Modulation of Internal Gravity Waves in a Multi-scale Model for Deep Convection on Mesoscales

DANIEL RUPRECHT * AND RUPERT KLEIN

FB Mathematik, Freie Universität Berlin, Germany

ANDREW J. MAJDA

Courant Institute of Mathematical Sciences, New York University

*Corresponding author address: Daniel Ruprecht, FB Mathematik, Freie Universität Berlin, Arnimallee 6, 14195 Berlin, Germany.

E-mail: ruprecht@zib.de
ABSTRACT

Starting from the conservation laws for mass, momentum, and energy together with a three species, bulk micro-physics model, a model for the interaction of internal gravity waves and deep convective hot towers is derived using multi-scale asymptotic techniques. From the leading order equations, a closed model for the large-scale flow is obtained analytically by applying horizontal averages conditioned on the small-scale hot towers. No closure approximations are required besides adopting the asymptotic limit regime which the analysis is based on. The resulting model is an extension of the anelastic equations linearized about a constant background flow. Moist processes enter through the area fraction of saturated regions and through two additional dynamic equations describing the coupled evolution of the conditionally averaged small-scale vertical velocity and buoyancy. A two-way coupling between the large-scale dynamics and these small-scale quantities is obtained: moisture reduces the effective stability for the large-scale flow, and micro-scale up- and downdrafts define a large-scale averaged potential temperature source term. In turn, large-scale vertical velocities induce small-scale potential temperature fluctuations due to the discrepancy in effective stability between saturated and non-saturated regions.

The dispersion relation and group velocity of the system are analyzed and moisture is found to have several effects: it (i) reduces vertical energy transport by waves, (ii) increases vertical wavenumbers but decreases the slope at which wave packets travel, (iii) introduces a new lower horizontal cut-off wavenumber besides the well-known high wavenumber cut-off, and (iv) moisture can cause critical layers. Numerical examples reveal the effects of moisture on steady-state and time-dependent mountain waves in the present hot-tower regime.
1. Introduction

Internal gravity waves are one prominent feature of atmospheric flows on lengthscales from approx. 10 to 100 km and are responsible for a number of important effects. As Bretherton and Smolarkiewicz (1989) and Lane and Reeder (2001) show, convecting clouds emit gravity waves which alter their environment, rendering it favorable for further convection by reducing convective inhibition (CIN). Chimonas et al. (1980) investigate a feedback mechanism between saturated regions and gravity waves that can trigger convection. They hypothesize that gravity waves contribute to the organization of individual convective events into larger scale structures like squall lines. Vertically propagating gravity waves are associated with vertical transport of horizontal momentum. The dissipation of these waves in the stratosphere exerts a net force on middle atmospheric flows known as “gravity-wave drag” (GWD), see, e.g., Sawyer (1959); Lindzen (1981). McLandress (1998) demonstrates the necessity of including the effects of GWD in “global circulation models” (GCM) to obtain realistic flows. Joos et al. (2008) find that gravity waves are also important for the parameterization of cirrus clouds. Because GCMs have a spatial resolution of 100 – 200 km, gravity waves cannot be resolved in these models and their effects have to be parameterized. Kim et al. (2003) provide an overview of concepts of GWD parameterizations in GCMs.

Moisture in the atmosphere significantly affects the propagation of internal waves. Barcilon et al. (1979) propose a model for steady, hydrostatic flow over a mountain with reversible moist dynamics. This model distinguishes between saturated and non-saturated regions by a switching function that depends on the vertical displacement of a parcel: if the parcel is displaced beyond the lifting condensation level, it is treated as saturated and
the dry stability frequency is replaced by the reduced moist stability frequency. Barcilon 
et al. (1980) extend the model to non-hydrostatic flows with irreversible condensation, and 
Barcilon and Fitzjarrald (1985) to nonlinear, steady flow. These authors find that moisture 
can significantly reduce the mountain drag which is closely related to the wave drag. Jusem 
and Barcilon (1985) employ a nonlinear, non-steady, non-hydrostatic anelastic model that 
explicitly includes the mixing ratios of liquid water and vapor to define heating source terms 
for the potential temperature. Besides finding again that moisture can reduce drag, they 
also find that moisture does reduce the wave intensity and increases the vertical wavelength. 
While the first result is also found in the present paper, instead of an increased vertical 
wavelength, we observe an increase of the vertical wavenumber by moisture, corresponding 
to a reduced vertical wavelength.

