A Simple Dynamical Model with Features of Convective Momentum Transport

Andrew J. Majda and Samuel N. Stechmann *

Department of Mathematics and
Center for Atmosphere–Ocean Science,
Courant Institute, New York University, New York, New York

Article submitted to the
Journal of the Atmospheric Sciences
April 2, 2008

Revised June 3, 2008

*Corresponding author address: Samuel N. Stechmann, Courant Institute, New York University, 251 Mercer Street, New York, NY 10012.
E-mail: stechman@cims.nyu.edu
ABSTRACT

Convective momentum transport (CMT) plays a central role in interactions across multiple space and time scales. However, due to the multiscale nature of CMT, quantifying and parameterizing its effects is often a challenge. Here a simple dynamic model with features of CMT is systematically derived and studied. The model includes interactions between a large scale zonal mean flow and convectively coupled gravity waves, and convection is parameterized using a multicloud model.

The moist convective wave–mean flow interactions shown here have several interesting features that distinguish them from other classical wave–mean flow settings. First an intraseasonal oscillation of the mean flow and convectively coupled waves (CCWs) is described. The mean flow oscillates due to both upscale and downscale CMT, and the CCWs weaken, change their propagation direction, and strengthen as the mean flow oscillates. The basic mechanisms of this oscillation are corroborated by linear stability theory with different mean flow background states.

Another case is set up to imitate the westerly wind burst phase of the Madden–Julian Oscillation (MJO) in the simplified dynamic model. In this case, CMT first accelerates the zonal jet with strongest westerly wind aloft, and then there is deceleration of the winds due to CMT; this occurs on an intraseasonal time scale and is in qualitative agreement with actual observations of the MJO. Also, in this case, a multiscale envelope of convection propagates westward with smaller scale convection propagating eastward within the envelope. The simplified dynamic
model is able to produce this variety of behavior even though it has only a single horizontal direction and no Coriolis effect.
1. Introduction

Convective momentum transport (CMT) is the process of conversion of (moist) convective available potential energy to horizontal kinetic energy in the flow field. The significance of CMT was first established in explaining the organization of cumulus convection on mesoscales (LeMone 1983; LeMone et al. 1984), and important studies on the parameterization of CMT have followed. For instance, Wu and Yanai (1994) developed a parameterization of CMT based on a cumulus mass flux spectrum (Wu et al. 2007). Wu and Moncrieff (1996) showed in a cloud-resolving model (CRM) simulation that the generation of kinetic energy by CMT-generated shear is comparable to buoyancy generation and dominates the total buoyancy; thus, they argued for the need to represent CMT in convective parameterizations.

Besides these results on CMT due to mesoscale convection, CMT is also thought to play a central role at both the synoptic and planetary scales in multiscale interactions in the tropical atmosphere. The following are several examples that illustrate the importance of CMT through this range of larger scales.

It is well known from observations that the zonal winds in the Tropics oscillate on intraseasonal time scales due to the Madden–Julian oscillation (MJO) (Madden and Julian 1972; Madden and Julian 1994). Many details of the structure of the MJO have been examined in statistical composites of reanalysis data (Hendon and Salby 1994; Kiladis et al. 2005) and in observations of two individual MJO events that occurred during the Tropical Ocean Global Atmosphere Coupled Ocean–Atmosphere Response Experiment (TOGA COARE) (Lin and Johnson 1996; Yanai et al. 2000; Houze et al. 2000). Yanai et al. (2000) showed the dominant effect in the MJO of conversion of available potential energy, generated by con-
vective heating, to kinetic energy. Tung and Yanai (2002a,b) showed that on average CMT is downscale (damping on the large scales) but also that the fluctuations about the mean are very large with intense bursts of upscale transport (amplification on the large scales). In the westerly wind burst regime of the MJO, they found first upscale then downscale CMT. A well-known theoretical model for the MJO that roughly agrees with observations is a first baroclinic Kelvin–Rossby wave (Houze et al. 2000), but this paradigm does not account for important features of the observational record such as the midlevel westerly jet of the westerly wind burst and the horizontal quadrupole vortices. These latter features have been captured in more refined diagnostic models of the MJO that suggest they are a result of CMT from synoptic scale waves (Majda and Biello 2004; Biello and Majda 2005).

CMT on other scales has been posited to play a key role in the “MJO-like” structures in “superparameterization” computer simulations (Grabowski 2002, 2003, 2004; Moncrieff 2004). Further evidence of the dynamic role of CMT in those MJO-like structures is that the MJO-like wave develops when the small scale 2D models are oriented in the east–west direction, favoring zonal CMT; but the MJO-like wave does not develop when the small scale 2D models are oriented in the north–south direction (Grabowski 2002).

CMT also appears to be important in superclusters and CRM simulations of superclusters. The approximate self-similarity of tropical convection – from mesoscale cloud clusters to synoptic scale superclusters to the planetary scale MJO – has been documented in observations (Mapes et al. 2006), and a framework for this self-similarity has been developed with CMT playing a central role in the multiscale interactions (Majda 2007b). In CRM simulations of superclusters on large one-dimensional horizontal periodic domains of order 10,000 km, the importance of CMT varies because different studies have used large-scale
momentum damping of different strengths. For instance, CMT plays an active role in the simulations of Grabowski and Moncrieff (2001), but CMT is inactive in the simulations of Tulich et al. (2007), which have much stronger momentum damping. In addition, the pioneering CRM studies of Held et al. (1993) on a smaller mesoscale periodic domain of 640 km displayed an oscillation of the zonal wind that was called “QBO-like” because it resembled the quasi-biennial oscillation (QBO) of the tropical stratosphere. This QBO-like oscillation was shut down when the domain-mean momentum was damped. In their discussion section, it was asked, Is the oscillation in the troposphere due to interactions with the stratosphere, or is it due to CMT? The other results in this paragraph suggest that CMT might play a central role.

A common theme from the CMT examples above is that upscale transports from CMT can alter the large scale mean flow on both synoptic and planetary scales. Much has been learned about this process through models that diagnose the waves that generate the CMT (Moncrieff 1992, 2004; Majda and Biello 2004; Biello and Majda 2005). The goal here is to develop a simple dynamical model which demonstrates the nonlinear interaction between a large scale mean flow and convectively coupled waves (CCWs) with relevance for the synoptic and planetary scales. The important features to capture in such a dynamical model are that the mean flow can respond to the CCWs through upscale and downscale CMT, and, simultaneously, the nature of the mean flow determines the character of the convection and waves; thus, there are two-way interactions which need to be modelled. Convectively coupled Kelvin waves and two-day waves are a prominent feature of the observational record on equatorial synoptic scales of order 1,500 to 6,000 km (Wheeler and Kiladis 1999; Yang et al. 2007a,b,c); A prominent observed feature of these CCWs is a vertical tilt which means
that they can transport momentum to larger scales (see section 2e below for an elementary explicit demonstration). Here we utilize a recently developed multi-cloud model for CCWs with crude vertical resolution (Khouider and Majda 2006c, 2007, 2008b) to provide a base dynamical model for the CCWs. This model includes the effect of three cloud types—deep convective, stratiform, and congestus—and, like the observations, its CCWs have vertical tilts, which are crucial for CMT. The complete dynamical model involves a one-dimensional horizontal periodic domain with a large-scale, spatially independent, mean flow. The mean flow responds to the waves through CMT and upscale moisture and temperature fluxes, and, in turn, it alters the CCWs primarily through advection by the large scale winds and mean moist thermodynamic state. While it is well-known that the properties of mesoscale convective systems are determined by the environmental shear and thermodynamic conditions (Barnes and Sieckman 1984; Dudhia et al. 1987; Nicholls et al. 1988; LeMone et al. 1998; Lucas et al. 2000), it is less well-understood how CCWs can have different properties depending on the large scale environment (Wheeler and Kiladis 1999; Roundy and Frank 2004; Yang et al. 2007a,b,c). This is an important feature that the dynamical model developed here attempts to capture in addition to CMT. As shown below, this model has important features that distinguish it from models of classical wave–mean flow interactions [such as the QBO (Baldwin et al. 2001) and midlatitude baroclinic instability (Vallis 2006)].

