{ess}\liminf_{t\to0^+}\int_B \phi\psi_\infty h(t)dR \geq \int_B \phi\psi_\infty h_0 dR$. Since, β is convex and $|\beta(h)| \leq Ch$, we deduce that $\lim_{t\to0^+}\beta(h(t)) \geq \beta(\lim_{t\to0^+}h(t)) \geq \beta(h_0)$. To prove the rest of the proposition, we write $\mathrm{div}_R(A(t,R)) = \frac{1}{\psi_\infty}\mathrm{div}_R(\psi_\infty A(t,R)) + \nabla\mathcal{U}.A$ and notice that by Proposition B.5, we deduce that $\psi_\infty A.\mathbf{n} = 0$ on ∂B. Hence, we get that

$$\psi_\infty\partial_t h - \mathrm{div}_R(\psi_\infty(\nabla h + A(t,R))) \geq \psi_\infty[B(t,R) + \nabla\mathcal{U}.A] \in L^1((0,T)\times B). \tag{154}$$

The rest of the proof is similar to the proof of Theorem E.1 in [60]. We regularize h in t by convolution and hence we can assume that h is smooth in t. Therefore, we have just to prove that if $-\mathrm{div}_R\Big(\psi_\infty(\nabla h + A(t,R))\Big) \geq H(R)$ with $H(R) \in L^1(B)$, $h \in \mathcal{H}^1_k$ and $\psi_\infty\nabla h.\mathbf{n} \geq 0$, then

$$-\mathrm{div}_R\Big(\psi_\infty(\nabla\beta(h) + \beta'(h)A)\Big) + \psi_\infty\beta''(h)\nabla h.(\nabla h + A) \geq \beta'(h)H. \tag{155}$$

To check this, we use a weak formulation, namely for $\phi \in C^\infty(\overline{B})$, $\phi \geq 0$, we have

$$\int_B \psi_\infty(\nabla h + A(t,R)).\nabla\phi\,dR \geq \int_B H(R)\phi(R)\,dR.$$

By density, this also holds for $\phi \in L^\infty \cap \mathcal{H}^1_k$. We use $\tilde{\phi} = \beta'(h)\phi$ as a test function, hence

$$\int_B \psi_\infty \Big[\beta''(h)\nabla h.(\nabla h + A)\phi + (\nabla\beta(h) + \beta'(h)A)\nabla\phi\Big]\,dR \geq \int_B \beta'(h)H\phi\,dR,$$

which is the weak formulation of (155) with the boundary condition $\psi_\infty(\nabla\beta(h)+\beta'(h)A).\mathbf{n} \geq 0$ on ∂B, which is equivalent to $\psi_\infty\nabla\beta(h).\mathbf{n} \geq 0$ on ∂B. □

Corollary B.7. *Assume that* $h \in C([0,T);\mathcal{L}^2_k) \cap L^2((0,T);\mathcal{H}^1_k)$ *solves*

$$\left\{ \begin{array}{lcl} \partial_t h + \mathcal{L}h & = & \mathrm{div}_R(A(t,R)) + B(t,R), \\ \psi_\infty \nabla h.\mathbf{n} = 0, & & h(t=0,R) = h_0(R), \end{array} \right. \tag{156}$$

in the weak sense, namely for all $\phi \in C^\infty([0,\infty)\times\overline{B};\mathbb{R})$ *compactly supported in* $[0,\infty)\times\overline{B}$, $\phi \geq 0$, (152) *holds with an equality. Hence, for* $\beta \in C^\infty(\mathbb{R})$ *such that* β' *and* β'' *are bounded,* $\beta(h) \in C([0,T);\mathcal{L}^2_k) \cap L^2((0,T);\mathcal{H}^1_k)$ *solves* (153) *with the inequalities replaced by equalities.*

The proof is the same as that of Proposition B.6. We just have to replace the $\leq$ sign by $=$ in (154) and (155). The only difference is that we can deduce the equality $\beta(h)(t=0,R) = \beta(h_0)(R)$ by using that $h \in C([0,T);\mathcal{L}^2_k)$ and hence that $\lim_{t\to 0^+} h(t) = h_0$ in $\mathcal{L}^2_k$. In particular we do not need that β is convex.

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