Durran and Klemp (1983) employ a fully compressible model combined with prognostic 
equations for water vapor, rain water, and cloud water to simulate moist mountain waves. 
They also find that moisture reduces the vertical flux of momentum and the amplitude of the 
generated wave patterns. Further, they observe an increase in vertical wavelength for nearly 
hydrostatic waves. Attenuation of gravity waves by moisture and an increase of vertical 
wavelength is also found in the analysis of wave propagation in a fully saturated atmosphere 
in Einaudi and Lalas (1973).

Although there is extensive literature dealing with the parameterization of drag from 
convectively generated waves, there are very few attempts to include the effect of moisture 
in parameterizations of orographic waves. In their review, Kim et al. (2003) mention only 
the work of Surgi (1989), investigating the introduction of a stability frequency modified by 
moisture into the orographic GWD parameterization.
Klein and Majda (2006) derive a multi-scale model for the interaction of non-hydrostatic internal gravity waves with moist deep convective towers from the conservation laws of mass, momentum, and energy combined with a classical bulk micro-physics scheme. In agreement with the regime of non-rotating, non-hydrostatic gravity waves described by Gill (1982), Ch. 8, the characteristic horizontal and vertical scales for the gravity waves are assumed comparable to the pressure scale height, $h_{\text{sc}} \sim 10$ km. LeMone and Zipser (1980) provide an indication for the characteristic horizontal scales of the narrow deep convective towers: they analyze data obtained during the GATE experiment and find that the median diameter of “convective events” related to tropical cumulonimbus clouds is about 900 m, see also Stevens (2005). Thus a horizontal “micro”-scale of 1 km is used as the second horizontal lengthscale in the multi-scale ansatz to describe the tower-scale dynamics. The assumed timescale of 100 s is compatible with the typical value of 0.01 s$^{-1}$ for the stability frequency in the troposphere.

By using an asymptotic ansatz representing these scales, this paper presents the derivation of a reduced model for modulation of internal waves by moisture. For a first reading, one can study the summary of the model in 1a, skip the technical derivation in 2 and go immediately to the new phenomena and application of the model in sections 3 and 4. The reduced model allows for an analytical investigation, concisely revealing a number of mechanisms by which moisture does affect internal waves. It confirms already known facts but also allows to hypothesize new effects that might be important for improved parameterizations of internal waves. Further, the model equations themselves might serve as a starting point for the development of such parameterizations.

The used asymptotic ansatz is the one introduced by Klein and Majda (2006) with
the slight modification of adding a constant, horizontal background flow. Using weighted averages, the obtained leading order multi-scale equations are converted into a closed system of equations for the gravity wave scale only. In this system, the effective vertical mass fluxes are obtained analytically so that no additional physical closure assumptions beyond those made in adopting the asymptotic scaling regime are required. The resulting equations are an extension of the anelastic equations, linearized around a moist adiabatic, constant wind background flow. Mathematical analysis of these equations reveals, among other effects, that moisture introduces a lower horizontal cut-off wavenumber, the existence of which is, to the authors’ knowledge, a new hypothesis. The essential moisture-related parameter in the model is the area fraction of saturated regions on the micro-scale, reminiscent of a smaller scale version of the “cloud cover fraction”, a parameter routinely computed and used in GCMs. Jakob and Klein (1999) discuss this parameter in the context of microphysical parameterizations in the ECMWF\(^1\) model. They find that a uniform value for cloud cover over one cell is not sufficient and divide the cell into a number of sub-columns so as to approximately represent a spatially inhomogeneous distribution of cloud cover inside a cell. Trying to link the saturated area fraction arising in the present model to such decompositions of cloud cover might be a promising ansatz to include moisture effects in GWD parameterizations in a systematic way.

\(^1\)European Centre for Medium-Range Weather Forecast
a. **Summary of the model**

The new model for gravity wave–convective tower interactions consists of equations describing linearized, anelastic moist dynamics plus two equations for the conditionally averaged tower-scale dynamics. \( u \) is the horizontal velocity, \( \bar{w} \) the averaged vertical velocity and \( \bar{\theta} \) the average potential temperature. \( w' \) and \( \theta' \) are conditional averages of deviations from \( \bar{w} \) and \( \bar{\theta} \) within the deep convective clouds. Here \( \pi \) is the Exner function, \( \rho^{(0)} \) the leading order background density and \( \theta_z^{(2)} \) the background stratification while \( u^\infty \) is a constant, horizontal background velocity. The source term \( \bar{C}_- \) is a constant cooling term related to evaporating rain in non-saturated areas, and \( \sigma \) is the saturated area fraction mentioned earlier. See section 2 for details.