The rest of the paper is organized as follows. The dynamic model is systematically derived and described in section 2. The simplest scenario with a regular intraseasonal oscillation of the mean flow is described in section 3, and cases with irregular oscillations and a climate base state are described in section 4. Discussion and conclusions follow in sections 5 and 6.
2. The dynamic model

The model used here consists of two parts. The first part describes the mean variables and the second part describes the waves. Conceptually, the model then takes the form (using the zonal velocity as an example)

\[
\frac{\partial \bar{U}}{\partial T} + \frac{\partial}{\partial z} \langle w'u' \rangle = 0 \tag{1}
\]

\[
\frac{\partial u'}{\partial t} + \bar{U} \frac{\partial u'}{\partial x} + w' \frac{\partial \bar{U}}{\partial z} + \frac{\partial p'}{\partial x} = S'_{u,1} \tag{2}
\]

where the notation is standard and described below with \( u' \) describing the smaller scale fluctuations and \( \bar{U} \) the large scale mean. Key interactions are (i) eddy flux convergence of wave momentum \( \partial_z \langle w'u' \rangle \) feeding the mean flow \( \bar{U} \), and (ii) advection of the waves \( u' \) by the mean flow \( \bar{U} \). The time scale \( T = \epsilon^2 t \) for the changes of the zonal mean flows in (1) is longer than that for the waves and is explained below. At this stage of the discussion, \( S'_{u,1} \) is a source of momentum for the smaller scales which will be described in more detail below.

This section is organized as follows. The starting point for the model is the multicloud model of Khouider and Majda (2008b) with advection, which is described in section 2a. The equations for the mean and wave variables are obtained from this model in sections 2b and 2c, respectively. In section 2d, a systematic derivation of these wave–mean equations is presented using multiscale asymptotics. A discussion of CMT and wave tilts is given in section 2e, and the numerical methods for the wave–mean equations are described in section 2f. Here only central or illustrative components of the model are described in order to streamline the presentation; details and complex formulas are relegated to appendices.
a. Multicloud model with advection

The starting point for the model is the multicloud model of Khouider and Majda (2008b) with advection terms added. The multicloud model is so-named because it provides a simple dynamical framework which parameterizes the effect of three cloud types, deep convective, stratiform, and congestus clouds, which are prominent in the observational record (Johnson et al. 1999). The original form of this model was described by Khouider and Majda (2006c), and the detailed structure of the CCWs in the model, including fidelity with the observations, has been documented extensively through linear theory (Khouider and Majda 2006c,b, 2008b,a) and in nonlinear simulations (Khouider and Majda 2007, 2006a, 2008b). A key feature of the multicloud models in agreement with observations is the westward (eastward) tilt with height in an eastward- (westward-) propagating CCW which allows for nonzero CMT onto the larger scales. This is a crucial feature for the dynamical models developed here.

In the multicloud model, the dynamical variables have a crude vertical structure that includes two vertical baroclinic modes:

\[
\begin{align*}
u(x, z, t) &= u_1(x, t) \sqrt{2} \cos(z) + u_2(x, t) \sqrt{2} \cos(2z) \\
\theta(x, z, t) &= z + \theta_1(x, t) \sqrt{2} \sin(z) + \theta_2(x, t) 2\sqrt{2} \sin(2z), \quad 0 \leq z \leq \pi,
\end{align*}
\]

where \(z\) has been nondimensionalized so that \(z = 0\) at the surface and \(z = \pi\) at the tropopause. Notice that the total potential temperature also includes a linear background stratification (which has been nondimensionalized). The setup used here is two-dimensional \((x-z)\) above the equator, so rotational effects of the Coriolis terms are ignored. The variables are nondimensionalized using the standard equatorial reference scales listed in Table 1 [see also, for example, Majda (2007b)]. Thus, the basic spatial scale in the dynamic model is the
equatorial synoptic scale of order 1500 km.

Here the multicloud model of Khouider and Majda (2008b) is used with nonlinear advection terms added as done by Stechmann et al. (2008). The full set of equations is shown in appendix A. To illustrate the main features of the multicloud model with advection, consider the equations for $u_2$ and $\theta_2$, the second baroclinic velocity and potential temperature:

\[
\frac{\partial u_2}{\partial t} - \frac{\partial \theta_2}{\partial x} = -\frac{1}{\tau_u} u_2 - 2\sqrt{2} \bar{U}_3 \frac{\partial u_1}{\partial x}
\]

\[= S_u + A_u \] (5)

\[
\frac{\partial \theta_2}{\partial t} - \frac{1}{4} \frac{\partial u_2}{\partial x} = H_c - H_s - R_2 - \frac{1}{2\sqrt{2}} \left[ (u_1 - \bar{U}_3) \frac{\partial \theta_1}{\partial x} - (\theta_1 - 9\bar{\Theta}_3) \frac{\partial u_1}{\partial x} + 8\bar{\Theta}_4 \frac{\partial u_2}{\partial x} \right]
\]

\[= S_\theta + A_\theta \] (7)

where the nonlinear advection terms and source terms are denoted by $A$ and $S$, respectively. The source terms $H_c$ and $H_s$ represent heating from congestus and stratiform clouds, and $R_2$ represents radiative cooling of the second baroclinic mode. In (7), the terms in brackets on the right hand side are the projection of nonlinear advection terms $u \partial_x \theta + w \partial_z \theta$ onto the second baroclinic mode, and the linear advection term on the left hand side is due to the background stratification. Advection terms involving the third and fourth baroclinic modes are shown for zonally averaged variables $\bar{U}_3$, $\bar{\Theta}_3$, and $\bar{\Theta}_4$. See appendix A and Khouider and Majda (2008b) for more details of the multicloud model, and see Stechmann et al. (2008) for more details on the nonlinear advection terms.

The equations for the mean flow and the CCWs are derived below by essentially averaging the multicloud model with advection (although there are a few caveats). The variables are split into mean and fluctuating components by using a space-time average discussed below:

\[
u_2(x, t) = \bar{U}_2(T) + u'_2(x, t),
\]

9
with similar expressions for the other variables, and where $T = \epsilon^2 t$ is a slow (intraseasonal) time scale as also explained below.

b. Equations for mean variables

To obtain equations for the mean variables, a space-time average is applied to the multicloud model with advection shown in (5)–(8) and in appendix A. The averaging involves a spatial average (denoted $\bar{f}$) over the periodic domain and a time average (denoted $\langle f \rangle$) that is explained below. The space-time average of, for instance, $u_2$ will be denoted by $\bar{U}_2$. Using $\bar{U}_2$ and $\Theta_2$ as examples, the equations (5)–(8) are averaged to give

$$\frac{\partial \bar{U}_2}{\partial T} = \langle A_u \rangle$$

(10)

$$\frac{\partial \Theta_2}{\partial T} = \langle S_\theta \rangle + \langle A_\theta \rangle,$$

(11)

As mentioned above, there is a caveat that makes these equations differ from a simple space-time average of (5)–(8): the large scale average of the momentum source terms, $\langle S_u \rangle = -\bar{U}_2/\tau_u$, is not included. The reason for this is that the momentum drag $-u_2/\tau_u$ is a parameterization of unresolved CMT, but the CMT affecting $\bar{U}_2$ is resolved by the eddy flux divergence, $\langle A_u \rangle$, so there is no need for a parameterization.

Another caveat is that mean variables are included for the third and fourth baroclinic modes ($\bar{U}_3$, $\bar{\Theta}_3$, and $\bar{\Theta}_4$), but the fluctuations of these modes ($u_3', \theta_3'$, and $\theta_4'$) will not be represented in the multicloud model for the CCWs. In fact, since the multicloud model includes wave momentum in only the first two baroclinic modes, it is only possible for the first four baroclinic modes to be affected by the advection terms $\langle A_u \rangle$ and $\langle A_\theta \rangle$. To illustrate the explicit form of the averaged advection terms, $\langle A_u \rangle$ and $\langle A_\theta \rangle$, consider the form of $\langle A_u \rangle$.
for $\bar{U}_j$, $j = 1, 2, 3$. The mean flow equations for the different baroclinic modes are

\[
\frac{d\bar{U}_1}{dT} = -\frac{1}{\sqrt{2}} \left\langle u'_2 \frac{\partial u'_1}{\partial x} - \frac{1}{2} u'_1 \frac{\partial u'_2}{\partial x} \right\rangle \tag{12}
\]
\[
\frac{d\bar{U}_2}{dT} = 0 \tag{13}
\]
\[
\frac{d\bar{U}_3}{dT} = -\frac{3}{\sqrt{2}} \left\langle -u'_2 \frac{\partial u'_1}{\partial x} - \frac{1}{2} u'_1 \frac{\partial u'_2}{\partial x} \right\rangle \tag{14}
\]

The mean thermodynamic variables satisfy similar equations that are obtained by space-time averaging the multicloud model equations shown in appendix A.

While one might also expect $\bar{U}_4$ to be needed here, it turns out that waves with only first and second baroclinic components can excite $\Theta_4$ but not $\bar{U}_4$. This is because the incompressibility equation leads to $w_2(x, t) \sin(2z) = -\partial_x u_2(x, t) \sin(2z) / 2$, from which it follows that $\bar{w}'_2 u'_2 = -\bar{u}'_2 \bar{u}'_2 / 2 = 0$. On the other hand, $\bar{\Theta}_4$ is needed because $\bar{w}'_2 \bar{\theta}'_2 \neq 0$.

c. Equations for convectively coupled waves

The equations for the synoptic scale fluctuating CCWs are obtained by decomposing each variable into mean and fluctuation parts as, for instance, $u_2(x, t) = \bar{U}_2(T) + u'_2(x, t)$. The wave equations are then obtained by subtracting the mean equations [such as (13)] from the full equations [such as (5)]. For example, the equation for $u'_2$ is

\[
\frac{\partial u'_2}{\partial t} - \frac{\partial \theta'_2}{\partial x} = -\frac{1}{\tau_u} u'_2 - 2\sqrt{2}\bar{U}_3 \frac{\partial u'_1}{\partial x} \tag{15}
\]

where the caveat regarding momentum damping was respected. Note that $u_3$, $\theta_3$, and $\theta_4$ have mean components but no wave components. Also note that $\bar{U}_4$ is not needed here for the reason given at the end of section 2b.
d. Asymptotic description of convectively coupled wave–mean flow interactions

The model described above includes two parts: a model for the mean variables such as \( \bar{U}_2 \), and a model for the waves such as \( u'_2 \). The main interactions between the waves and means occurs through (i) eddy flux divergences feeding the mean variables, and (ii) advection of the waves by the mean variables. This model can also be described in a framework of multiscale asymptotics (Majda and Klein 2003; Majda 2003, 2007b,a; Majda and Xing 2008). A partial description of the asymptotic model is given here for the zonal velocity \( u \), and a full derivation is given in appendix B. These asymptotic equations serve as motivation for the form of the model in sections 2c and 2b. They also serve as a partial explanation for the evolution of the mean flow on intraseasonal time scales, which is described later in sections 3 and 4.

Assume the zonal velocity \( u \) consists of a mean flow \( \bar{U} \) and waves \( u' \):

\[
u = \bar{U}(z, \varepsilon^2 t) + \varepsilon u'(x, z, t, \varepsilon^2 t) + O(\varepsilon^2),
\]

where \( \varepsilon \ll 1 \) is a small parameter, and \( x \) and \( t \) have been nondimensionalized by the reference values in Table 1. The parameter \( \varepsilon \) is essentially the Froude number as in Majda (2007b). Thus, the zonal mean flow can be large (up to 50 m s\(^{-1}\)), while the waves are weaker (of order 5 m s\(^{-1}\)). The mean flow \( \bar{U} \) is a function of height \( z \) and a long time scale \( \varepsilon^2 t \), and it is a zonally averaged flow. The waves \( \varepsilon u' \) are also functions of the zonal coordinate \( x \) and the synoptic time scale \( t \), and they are \( O(\varepsilon) \) in magnitude. Other variables (pressure \( p \), potential temperature \( \theta \), etc.) are assumed to have similar expansions, and \( u \) is assumed to evolve as

\[
\frac{\partial u}{\partial t} + \bar{u} \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = S_u,
\]

where

\[
u = \bar{U}(z, \varepsilon^2 t) + \varepsilon u'(x, z, t, \varepsilon^2 t) + O(\varepsilon^2),
\]
where $S_u$ is a momentum source. With these assumptions, the equations for wave–mean flow interaction from (1)–(2) can be derived using multiscale asymptotics. The notation $f = \bar{f} + f'$ is a decomposition into zonal mean and fluctuation, and the angle brackets $\langle f \rangle$ denote an average over the synoptic time scale $t$. (The time average is described in a practical setting in section 2f and in a theoretical setting in appendix B.) The mean flow $\bar{U}$ evolves on an intraseasonal time scale $T = \epsilon^2 t$, and it is altered by eddy flux divergence of wave momentum, $\partial_z \langle w'u' \rangle$. The waves $u'$ evolve on the synoptic time scale $t$ (and are modulated on the intraseasonal time scale $T = \epsilon^2 t$) and are advected by the large scale mean flow $\bar{U}$. Thus there are two-way feedbacks between the waves and the mean flow. See appendix B for details of the derivation and the equations for other variables. See Majda and Xing (2008) for another example for squall line dynamics of multiscale asymptotics with a non-low Froude number velocity contribution from $\bar{U}$.

Note that a choice of $\epsilon \approx 0.1$ implies that $T = \epsilon^2 t$ is an intraseasonal time variable. To see this, recall that $t$ is nondimensionalized by the reference scale $T_E \approx 8$ hours (see Table 1). Thus $t$ has magnitude $O(1)$ on the time scale $T_E \approx 8$ hours, and it has magnitude $O(\epsilon^{-2})$ on the time scale $\epsilon^{-2} T_E \approx 30$ days. The variable $T = \epsilon^2 t$ then has magnitude $O(1)$ on the intraseasonal time scale $\epsilon^{-2} T_E \approx 30$ days, so it is an intraseasonal time variable. In addition, the choice $\epsilon \approx 0.1$ also implies low Froude number dynamics for the CCWs in the ansatz in (16), and this scaling is corroborated by numerical simulations in subsequent sections with the model (see Figure 4).
Earlier studies of CMT have emphasized the importance of wave tilts for upscale momentum transport (Moncrieff 1992; Majda and Biello 2004; Biello and Majda 2005). Since the multi-cloud model used here for the CCWs includes two baroclinic modes, the CCWs are tilted with height and can transport momentum upscale or downslope. Recall from earlier discussion that the CCWs in the multi-cloud model, like the observations, have a westward (eastward) tilt with height in an eastward- (westward-) propagating CCW (Khoudi and Majda 2006c,a, 2007, 2008b,a).

A simple kinematic illustration of these effects can be made with a weak temperature gradient (WTG) model, which has been derived, for instance, in the multiscale BMESD model of Majda (2007b). This model applies on equatorial synoptic scales of order 1500 km. In this model, there is a balance ($w' = S'_\theta$) between the vertical velocity $w'$ and the potential temperature source $S'_\theta$, which represents convective heating. As a simple model of $S'_\theta$ for a tilted wave, consider a two-dimensional ($x$-$z$) setup and a heat source with two phase-lagged baroclinic modes: $S'_\theta = k \cos[kx - \omega t] \sqrt{2} \sin(z) + \alpha k \cos[k(x + x_0) - \omega t] \sqrt{2} \sin(2z)$. Two key parameters here are $\alpha$, the strength of the second baroclinic heating, and $x_0$, the lag between the heating in the two vertical modes. The vertical velocity is then given by WTG balance, $w' = S'_\theta$, and the zonal velocity is given by the continuity equation, $u'_x + u'_z = 0$:

\begin{align*}
    u'(x, z, t) &= -\sin[kx - \omega t] \sqrt{2} \cos(z) - 2\alpha \sin[k(x + x_0) - \omega t] \sqrt{2} \cos(2z) \\
    w'(x, z, t) &= k \cos[kx - \omega t] \sqrt{2} \sin(z) + \alpha k \cos[k(x + x_0) - \omega t] \sqrt{2} \sin(2z)
\end{align*}

While they will not be needed for the exposition here, the thermodynamic variables in this model are given by the balance equations $\rho' = \Phi'_s$ and $\theta' = \rho'_z$, where $\Phi'_s$ is the velocity
potential for momentum sources.