**Linearized, anelastic moist dynamics:**

\[
\begin{align*}
    u' &+ u^\infty u_x + \pi_x = 0 \\
    \bar{w}' &+ u^\infty \bar{w}_x + \pi_z = \bar{\theta} \\
    \bar{\theta}' &+ u^\infty \bar{\theta}_x + (1 - \sigma) \theta_z^{(2)} \bar{w} = \theta_z^{(2)} w' + \bar{C}_- \\
    (\rho_0 \bar{u})_x + (\rho_0 \bar{\omega})_z &= 0
\end{align*}
\]

(1)

**Averaged tower-scale dynamics:**

\[
\begin{align*}
    w' &+ u^\infty w'_x = \theta' \\
    \theta' &+ u^\infty \theta'_x + \sigma \theta_z^{(2)} w' = \sigma (1 - \sigma) \theta_z^{(2)} \bar{w} - \sigma \bar{C}_-
\end{align*}
\]

(2)

Moisture affects the large-scale dynamics given by (1) in two ways. It reduces the effective stability of the atmosphere by a factor of \( 1 - \sigma \), representing the effect that if a parcel rises and starts condensating water, the release of latent heat will effectively reduce the restoring
force the parcel experiences. Because of the short timescale in this model, the only conversion mechanism between moist quantities that has a leading order effect is evaporation of cloud water into vapor and condensation of vapor into cloud water in fully saturated regions. As a consequence, $\sigma$ itself does not change with time in the present model except for being advected by the mean flow. In fact, for the scalings assumed in the present derivation, the physical effect of non-saturated rising parcels eventually becoming saturated when lifted sufficiently is not present. An extension of the model to capture this effect is the subject of current work and beyond the scope of the present paper. See also our comments in section 3e.

The release and consumption of latent heat by averaged small-scale up- and downdrafts in saturated areas is described by the source term $\theta^{(2)}_z w'$ in (1)3. A positive $w'$ results in a positive contribution to $\bar{\theta}$, modelling latent heat release whereas a negative $w'$ models latent heat consumption. Yet, the micro-scale model not only provides the source term for the large-scale dynamics but is also affected by them in return through the $\bar{w}$ source term on the right hand side of (2)2. Finally, for the chosen time and length scales, the mass conservation equation reduces to the anelastic divergence constraint (1)4.

Note that if all moisture related terms vanish, i.e., for $\sigma = 0$, $\bar{C}_- = 0$ and $w'(\tau = 0) = \theta'(\tau = 0) = 0$, the system in (1) reduces to the anelastic equations linearized around a constant-wind background flow with stable stratification, $\theta^{(2)}_z > 0$, see, e.g., Davies et al. (2003),
\[ \begin{align*}
\ddot{u}_x + u^\infty \dot{u}_x + \pi_x &= 0 \\
\ddot{w}_x + u^\infty \dot{w}_x + \pi_z &= \ddot{\theta} \\
\ddot{\theta}_x + u^\infty \dot{\theta}_x + \theta^{(2)}_x \dot{w} &= 0 \\
(\rho_0 \ddot{u})_x + (\rho_0 \ddot{w})_z &= 0.
\end{align*} \]

2. Derivation of the model

This section provides the derivation of the model from eqs. (1), (2). The present modification of the original asymptotic regime considered by Klein and Majda (2006) is developed in section 2a together with a justification for the particular scaling of the constant horizontal background wind velocity. Section 2b describes the closure of the leading order equations by weighted averages.