With this form of $u'$ and $w'$, the eddy flux divergence is

$$
\partial_z \langle w' u' \rangle = \frac{3\alpha k}{2} \sin(kx_0)[\cos(z) - \cos(3z)]
$$

(20)

Notice that a wave with first and second baroclinic components generates CMT that affects the first and third baroclinic modes (Majda and Biello 2004; Biello and Majda 2005). The third baroclinic mode was not included in earlier work with the multicloud model, and a third baroclinic wave momentum $u'_3(x, t)$ for the fluctuations is still not included here. However, a third baroclinic mode mean flow, $\bar{U}_3(T)$, is included in (14) in the model used here in order to capture the large scale effect of CMT, and it will play an important role in the dynamics. Also notice that (20) is nonzero as long as $\alpha \neq 0$ (i.e., there are both first and second baroclinic mode contributions) and $x_0 \neq 0$ (i.e., there is a phase lag between the first and second baroclinic modes). The CCWs in the multi-cloud model typically have this structure (Khouider and Majda 2006c, 2008b).

f. Numerical methods

The model used here involves dynamics on two time scales. The waves $u'$ evolve on a fast time scale $t$ as

$$
\frac{\partial u'}{\partial t} = f(u', \bar{U}),
$$

(21)

and the mean flow $\bar{U}$ evolves on a slow time scale $T = \epsilon^2 t$ as

$$
\frac{\partial \bar{U}}{\partial T} = g(u', \bar{U}),
$$

(22)
To solve such a system of equations numerically, two different time steps, $\Delta t$ and $\Delta T$, are chosen with $\Delta t \ll \Delta T$. Here $\Delta T = 10\Delta t$ is used. First the waves $u'(x, t_0)$ are stepped forward in time with time step $\Delta t$ to obtain $u'(x, t_0 + \Delta t), u'(x, t_0 + 2\Delta t)$, etc., while the mean flow $\bar{U}(t_0)$ is held frozen. Since $g(u', \bar{U})$ in (22) involves time and space averages such as $\langle w'u' \rangle$, these averages are calculated over the long time interval $\Delta T = 10\Delta t$, after which the mean flow variables $\bar{U}(t_0)$ are updated to $\bar{U}(t_0 + \Delta T)$ using (22). Then the cycle is repeated with the waves $u'$ being stepped forward with time step $\Delta t$ while the mean flow $\bar{U}(t_0 + \Delta T)$ is held fixed. See Grabowski (2004) and Majda (2007a) for other examples and references for this technique.

### 3. Regular intraseasonal oscillations of the mean flow – the simplest scenario

The case shown in this section involves the model in its simplest setup. The results will show an intraseasonal oscillation of the mean variables along with modulations of the CCWs on the same time scale. The CCWs change their propagation direction as the mean variables change, and their CMT (both upscale and downscale) causes the mean variable oscillations. This case uses a trivial climatological base state so that $\bar{U}$ oscillates about the state $\bar{U} = 0$. This case also has a symmetric mean flow evolution between the first and third baroclinic modes with $\bar{U}_1(T) = -\bar{U}_3(T)$ and $\bar{U}_2(T) = 0$ for all times.
a. Life cycle of the basic oscillation

Figures 1 illustrates one transition of the mean flow $\bar{U}(z, T)$ and CCWs. At time $t = 550$ days, there is a low-level easterly jet and a mid-level shear with $\partial \bar{U} / \partial z > 0$. At this time, the CCWs have not yet developed coherently. By $t = 555$, an eastward-propagating CCW has developed. Thus eastward-propagating CCWs are favored by this mean flow with a low-level easterly jet (and this will be corroborated by linear stability results shown below). The CCW reaches its peak amplitude at $t = 555$ to 560 days. At this time, the low-level easterly jet has weakened, and it transitions to a low-level westerly jet by $t = 570$ days. The eastward-propagating CCW tilts from east to west with height. Thus, according to the discussion in section 2e, the CMT of the eastward-propagating CCW first damps the low-level easterly jet and then generates a low-level westerly jet. The amplitude of the low-level westerly jet is then increased in this stage due to upscale CMT from the CCW. A low-level westerly jet, however, is unfavorable for the eastward-propagating CCWs (this will be corroborated by linear stability results shown below). As long as the CCW propagates eastward, its amplitude continues to weaken, and its CMT continues to strengthen the low-level westerlies, even when its amplitude has weakened. Thus, the CCW weakens until $t = 600$ days, after which a westward-propagating CCW forms, and the next transition repeats in the same fashion.

The multi-cloud model does not resolve squall lines in detail, yet it is important that the above CCW–mean flow dynamics are broadly consistent with known properties of these mesoscale features. Note that the low-level easterly jet at $t = 555$ to 600 days should also favor westward-propagating squall lines if squall lines were resolved here (Barnes and
Sieckman 1984; Dudhia et al. 1987; Nicholls et al. 1988; LeMone et al. 1998; Lucas et al. 2000). In addition, observations and simulations often show CCWs propagating in the opposite direction of the mesoscale convective systems within them (Nakazawa 1988; Grabowski and Moncrieff 2001; Tulich et al. 2007). Thus the favored propagation direction of the CCWs (eastward) is consistent with what would be expected from observations of squall lines and simulations of multiscale CCWs with the large scale wind $\bar{U}_3(T)$ determining the sign and strength of the low-level shear.

Fig 2 shows the evolution of the large scale moisture, $\bar{Q}$, and boundary layer equivalent potential temperature, $\bar{\Theta}_{eb}$, presented as deviations from the basic mean sounding which has been taken as a radiative–convective equilibrium (Khouider and Majda 2006c, 2008b). These variables oscillate with a period of roughly 50 days as the CCWs intensify and weaken along with the mean flow. Notice that the oscillation amplitude of the fluctuations of $\bar{Q}$ and $\bar{\Theta}_{eb}$ is roughly 0.1 K, which is small compared to the amplitude of $q'$ and $\theta'_{eb}$ (roughly 1 K, not shown). All of these values are small compared with the mean thermodynamic base state and reflect the validity of the asymptotic model.

b. Nonlinear propagation dynamics and linear theory

The nonlinear results of Figures 1–2 can be corroborated by linear stability theory, i.e., either eastward- or westward-propagating waves are favored depending on the mean flow $\bar{U}$. Linear theory results with a resting background state $\bar{U} = 0$ have been reported for the multicloud model in several earlier papers (Khouider and Majda 2006c,b, 2008b). Here advection terms have been added to the multicloud model, and results are shown for linear
stability theory for the equations in appendix A with a background wind shear. The back-
ground shear will be chosen every 10 days from \( t = 550 \) days to 600 days, which are the
times of the snapshots from Figure 1a.

Linear stability results are shown in Table 2, and results for the case of \( t = 550 \) days
are also shown in Figure 3. At \( t = 550 \) days, when there is a low-level easterly jet, the
maximum growth rate of the eastward-propagating CCW is more than twice as large as that
of the westward-propagating CCW. After \( t = 570 \) days, when a low-level westerly jet has
replaced the low-level easterlies, the westward-propagating CCW has a larger growth rate.
This corroborates the nonlinear simulation results shown in Figure 1.

Figure 4 shows snapshots of the velocity field, including both the wave and mean flow,
at times \( t = 560 \) and 580 days. At \( t = 560 \) days, there is strong low-level, \emph{front} inflow into
the wave, partly due to the low-level easterly jet of the mean flow. At \( t = 580 \) days, there is
a strong \emph{rear}-inflow jet at low-levels and weak \emph{front} inflow into the wave; this is partly due
to the low-level westerly jet of the mean flow at this time. This change in the inflow into the
convective region likely plays a role in the strength of the CCW, and it is clear that changes
in the mean flow play a major role in the changes in the inflow.