a. Multiple scales ansatz

In the derivation by Klein and Majda (2006), three primary dimensionless parameters occur: the Mach-number \( M \), describing the ratio of a typical flow velocity \( u_{\text{ref}} \) to the speed of sound waves, the barotropic Froude number \( \overline{Fr} \), describing the ratio of a typical flow velocity to the speed of external gravity waves and the bulk micro-scale Rossby radius \( R_{0B} \), providing a measure of the importance of rotational effects for flows on the bulk micro-scale. These parameters are defined as

\[
M = \frac{u_{\text{ref}}}{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}, \quad \overline{Fr} = \frac{u_{\text{ref}}}{\sqrt{gh_{\text{sc}}}}, \quad R_{0B} = \frac{u_{\text{ref}}}{l_{\text{bulk}}\Omega}
\]
where \( h_{sc} \) is the pressure scale height, \( l_{bulk} \) is the lengthscale of the bulk micro-physics, \( p_{ref} \) and \( \rho_{ref} \) are typical values for pressure and density, \( \Omega \) is the rate of earth’s rotation and \( g \) the gravity acceleration. Following Majda and Klein (2003), these parameters are related to a universal expansion parameter \( \varepsilon \) in the following distinguished limit

\[
M \sim \frac{\varepsilon^2}{\Omega} \sim \varepsilon^2, \quad \text{Ro}_B \sim \varepsilon^{-1} \quad \text{as} \quad \varepsilon \to 0.
\]

The scaling of a fourth dimensionless parameter, the baroclinic Froude number, will be discussed shortly in the context of eq. (10).

The starting point of the model development are the conservation laws for mass, momentum, and energy combined with a slightly modified version of the bulk micro-physics model from Grabowski (1998). The prognostic quantities in the original equations are the horizontal velocity \( u \), vertical velocity \( w \), density \( \rho \), pressure \( p \), potential temperature \( \theta \), and the mixing ratios of water vapor \( q_v \), cloud water \( q_c \), and rain water \( q_r \). The scales considered in the derivation are a timescale of \( t_r \approx 100 \text{s} \), a vertical lengthscale equal to the pressure scale height \( h_{sc} \approx 10 \text{km} \) and two horizontal scales \( l \approx 10 \text{km} \) and \( l_{bulk} \approx 1000 \text{m} \). As discussed in the introduction, these scales correspond to a combination of the regimes of non-hydrostatic gravity waves and deep convective towers. To resolve them, new coordinates are introduced by rescaling the “universal” coordinates \( x \) and \( t \), which resolve the reference lengthscale of \( l_{ref} \approx 10 \text{ km} \) and timescale of \( t_{ref} \approx 1000 \text{ s} \), by powers of \( \varepsilon \). The new coordinates resolving the short scales are

\[
\tau = \frac{t}{\varepsilon}, \quad \eta = \frac{x}{\varepsilon}.
\]

The model distinguishes between saturated and non-saturated regions by a switching func-
tion $H_{q_v}$, defined as

$$H_{q_v}(q_v) = \begin{cases} 
1 & : q_v^{(0)} \geq q_{vs}^{(0)} \quad \text{(saturated at leading order)} \\
0 & : q_v^{(0)} < q_{vs}^{(0)} \quad \text{(non-saturated at leading order)} 
\end{cases},$$

(7)

where $q_v^{(0)}$ is the leading order water vapor mixing ratio and $q_{vs}^{(0)}$ is the leading order saturation water vapor mixing ratio, computed from Bolton’s formula for the saturation vapor pressure. See Emanuel (1994) for the original formula and Klein and Majda (2006) for the derivation of an asymptotic expression for $q_{vs}$. For the warm micro-physics considered here saturated regions and clouds coincide and $H_{q_v}$ is the leading-order characteristic function for cloudy patches of air which equals unity inside clouds and zero between them.

We modify the ansatz for the horizontal velocity by introducing a constant background velocity $u^\infty$. To avoid inconsistencies in the derivation, we also add a second coordinate $\tau'$ corresponding to the timescale set by advection of flows with $u^\infty$-velocity over the short, tower-scale distances resolved by the $\eta$ coordinate. The terms related to $\tau'$ will eventually drop out by sublinear growth conditions and do not appear in the final model. In terms of $\varepsilon$ the new coordinate is

$$\tau' = \frac{t}{\varepsilon^2}. \quad (8)$$

All quantities depending on $\eta$ also depend on $\tau'$. The horizontal velocity is assumed to be independent of the small horizontal coordinate $\eta$, so we use an ansatz

$$u(x, z, t; \varepsilon) = \varepsilon^{-1} u^\infty + u^{(0)}(x, z, \tau) + \mathcal{O}(\varepsilon).$$

(9)