In Figure 4a for \( t = 560 \) days, \( \bar{U}_3 > 0 \) and there are thus enhanced low-level easterlies in
the mean flow which increase the second baroclinic inflow, wave tilt, and potential congestus
preconditioning in this eastward-propagating CCW. On the other hand, in Figure 4b for
\( t = 580 \) days, \( \bar{U}_3 < 0 \) and there are thus enhanced low-level westerlies which decrease the sec-
ond baroclinic inflow, wave tilt, and congestus preconditioning of the eastward-propagating
CCW. This intuition is confirmed by the detailed structure of the waves from linear stability
analysis reported next. The results above showed that at the most unstable wavelength, the
eastward-propagating CCW is favored with the largest growth rate for $\bar{U}_3 > 0$, while the westward-propagating CCW is favored for $\bar{U}_3 < 0$. In Figure 5, bar diagrams from linear theory show the relative strength of wave components of the most unstable eigenvectors (Khouider and Majda 2006c, 2008b) for $\bar{U}_3 > 0$ at the maximum amplitude for the mean flow. These confirm the above intuition with order 20% larger congestus heating, $H_c$, and second baroclinic wave components, $u_2$ and $\theta_2$, in the favored eastward-propagating CCW compared with the less-favored westward-propagating CCW while the first baroclinic components, $u_1$ and $\theta_1$, remain unchanged. This decrease in second baroclinic mode amplitude relative to first baroclinic mode amplitude amounts to a decrease in the vertical tilts of the CCW, as discussed above in section 2e. The other important quantity for wave tilts is the lag between first and second baroclinic mode variables, which is the same for the eastward- and westward-propagating waves shown in Figure 5b,d.

c. Sensitivity studies

To investigate the sensitivity of the oscillation to the domain size, the numerical experiment presented above was carried out using four different periodic domain sizes: 8,000, 6,000, 4,000, and 2,000 km. The results already shown used a 6,000 km domain width. Table 3 shows how the oscillation changes as the domain size changes. The amplitude of the oscillation is measured by the amplitude of the mean flow jet max, which attains its maximum value for the 6,000 km domain width. The oscillation time increases as the domain width decreases, ranging from 34 days for 8,000 km to 100 days for 2,000 km. With a 6,000 km domain width, the oscillation time is 53 days, as shown in Figures 1–2 and Table
2. Note from Table 3 that the amplitude of the mean oscillation is not a monotone function of domain width. This reflects the strength of the CCWs, which is a complex function of the mean flow itself as shown above in section 3b.

In summary, the oscillation time scale is intraseasonal for a range of synoptic scale domain widths. Why is the time scale intraseasonal? One answer to this question is that the model used here has the same form as an asymptotic model that is derived under the assumption of a separation of time scales; that is, the asymptotic derivation in section 2d predicts self-consistently a separation of time scales between the synoptic scale waves and the intraseasonal mean variables, and the ansatz in (16) is also consistent with the amplitude of the CCWs in Figure 4. Note that in standard fashion, to check the validity of the asymptotic expansion implied by (16), the fluctuating components need to remain low Froude number as illustrated in Figure 4, but the magnitude of the large scale flow $\bar{U}$ just needs to be order one or less in magnitude to retain asymptotic validity. In the present example, the mean flow $\bar{U}$ is also low Froude number throughout the regular oscillation period. Similar remarks apply to the temperature and moisture fluctuations in Figure 2.

In order to test the sensitivity of these results to the convective parameterization, we increased the adjustment time for deep convection from the standard value $\tau_{\text{conv}} = 1$ hour to $\tau_{\text{conv}} = 1.5$ hours (see appendix A). For 4,000 and 6,000 km domains, the oscillation of the mean and character of the CCW dynamics is very similar to the standard case. For a larger domain of 8,000 km, the basic oscillation was sustained for the first 200 days of the simulation and then spontaneously decayed when the single propagating CCW changed to two oppositely propagating CCWs. These oppositely propagating CCWs became a standing wave pattern, and, at the same time, the mean wind decayed to zero. With a 10,000 km
domain, the standard basic oscillation of the mean wind was restored with a pair of CCWs (essentially wavenumber 2) propagating in the same direction as in the basic oscillation during each organized phase of the waves. There are caveats in utilizing such large scale domains for the model as discussed in section 5. All of these facts reflect the sensitive dependence on parameters in turbulent chaotic dynamical systems. The next section describes other examples with nonzero climatological base states and further interesting dynamical phenomena.

Another parameter that was varied in sensitivity studies is the ratio between the time steps of the fast time scale \( t \) and the slow time scale \( T = \epsilon^2 t \). All simulations reported here use the standard value \( \Delta T = 10\Delta t \) as described in section 2f. The simulation described in Figures 1–5 was also repeated using \( \Delta T = 5\Delta t, 20\Delta t, \) and \( 50\Delta t \) (not shown), and no differences were seen among the simulations.

4. Irregular intraseasonal oscillations and multi-scale waves with a climate base state

In the previous section, the model had a regular oscillation about a resting state \( \bar{U} = 0 \). If a different initial condition is chosen, an oscillation can develop about a climatological background state with \( \bar{U} \neq 0 \). Two other climatological regimes are considered in this section. The first regime is similar to the westerly wind burst stage of the MJO, and the second regime is similar to the westerly onset stage of the MJO (Lin and Johnson 1996; Houze et al. 2000; Tung and Yanai 2002a,b; Majda and Biello 2004; Biello and Majda 2005).
a. Westerly wind burst stage

One case with strong low-level westerlies is shown in Figure 6. The mean flow oscillates about a climate base state that is mostly first baroclinic, i.e. the \( \cos z \) term dominates, but CMT causes the maximum low level winds to shift aloft to \( z = 3 \) or \( 4 \) km as occurs from \( t = 1040 \) to \( 1070 \) days. This phase in the cycle of the zonal winds in the simple dynamical model strongly resembles the one for the zonal winds in the westerly wind burst stage of the MJO from the observational record (Lin and Johnson 1996; Tung and Yanai 2002a,b). First at time \( t = 1040 \), the shear is entirely first baroclinic (see Figure 7) with the maximum of the westerlies at the base of the troposphere as in the westerly onset stage which is dominated by deep convection. Tung and Yanai (2002a,b) use the diagnostic

\[
\frac{U(z,T)}{|U|} \frac{\partial U}{\partial t} > 0 \quad ( < 0)
\]

(23)

to denote acceleration (deceleration) of the zonal jet where \( \partial U/\partial t \) is measured from turbulent transports in the observations. In the westerly wind burst phase of the MJO, they find first a phase of acceleration of the zonal winds in the lower troposphere due to CMT which is followed by a phase of deceleration of these westerly winds (Tung and Yanai 2002b). This is exactly what happens in the simple model due to CMT as shown in the upper panels of Figure 6. The zonal winds in the lower troposphere first accelerate between \( t = 1040 \) to \( 1070 \) days where a strong westerly wind burst develops aloft, as in the observations, and then decelerate at the times beyond \( t = 1070 \) days. Since the first baroclinic shear is large in the present situation with \( |U_1| \gg |U_3| \), the diagnostic of Tung and Yanai (2002a,b) for the low level westerly winds for \( 1040 \) days \( \leq t \leq 1070 \) days is essentially \( U_1 \partial U_1/\partial t > 0 \) \((< 0)\) for acceleration (deceleration) of the low-level westerly winds; the graph of \( U_1 \) in Figure
7b clearly exhibits the same acceleration for $1040 \text{ days} \leq t \leq 1070 \text{ days}$ followed by the deceleration shown in Figure 6c. What happens in the simple dynamical model between times $t = 1040$ and $1070 \text{ days}$ is a coherent eastward-propagating CCW (see Figure 6) which affects the zonal mean flow through CMT as discussed earlier in section 3 and drives the acceleration of the westerly zonal wind. Masunaga et al. (2006) has noted the prominent occurrence in observations of eastward-propagating convectively coupled Kelvin waves in the westerly wind burst phase of the MJO. This occurs, for instance, as the CCW propagates eastward from $t = 1040$ to $1070 \text{ days}$. [This is also the same role played by eastward-propagating superclusters in a recent diagnostic multiscale model of the MJO (Majda and Biello 2004; Biello and Majda 2005).] Note that this analogous behavior occurs in this simple dynamical model even though it is one-dimensional horizontally and without Coriolis effects.

The most striking feature of Figure 6 is the occurrence of multiscale waves with envelopes propagating westward with smaller scale convection propagating eastward within the envelope. These multiscale waves appear in the transition phases between instances of coherent CCWs propagating in opposite directions. At these stages the wave patterns resemble those in the CRM simulations of Grabowski and Moncrieff (2001). The occurrence of both coherent and scattered convection is also reminiscent of the CRM simulations in Grabowski et al. (1996), although their results were on smaller scales, and their mean variables were prescribed, not dynamic.

Figure 7 shows the evolution of $\bar{U}$, which indicates how CMT changes with time. The most rapid changes in $\bar{U}$ occur in the presence of intense westward-propagating CCWs, whereas the CMT is nearly zero (i.e., $\bar{U}$ is slowly changing) while the multiscale waves are present. This demonstrates that waves are most effective at generating CMT when they are
both coherent and intense.