Although this scaling would formally suggest dimensional values for $u^\infty$ of about 100 m s$^{-1}$, a value of $u^\infty = 0.1$ corresponding to 10 m s$^{-1}$, will be used throughout this paper. The reason for this apparent inconsistency between the asymptotic scaling of $u^\infty$ and the actual value
used for it is the following: as shown by Klein (2009), the inverse timescales of advection and internal waves for flows on a lengthscale $h_{sc}$ with a typical velocity of $u/u_{ref} = O(1)$ and a background stratification $\bar{\theta}$, using the distinguished limit (5), are

\[
\begin{align*}
\text{advection} & \quad t_{\text{ref}}^{-1} : \quad \frac{u_{\text{ref}}}{h_{sc}} & : & & 1 \\
\text{internal waves} & \quad t_{\tau}^{-1} : \quad N & : & & \frac{1}{\varepsilon^2} \sqrt{\frac{h_{sc}}{\bar{\theta}} \frac{d\bar{\theta}}{dz}}.
\end{align*}
\]

Thus, except for very weak stratifications of order $O(\varepsilon^4)$, the advection timescale and the timescale of internal waves are asymptotically separated in the limit $\varepsilon \to 0$. To retain both effects, i.e., advection and internal gravity waves, in the leading order equations for a stratification of order $O(\varepsilon^2)$ as will be used here according to (15), the inverse advection timescale has to be of order $\varepsilon^{-1}$. For the $O(\varepsilon^{-1})$ scaling of $u^\infty$ in (9), (10) becomes $t_{\text{ref}}^{-1} \sim \varepsilon^{-1}u_{\text{ref}}/h_{sc} \sim \varepsilon^{-1}$ as required, and both timescales are of the same order.

One can also see that if the employed timescale is $t_{\tau}$, a $O(\varepsilon^{-1})$ scaling of $u$ is necessary to address the non-hydrostatic regime by analyzing the scaling of the baroclinic Froude number (11). It denotes the ratio between advection and wave speed and indicates the importance of non-hydrostatic effects. For a stratification of order $O(\varepsilon^2)$, $t_{\tau}^{-1} \sim \varepsilon^{-1}$ while $t_{\text{ref}} \sim 1$ according to (10). Thus the scaling of $Fr_{\text{baroclinic}}$ reads

\[
Fr_{\text{baroclinic}} = \frac{u}{N t_{\text{ref}}} \sim \frac{u}{t_{\tau}^{-1} h_{sc}} \sim \varepsilon \frac{u}{t_{\text{ref}}^{-1} h_{sc}} = O \left( \frac{\varepsilon u}{u_{\text{ref}}} \right) \]

If $u^\infty = 0$, then $u/u_{\text{ref}} = O(1)$ and $Fr_{\text{baroclinic}}$ is of order $O(\varepsilon)$, so that non-hydrostatic effects would not be contained in the leading order equations.

The scale separation revealed in (10) between internal waves for stratifications of order $O(\varepsilon^2)$ and advection based on the reference velocity $u_{\text{ref}}$ in the limit $\varepsilon \to 0$ is, however, ob-
secured for finite values of \( \varepsilon \approx 0.1 \), which are typical for realistic atmospheric flows: let \( k \) and \( m \) denote the horizontal and vertical wavenumber of an internal wave. For non-hydrostatic waves, these are of comparable magnitude. The lengthscale \( h_{sc} \) provides a reference value for both wavenumbers of

\[
k_{\text{ref}} = \frac{2\pi}{h_{sc}} = 2\pi \cdot 10^{-4} \text{ m}^{-1}.
\]  

(12)

Now, according to the dispersion relation for internal waves, see e.g., Bühler (2009), the horizontal phase velocity is

\[
c_{\text{phase}} = \frac{N}{\sqrt{k^2 + m^2}}.
\]  

(13)

Using a typical value for the stability frequency of \( N = 0.01 \text{ s}^{-1} \), we obtain a dimensional value for the phase velocity of

\[
c_{\text{phase, dim}} \approx \frac{10^{-2} \text{ s}^{-1}}{\sqrt{2} \cdot 2\pi \cdot 10^{-4} \text{ m}^{-1}} \approx 11 \text{ m s}^{-1}.
\]  

Thus, while a scaling of \( u^\infty \) of order \( \mathcal{O}(\varepsilon^{-1}) \) is required to retain advection in the limit \( \varepsilon \to 0 \), a value of \( u^\infty = 0.1 \), corresponding to dimensional values of about 10 m s\(^{-1}\), agrees very well with the timescale of internal waves actually obtained with realistic values of \( \varepsilon_{\text{actual}} = 0.1 \).