Figure 8 and Table 4 show linear stability theory that corroborates these results. For the mean flow with strongest westerlies aloft near $z = 3–4$ km, which occurs around $t = 1000$ and 1070 days, the most unstable waves are westward-propagating with wavelengths of 1200–1500 km. However, there is also a wide band of unstable eastward-propagating waves at smaller scales. This is consistent with the multiscale waves that appear at these times. The smaller scale band of unstable modes is less pronounced at time $t = 1030$, but the eastward- and westward-propagating waves have similar growth rates. Consistent with this, there are multiscale waves at this time in Figure 7.

b. Westerly onset stage

Figure 9 shows three cases similar to the westerly onset stage of the MJO. The three cases differ in the strength of $\bar{U}_2$, which takes the values 3, 4, and 5 m s$^{-1}$. Each case has a mid-level easterly jet and westerlies at low levels. The case in Figure 9ab uses $\bar{U}_2 = 3$ m s$^{-1}$, and the mean wind variables $\bar{U}_1$ and $\bar{U}_3$ show an irregular oscillation. The transition times of the mean wind vary widely from roughly 50 to 100 days. The longer, 100-day transitions appear to coincide with waves of wavenumber 2 (not shown). This is consistent with the results above, which showed the strongest CMT coming from intense, coherent CCWs and weaker CMT coming from multiscale CCWs.

Figures 9c-f show the cases with $\bar{U}_2 = 4$ and 5 m s$^{-1}$. These cases show small amplitude oscillations about a mid-level jet and decay to a mid-level jet, respectively. This type of behavior is characteristic of a Hopf bifurcation (Verhulst 1990). The waves in these two
cases propagate both eastward and westward at the same time (not shown), with slight changes in their amplitudes for the case in Figures 9cd.

5. Discussion

The results shown in sections 3 and 4 demonstrate why the multicloud model and the mean wind component $\bar{U}_3$ are needed in this convective wave–mean interaction model. One reason is that the multicloud model includes two vertical baroclinic modes, so its CCWs have vertical tilts with nontrivial eddy flux divergence $\partial_z \langle w'u' \rangle$; in this way the waves can transport momentum upscale to alter the zonal mean flow in $\bar{U}_1$ and $\bar{U}_3$. Another reason is that the mean flow component $\bar{U}_3$ affects which waves of the multicloud model are favored – either eastward- or westward-propagating. Furthermore, as demonstrated in section 3, when $\bar{U}_3 > 0$ so that there is an easterly jet shear in the lower troposphere, eastward-propagating CCWs are favored; such an easterly jet shear favors westward-propagating squall line clusters. Thus, the dynamic model for CCW–mean flow interactions on the equatorial synoptic scale is broadly consistent with the fact that from observations (Nakazawa 1988) and CRM simulations (Grabowski and Moncrieff 2001) that embedded squall line clusters propagate opposite to the direction of propagation of CCWs, even though squall lines are not resolved in the present model.

Obtaining the MJO in GCM simulations is a major multiscale challenge, and the results in sections 3 and 4 for the simple dynamical model give insight into the large scale impact of CMT in the MJO even though the simple model has a single horizontal dimension and no Coriolis effect. As discussed in detail in section 4a for the westerly wind burst scenario, the
simultaneous occurrence of eastward-propagating CCWs and first the acceleration followed by the subsequent deceleration of the low-level zonal winds with the strongest westerly winds aloft is analogous to what actually occurs in the observational record during the westerly wind burst phase of the MJO (Tung and Yanai 2002a,b). The results in sections 3 and 4 further support the results of the the multi-scale diagnostic model of Majda and Biello (2004) and Biello and Majda (2005), which suggest that upscale CMT from CCWs is crucial for obtaining the full structure of the observed MJO. For instance, in Figure 6, CMT from eastward-propagating CCWs raises the height of maximum westerlies from the surface to $z = 3$ or $4 \text{ km}$. This is also the role of the superclusters in the multiscale diagnostic model of Majda and Biello (2004) and Biello and Majda (2005). The results in sections 3 and 4 also suggest that a mean wind varying on intraseasonal time scales could favor or suppress the formation of certain types of CCWs. It is thus possible that the MJO evolves cooperatively with the CCWs within its envelope, and a successful simulation of the MJO in GCMs may require an accurate representation of CCWs as well (Lin et al. 2006).

The behavior of CMT in the simple dynamical model also has direct contact with the behavior found in the CRM simulations of Grabowski and Moncrieff (2001). These CRM simulations occur on a planetary scale domain of size 20,000 km in a single horizontal dimension without Coriolis effects and show the clear development of a wave train of CCWs moving eastward with embedded westward-propagating squall lines for a time period of 40 days. Figure 16 of that paper demonstrates large synoptic scale CMT present in these wave trains of CCWs; the simple model developed here captures the effect of CMT of such synoptic scale CCWs on the zonal winds and their two-way interactions on much longer time scales than these simulations and for a synoptic scale localized wave rather than a planetary
scale wave packet. On the other hand, Figure 17 of that paper demonstrates that the planetary scale CMT averaged over the global 20,000 km domain is much weaker due to the large areas where there is suppressed convection with no wave activity. These differences point toward the need to generalize the expansion in (16) to include wave packets modulated on the planetary scale in (16) to study the local synoptic scale transfer of CMT (Majda and Biello 2004; Biello and Majda 2005; Biello and Majda 2008). It is also interesting to note that the damping of Grabowski and Moncrieff (2001) was weaker (with a time scale of 1 day), and CMT was found to play an active role in the formation of CCWs, whereas the damping in the similar simulations of Tulich et al. (2007) was much stronger (with a time scale of 4 hours), and CMT was not found to play a role in the CCW dynamics; this is possibly because upscale CMT was not able to overcome the intense prescribed momentum damping. It is possible that a multiscale approach to CMT like the one in sections 2–4 could be useful for simulations of such multiscale waves.

The results in this paper may have implications for other efforts to simulate multi-scale waves. Held et al. (1993) use a mesoscale periodic domain of only 640 km. When they allowed CMT to drive the domain-mean flow, their mean flow oscillation had a period of roughly 70 days, and the smaller scale convection changed its propagation direction as the mean flow changed directions. When they constrained the domain-mean CMT to be zero, the QBO-like oscillation was shut down. The present model cannot be applied directly to analyze these simulations for two reasons. First, the coherent behavior on the smaller scales is due to squall line clusters and as discussed in the first paragraph of this section, the simple dynamical model both does not resolve squall lines and furthermore the synoptic scale CCWs propagate in the opposite sense with opposite wave tilts as the squall lines.
Second, the mesoscale spatial domain of 640 km is too far below the equatorial synoptic scale of 1500 km to accurately trust the model and a different non-dimensionalization is important. Nevertheless, with these strong caveats, the results in the present paper favor a CMT explanation for this oscillation. However, a different but similar model on mesoscales needs to be developed to confirm this.

6. Conclusions

A simple model with features of CMT was derived and studied. The model included CCWs and zonally averaged mean variables, and it conceptually was of the form

\[
\frac{\partial \bar{U}}{\partial T} + \frac{\partial}{\partial z} \langle w'w' \rangle = 0
\]

\[
\frac{\partial u'}{\partial t} + \bar{U} \frac{\partial u'}{\partial x} + w' \frac{\partial \bar{U}}{\partial z} + \frac{\partial p'}{\partial x} = S'_{u,1}
\]

The convective wave–mean interactions were present in eddy flux divergences and in advection of the CCWs by the mean flow. The convective parameterization was the multicloud model of Khouider and Majda (2006c, 2008b), which captures realistic CCWs with vertically tilted structures. Another key feature was that momentum damping on synoptic scales was parameterized by a drag term \(-u'/\tau_u\), but CMT for the large scale mean flow was driven by the CMT from the resolved synoptic scale CCWs and can be either upscale or downscale. A systematic asymptotic derivation of the model was also given, and the intraseasonal time scale of the mean variables appears self-consistently.

The simplest scenario with the model involved regular intraseasonal oscillations of the mean variables and CCWs. Within one transition of the mean flow, the CCWs first transport
momentum downscale and then upscale, which reverses the sign of the mean flow. After this mean flow reversal, CCWs propagating in the other direction are favored; i.e., the CCWs decay and then re-appear propagating in the opposite direction. This mechanism was also corroborated by linear stability theory. The oscillation was shown to be intraseasonal over a range of domain sizes, and the maximum amplitude occurred for a 6,000-km domain width.