The reason is that the factor \( (\sqrt{2} \cdot 2\pi)^{-1} \) in (14) is of order \( \mathcal{O}(1) \) in the limit \( \varepsilon \to 0 \) but is comparable to \( \varepsilon_{\text{actual}} = 0.1 \).

For a reference velocity \( u^\ast_{\text{ref}} = h_{sc} N = 100 \text{ m s}^{-1} \), no separation of the internal wave time-scale and the advection time-scale occurs. In principle, an equivalent derivation can be conducted, if the governing equations are non-dimensionalized using \( u^\ast_{\text{ref}} \). This changes the distinguished limit (5) and the expansions of the horizontal and vertical velocity, avoiding a \( \varepsilon^{-1} \) scaling of the leading order \( u^\infty \). The small parameter then is the amplitude of wave-induced perturbations of the velocity field. The justification for setting \( u^\infty = 0.1 \) is required
in this derivation, too.

The potential temperature is expanded about a background stratification $\tilde{\theta}(z) = 1 + \epsilon^2 \theta^{(2)}(z)$ as

$$\theta(x, z, t; \epsilon) = 1 + \epsilon^2 \theta^{(2)}(z) + \epsilon^3 \theta^{(3)}(\eta, x, z, \tau, \tau') + O(\epsilon^4). \quad (15)$$

As discussed in Klein and Majda (2006), considering realistic values of convectively available potential energy (CAPE) constrains deviations of $\theta$ from a moist adiabat, showing that $\theta^{(2)}$ should satisfy the moist adiabatic equation

$$\theta^{(2)}_z = -\Gamma^{**} L^{**} q^{**}_{vs}(0) =: -\hat{L} q^{(0)}_{vs,z} \quad (16).$$

Expansions about a moist adiabatic background are also considered, for example, in Lipps and Hemler (1982). In (16) $p^{(0)}$ is the leading order of the pressure and $\Gamma^{**}, L^{**}$ and $q^{**}_{vs}$ are $O(1)$ scaling factors arising from the non-dimensionalization in Klein and Majda (2006).

The expansions of all other dependent variables are adopted from Klein and Majda (2006), except that variables depending on $\eta$ in their derivation now also depend on $\tau'$.

All quantities $\phi \in \{u, w, \theta, \pi, q_v, q_c, q_r\}$ are split below as

$$\phi = \bar{\phi} + \tilde{\phi} \quad (17)$$

where

$$\bar{\phi} = \lim_{\eta_0 \to \infty} \frac{1}{2\eta_0} \int_{-\eta_0}^{\eta_0} \phi(\eta) d\eta \quad (18)$$

denotes the small-scale average with respect to the $\eta$-coordinate, while $\tilde{\phi}$ denotes deviations from this average.

The description of the scalings and the ansatz are basically a repetition of what is done in Klein and Majda (2006), so the reader is referred to the original work for a detailed
discussion. The focus here is the derivation of a closed model from the resulting leading order equations and an analysis of the model’s properties. The derivation is presented in a $x$-$z$-plane here. This simplifies the notation and numerical examples presented below will be of this type, too. However, this is not an essential restriction.

The leading order equations for the averages resulting from this ansatz are

$$
\tilde{u}_r^{(0)} + u^\infty \tilde{u}_x^{(0)} + \pi_x^{(3)} = 0
$$

$$
\tilde{w}_r^{(0)} + u^\infty \tilde{w}_x^{(0)} + \pi_z^{(3)} = \tilde{\theta}^{(3)}
$$

$$
\tilde{\theta}_r^{(3)} + u^\infty \tilde{\theta}_x^{(3)} + \tilde{w}^{(0)}\theta_z^{(2)} = \hat{L} \left( H_{q_v} C_d^{(0)} - H_{q_v} C_d^{(0)} + (H_{q_v} - 1) C_e^{(0)} - (H_{q_v} - 1) C_e^{(0)} \right).
$$

The leading order equations for the moist species are split into separate equations for the saturated ($H_{q_v} = 1$) and the non-saturated ($H_{q_v} = 0$) cases.
(i) Saturated

\[- (\tilde{w}^{(0)} + \bar{w}^{(0)}) q_{v_{s,z}}^{(0)} = C_{d}^{(0)}\]

\[q_{r,\tau}^{(0)} + u^{\infty} q_{r,x}^{(0)} + u^{(0)} q_{r,\eta}^{(0)} = 0\]  

(21)

(ii) Non-Saturated

\[C_{ev}^{**} \left( q_{v_{s}}^{(0)} - q_{v}^{(0)} \right) \sqrt{q_{i}^{(0)}} = C_{ev}^{(0)}\]

\[q_{c_{,x}}^{(0)} + u^{\infty} q_{c_{,x}}^{(0)} + u^{(0)} q_{c_{,\eta}}^{(0)} = 0\]

\[q_{r_{,x}}^{(0)} + u^{\infty} q_{r_{,x}}^{(0)} + u^{(0)} q_{r_{,\eta}}^{(0)} = 0\]  

(22)

\[C_{ev}^{**}\) is again a \(O(1)\) scaling factor from the non-dimensionalization. \(q_{v_{s}}^{(0)}, q_{v}^{(0)}, q_{c}^{(0)}\) and \(q_{r}^{(0)}\) are the leading order mixing ratios of the saturation water vapor, water vapor, cloud water and rain water. Key-steps of the derivation can be found in the appendix.

b. Computing the mass flux closure

To obtain a closed set of equations, an equation for \(H_{q_{v}}C_{d}^{(0)}\) in (19) will be derived from the perturbation equations (20) and the leading order equations (21) and (22) obtained from the bulk micro-scale model. We stress that the closure is computed analytically and does not require the introduction of additional physical closure assumptions.

Multiply (21)_1 by \(H_{q_{v}}\) and average over \(\eta\) to get

\[-H_{q_{v}} \tilde{w}^{(0)} q_{v_{s,z}}^{(0)} - H_{q_{v}} \bar{w}^{(0)} q_{v_{s,z}}^{(0)} = H_{q_{v}} C_{d}^{(0)} .\]

(23)
Considering (22) and noting that in saturated regions \( q_v^{(0)}(z) = q_{vs}^{(0)}(z) \) trivially satisfies the same transport equation, we find

\[
q_v^{(0)}(x, z, \eta, \tau) = q_v^{(0)}(x - u^\infty \tau, z, \eta - \int_0^\tau u^{(0)}(x, z, t')dt', 0). \tag{24}
\]

Define

\[
\sigma(x, z, \tau) := \overline{H_{q_v}(x, z, \eta, \tau)}. \tag{25}
\]

As \( \int_0^\tau u^{(0)}(x, z, t')dt' \) is independent of \( \eta \), we get

\[
\sigma(x, z, \tau) = \overline{H_{q_v}(x, z, \eta, \tau)}
\]

\[
= H_{q_v}(x - u^\infty \tau, z, \eta - \int_0^\tau u^{(0)}(x, z, t')dt', 0)
\]

\[
= \sigma(x - u^\infty \tau, z, 0). \tag{26}
\]

It will turn out that the dynamics induced by moisture in this model can all be tied to this variable \( \sigma \). Considering the definition (7) of the switching function \( H_{q_v} \), we see that

\[
\sigma(x, z, \tau) = \lim_{\eta_0 \to \infty} \frac{1}{2\eta_0} \int_{-\eta_0}^{\eta_0} H_{q_v}(x, z, \eta', \tau) d\eta'
\]

\[
= \lim_{\eta_0 \to \infty} \frac{|\{\eta \in (-\eta_0, \eta_0) : q_v(x, z, \eta, \tau) \geq q_{vs}(z)\}|}{|(-\eta_0, \eta_0)|}. \tag{27}
\]

So for a fixed point \((x, z, \tau)\), \( \sigma \) is the area fraction of saturated regions on the \( \eta \)-scale. Using (25) and (16) we can write (23) as

\[
\hat{L}^{-1} \sigma \hat{w}^{(0)} \theta_z^{(2)} + \hat{L}^{-1} \overline{H_{q_v} \hat{w}^{(0)} \theta_z^{(2)}} = \overline{H_{q_v} C_d^{(0)}}. \tag{28}
\]