Other cases with climate base states yielded irregular intraseasonal oscillations. In a case with strong low-level westerlies, the mean flow oscillated about a climate base state that is mostly first baroclinic. At the same time, the CCWs had transitions between states of intense, coherent CCWs and multiscale envelopes of CCWs. It was shown that the intense, coherent CCWs had the strongest CMT. Another case showed oscillations with a mid-level easterly jet and westerlies at low levels. Depending on the strength of the mid-level jet, the dynamics can involve large oscillation amplitudes, small oscillation amplitudes, or decay to a steady mean flow with CCWs propagating both eastward and westward at the same time.

These results were discussed in the context of multiscale wave simulations such as CRM simulations of CCWs and GCM simulations of the MJO. It is suggested that multiscale methods may be needed to deal with the subtle issue of resolved CMT on multiple scales.

Acknowledgments.

The research of A. M. is partially supported by grants NSF DMS–0456713 and ONR N0014–07–1–0750. The authors thank Mitch Moncrieff and two anonymous reviewers for suggestions that improved the presentation of the results in this paper.
APPENDIX A

Multicloud Model with Advection

The multicloud model with advection is the following set of seven equations:

\[
\frac{\partial u_1}{\partial t} - \frac{\partial \theta_1}{\partial x} = -\frac{1}{\tau_u} u_1 - \frac{1}{2\sqrt{2}} \left[ 6u_2 \frac{\partial u_1}{\partial x} + (3u_1 + 5\bar{U}_3) \frac{\partial u_2}{\partial x} \right]
\]

(A1)

\[
\frac{\partial u_2}{\partial t} - \frac{\partial \theta_2}{\partial x} = -\frac{1}{\tau_u} u_2 - 2\sqrt{2} \bar{U}_3 \frac{\partial u_1}{\partial x}
\]

(A2)

\[
\frac{\partial \theta_1}{\partial t} - \frac{\partial u_1}{\partial x} = H_d + \xi_s H_s + \xi_c H_c - R_1
\]

\[
- \frac{1}{2\sqrt{2}} \left[ -2u_2 \frac{\partial \theta_1}{\partial x} + 4(u_1 - \bar{U}_3) \frac{\partial \theta_2}{\partial x} + 8\theta_2 \frac{\partial u_1}{\partial x} - (\theta_1 - 9\Theta_3) \frac{\partial u_2}{\partial x} \right]
\]

(A3)

\[
\frac{\partial \theta_2}{\partial t} - \frac{1}{4} \frac{\partial u_2}{\partial x} = H_c - H_s - R_2
\]

\[
- \frac{1}{2\sqrt{2}} \left[ -(u_1 - \bar{U}_3) \frac{\partial \theta_1}{\partial x} + (\theta_1 - 9\Theta_3) \frac{\partial u_1}{\partial x} - 8\Theta_4 \frac{\partial u_2}{\partial x} \right]
\]

(A4)

\[
\frac{\partial \theta_{eb}}{\partial t} = \frac{1}{h_b} (E - D) + \frac{1}{\pi h_b} \left[ 4\theta_2 \frac{\partial u_1}{\partial x} + \theta_1 \frac{\partial u_2}{\partial x} \right]
\]

(A5)

\[
\frac{\partial q}{\partial t} + \tilde{Q} \frac{\partial}{\partial x} (u_1 + \tilde{\lambda} u_2) = -P + \frac{1}{H_T} D - \frac{\partial}{\partial x} [q(u_1 + \tilde{\lambda} u_2)]
\]

(A6)

\[
\frac{\partial H_s}{\partial t} = \frac{1}{\tau_s} (\alpha_s P - H_s) + \left[ A_s u_1 \frac{\partial H_s}{\partial x} + \frac{1}{2} A_s H_s \frac{\partial u_1}{\partial x} \right]
\]

(A7)

The variables \( u_j \) are the \( j \)th baroclinic mode velocity, \( \theta_j \) are the \( j \)th baroclinic mode potential temperature, \( \theta_{eb} \) is the boundary layer equivalent potential temperature, and \( q \) is the vertically integrated water vapor. The source terms for these equations are

\[
H_c = \alpha_c \frac{\Lambda - \Lambda^*}{1 - \Lambda^*} Q_c
\]

(A8)

\[
H_d = \frac{1 - \Lambda}{1 - \Lambda^*} Q_d
\]

(A9)

\[
P = \frac{2\sqrt{2}}{\pi} (H_d + \xi_s H_s + \xi_c H_c)
\]

(A10)
\[ Q_d = \left[ \frac{\tilde{Q}}{\tau_{\text{conv}}} + \frac{1}{\tau_{\text{conv}}} (a_1\theta_{eb} + a_2q - a_0(\theta_1 + \gamma_2\theta_2 + \bar{\Theta}_3 + \gamma_4\bar{\Theta}_4)) \right]^+ \] (A11)

\[ Q_c = \left[ \frac{\tilde{Q}}{\tau_{\text{conv}}} (\theta_{eb} - a_0(\theta_1 + \gamma_2\theta_2 + \bar{\Theta}_3 + \gamma_4\bar{\Theta}_4)) \right]^+ \] (A12)

\[ \Lambda = \begin{cases} 
\Lambda^* & \text{for } \theta_{eb} - \theta_{em} < \theta^- \\
\Lambda^* + (1 - \Lambda^*) \frac{\theta_{eb} - \theta_{em}}{\theta^- - \theta^-} & \text{for } \theta^- < \theta_{eb} - \theta_{em} < \theta^+ \\
1 & \text{for } \theta^+ < \theta_{eb} - \theta_{em}
\end{cases} \] (A13)

\[ \theta_{em} = q + \frac{2\sqrt{2}}{\pi} (\theta_1 + \alpha_2\theta_2 + \alpha_3\bar{\Theta}_3) \] (A14)

\[ R_j = \frac{1}{\tau_\theta} \theta_j + Q_{R,j}^0, \quad j = 1, 2 \] (A15)

\[ \frac{1}{h_b} E = \frac{1}{\tau_e} (\theta_{eb} - \theta_{eb}) \] (A16)

\[ D = \frac{m_0}{P_D} (P_D + \mu_2 (H_s - H_c))^+ (\theta_{eb} - \theta_{em}). \] (A17)

The source terms \( H_c, H_d, \) and \( H_s \) represent heating from congestus, deep convective, and stratiform clouds, respectively. Radiative cooling is \( R_j \), evaporation is \( E \), downdrafts are \( D \).

These are the equations of the multicloud model of Khouider and Majda (2008b), with advection terms added using vertical mode projections as described by Stechmann et al. (2008), and with the following changes. Advection terms have been included with zonally averaged variables \( \bar{U}_3, \bar{\Theta}_3, \) and \( \bar{\Theta}_4 \), which are associated with third and fourth baroclinic mode vertical structures. Advection terms have also been included in the \( \theta_{eb} \) equation; these terms ensure conservation of the model’s vertically integrated moist static energy [see Khouider and Majda (2006c)], and they can be thought of as environmental downdrafts. Advection terms have also been added to the \( H_s \) equation to represent advection of stratiform clouds.

A few source terms have also been changed from Khouider and Majda (2008b). The congestus heating, \( H_c \), is treated diagnostically here by taking the limit \( \tau_c \to 0 \) in Khouider
and Majda (2008b). Also, the parameters $\gamma_2'$ in $Q_c$ and $\gamma_2$ in $Q_d$ take different values here: $\gamma_2' = 2$ and $\gamma_2 = 0.1$. Using a large value of $\gamma_2'$ emphasizes the second baroclinic mode and gives $Q_c$ the characteristics of a low-level convective available potential energy (CAPE) closure. This change in $\gamma_2'$ and the use of diagnostic $H_c$ were also necessary to damp small-scale instabilities that sometimes arise when nonlinear advection is added to the multicloud model. The other changes to the source terms are inclusions of $\bar{\Theta}_3$ and $\bar{\Theta}_4$ in (A11)–(A14) with new parameters $\gamma_3 = 0.3$, $\gamma_4 = 0$, $\gamma_3' = 0.5$, $\gamma_4' = 0.25$, and $\alpha_3 = 0.1$; this is done to include feedbacks from the mean thermodynamic variables on the waves. The two momentum drag sources from Khouider and Majda (2008b) have been combined here into a single term with time scale $\tau_u = 3$ days, and the radiative damping time scale used here is $\tau_\theta = 10$ days. The parameter $\tau_{\text{conv}}$ takes a value of 1 hour here instead of $\tau_{\text{conv}} = 2$ hours as it was in Khouider and Majda (2008b). This change in $\tau_{\text{conv}}$ reduces the wavelength of the most unstable waves from 4000 to 1500 km, thereby reducing artificial effects of the wave wrapping around the 6000 km periodic domain and interacting with itself. Without a background shear, reducing $\tau_{\text{conv}}$ increases the growth rate of the unstable waves (Khouider and Majda 2006c), and this is useful for strengthening the CCWs on spatial domains with smaller widths like 6,000 km. Besides these changes mentioned above, the parameter values used here are all the same as those in the standard case of Khouider and Majda (2008b).

The linearized multi-cloud model equations without background shear have been developed in mathematical detail elsewhere (Khouider and Majda 2006c, 2008b). It is straightforward to linearize the quadratic advection terms at a mean background shear to produce the complete linearized equations that have been used throughout this paper for linear stability analysis.
APPENDIX B

Asymptotic Derivation of the Model

A derivation using multiscale asymptotics is now given for the multiscale wave–mean model in section 2. This derivation will start from the hydrostatic Boussinesq equations:

\[ \partial_t u + \partial_x (u^2) + \partial_z (wu) + \partial_x p = S_u \]  
\[ \partial_t \theta + \partial_x (u\theta) + \partial_z (w\theta) + w = S_\theta \]  
\[ \partial_x p = \theta \]  
\[ \partial_x u + \partial_z w = 0 \]

A two-dimensional setup is used for simplicity here. These equations have been nondimensionalized using the scales shown in Table 1. The space and time scales \( x \) and \( t \) represent equatorial synoptic scales. The only spatial variable of the multiscale model is the synoptic scale variable \( x \), and there are two temporal variables: \( t \) represents the synoptic scale and \( T = \epsilon^2 t \) represents a longer intraseasonal time scale.

The derivation will involve both space and time averages. The zonal average of a function \( f(x, z, t, T) \) is defined as

\[ \bar{f}(z, t, T) = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} f(x, z, t, T) \, dx \]  

Using this average, any function can be split into a zonal average and fluctuation: \( f(x, z, t, T) = \bar{f}(z, t, T) + f'(x, z, t, T) \), where \( f' \) is defined as \( f - \bar{f} \). The temporal average over the synoptic scales is defined similarly as

\[ \langle f \rangle(x, z, T) = \lim_{\tilde{T} \to \infty} \frac{1}{2T} \int_{-\tilde{T}}^{\tilde{T}} f(x, z, t, T) \, dt \]
The ansatz for the velocity $u$ includes an $O(1)$ large-scale average $\bar{U}(z,T)$ and an $O(\epsilon)$ fluctuation $\epsilon u'(x,z,t,T)$, with similar expansions for the other variables:

$$u = \bar{U}(z,T) + \epsilon u'(x,z,t,T) + \epsilon^2 u_2 + O(\epsilon^3) \quad (B7)$$

$$\theta = \bar{\Theta}(z,T) + \epsilon \theta'(x,z,t,T) + \epsilon^2 \theta_2 + O(\epsilon^3) \quad (B8)$$

$$p = \bar{P}(z,T) + \epsilon p'(x,z,t,T) + \epsilon^2 p_2 + O(\epsilon^3) \quad (B9)$$

$$w = \epsilon w'(x,z,t,T) + \epsilon^2 w_2 + O(\epsilon^3) \quad (B10)$$

Notice that $\bar{w} = 0$ because the spatial average of the continuity equation (B4) yields $\partial_z \bar{w} = 0$.

To obtain the multiscale equations, this ansatz is inserted into (B1)–(B4) and terms are gathered at each order in $\epsilon$. The full temporal derivative of $f(t,\epsilon^2 t)$ must be expanded using the chain rule as $\partial_t f + \epsilon^2 \partial_T f$ to represent the two time scales of the ansatz. When all terms in the momentum equation (B1) are expanded, the leading order terms, at $O(\epsilon)$, are

$$\partial_t u' + \partial_x (2\bar{U} u') + \partial_z (w' \bar{U}) + \partial_x p' = S_{u,1}' \quad (B11)$$

This is an equation for the fluctuations $u'$. When all terms at $O(\epsilon^2)$ are collected, the result is

$$\partial_t u_2 + \partial_T \bar{U} + \partial_x (u'^2 + 2\bar{U} u_2) + \partial_z (w' \bar{w}) + \partial_x p_2 = S_{u,2}. \quad (B12)$$

When a zonal average is applied to this result, the $\partial_x$ terms vanish because $\partial_x f = 0$ for any bounded function $f$ by the definition (B5). The terms remaining after the zonal average are

$$\partial_t \bar{u}_2 + \partial_T \bar{U} + \partial_z (\bar{w}' \bar{w}') = 0. \quad (B13)$$

At this point a key step in the multiscale asymptotic procedure must take place: suppressing secular growth of the higher order terms (Majda 2003; Majda and Klein 2003; Majda
The ansatz in (B7)–(B10) assumed that $\epsilon^2 u_2$ has a magnitude of $O(\epsilon^2)$, and this must be maintained or else the asymptotic ordering in (B7)–(B10) would be destroyed. For a time-dependent equation like (B13) of the form $\partial \bar{u}_2/\partial t = F(t)$, secular growth in time is avoided if and only if $\langle F \rangle = 0$. Thus secular growth of $\bar{u}_2$ is avoided if

$$\partial_T \bar{U} = -\partial_z \langle w'u' \rangle$$  \hspace{1cm} (B14)$$

This is an equation for the large-scale variable $\bar{U}(z, T)$ evolving on the long time scale $T$. Note that the angle brackets $\langle \rangle$ are left off $\bar{U}$ to ease notation.

A similar derivation leads to equations for $\bar{\Theta}$ and $\theta'$, and the hydrostatic and continuity equations (B3)–(B4) are straightforward to separate into orders of $\epsilon$. The result includes a set of equations for the large-scale variables,

$$\partial_T \bar{U} = -\partial_z \langle w'u' \rangle$$  \hspace{1cm} (B15)$$

$$\partial_T \bar{\Theta} = -\partial_z \langle w'\theta' \rangle + \langle \bar{S}_{\theta,2} \rangle$$  \hspace{1cm} (B16)$$

$$\partial_z \bar{P} = \bar{\Theta}$$  \hspace{1cm} (B17)$$

and a set of equations for the fluctuations,

$$\partial_t u' + \bar{U} \partial_x u' + w' \partial_z \bar{U} + \partial_z p' = S_{u,1}'$$  \hspace{1cm} (B18)$$

$$\partial_t \theta' + \bar{U} \partial_x u' + w' \partial_z \bar{U} + w' = S_{\theta,1}'$$  \hspace{1cm} (B19)$$

$$\partial_z p' = \theta'$$  \hspace{1cm} (B20)$$

$$\partial_z u' + \partial_x w' = 0$$  \hspace{1cm} (B21)$$

This completes the derivation. These equations are then separated into vertical baroclinic modes, and a multicloud convective parameterization is added as shown in section 2.
REFERENCES


Khouider, B. and A. J. Majda, 2006c: A simple multicloud parameterization for convectively


Khouider, B. and A. J. Majda, 2008a: Equatorial convectively coupled waves in a simple

Khouider, B. and A. J. Majda, 2008b: Multicloud models for organized tropical convection:

Kiladis, G. N., K. H. Straub, and P. T. Haertel, 2005: Zonal and vertical structure of the

LeMone, M., 1983: Momentum transport by a line of cumulonimbus. J. Atmos. Sci., 40 (7),
1815–1834.

LeMone, M., G. Barnes, and E. Zipser, 1984: Momentum flux by lines of cumulonimbus over

LeMone, M., E. Zipser, and S. Trier, 1998: The role of environmental shear and thermo-
dynamic conditions in determining the structure and evolution of mesoscale convective

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Table 1. Physical parameters and reference scales.

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<th>Value</th>
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<th>Time (days)</th>
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<th>$k_*$</th>
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Table 3. Sensitivity of oscillation to domain width.

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