Now, an expression for

\[
w' := \overline{H_{q_v} \hat{w}^{(0)}} \tag{29}
\]

is required. Multiply (20)\(_{1,2} \) by \( H_{q_v} \) and average to get
\[ w'_r + u^\infty w'_x = \theta' \]
\[ \theta'_r + u^\infty \theta'_x + w'^{(2)}_z = \left[ (1 - \sigma) \hat{L}H_{q_c}C^{(0)}_d - \sigma \bar{C}_- \right] \]

where

\[ \theta' := \overline{H_{q_c} \tilde{\theta}^{(3)}} \]

and, using (22) and \( H_{q_v} (H_{q_v} - 1) = 0, \)

\[ \bar{C}_- (x, z, \tau) := \hat{L}C_{ev}^{**} (H_{q_v} - 1) \left( q^{(0)}_{vs} - q^{(0)}_v \right) \sqrt{q^{(0)}_r} \].

From (21) and (22) one can see that \( q^{(0)}_r \) is only advected with the background flow on the chosen short timescale. The same holds for \( q^{(0)}_v \), so that the evaporation source term \( \bar{C}_- \) is also only advected and can thus be computed once at the beginning of a simulation and then be obtained by suitable horizontal translations. Combining (28), (32) and (30) with (19) yields the final model (1), (2).

### 3. Analytical properties of the model

In this section we point out some analytical properties of the model. The dispersion relation and group velocity is computed and we find that moisture reduces the absolute value of the group velocity and changes its direction. For solutions with a plane-wave structure in the horizontal and in time, a Taylor-Goldstein equation for the vertical profiles is derived, revealing that moisture introduces a lower cut-off horizontal wavenumber and may cause critical layers. A way to assess the amount of released condensate is sketched, and a possible extension of the presented model to include nonlinear effects from dynamically changing area.
fractions, $\sigma$, will be explained briefly.

**a. Dispersion relation**

The leading order density for a near-homentropic atmosphere in the Newtonian limit $(\gamma \to 1) = \mathcal{O}(\varepsilon)$, see Klein and Majda (2006), reads

$$\rho(z) = \exp(-z)$$  \hspace{1cm} (33)

in non-dimensional terms. Thus, the anelastic constraint in (1) can be rewritten as

$$\bar{u}_x + \bar{w}_z - \ddot{w} = 0.$$  \hspace{1cm} (34)

Applying a standard plane wave ansatz would lead to a complex-valued dispersion relation, as in an atmosphere with decreasing density the amplitude of gravity waves grows with height. However, we do obtain a real valued expression by allowing for a vertically growing amplitude readily in the solution ansatz. Insert

$$\phi(x, z, \tau) = \hat{\phi} \exp(\mu \bar{z}) \exp(i(kx + mz - \omega \tau))$$  \hspace{1cm} (35)

with $\phi \in \{ \bar{u}, \bar{w}, \bar{\theta}, \pi, w', \theta' \}$ into (1) and (2) and assume, for the purpose of this section, $\bar{C}_- = 0$, i.e., the absence of source terms from evaporating rain, and that $\sigma$ is uniform in $x$. By successive elimination of the $\hat{\phi}$, we are left with roots

$$(\omega - u^\infty k)^2 = \omega^2_{\text{intr}} = 0$$  \hspace{1cm} (36)

and

$$(\omega - u^\infty k)^2 = \frac{k^2 - \sigma(\mu^2 - \mu - m^2) - \sigma(2\mu m - m)}{k^2 - (\mu^2 - \mu - m^2) - i(2\mu m - m)} \theta_z^{(2)}.$$  \hspace{1cm} (37)

18
The solution with $\omega_{\text{intr}} = 0$ corresponds to a vortical mode while the nonzero solutions are gravity waves. Choosing

$$\mu = \frac{1}{2}$$

results in the real-valued dispersion relation

$$(\omega - u^\infty k)^2 = \frac{k^2 + \sigma(m^2 + \frac{1}{4})}{k^2 + m^2 + \frac{1}{4}} \theta_z^{(2)}.$$  \hspace{1cm} (39)

For $\sigma = 0$, this is equal to the dispersion relation for the pseudo-incompressible equations derived in Durran (1989). Equation (39) can be rewritten as

$$\omega = u^\infty k + \omega_{\text{intr}} = u^\infty k \pm \sqrt{\frac{k^2 + \sigma(m^2 + \frac{1}{4})}{k^2 + m^2 + \frac{1}{4}}} \theta_z^{(2)}. \hspace{1cm} (40)$$

Here $\omega_{\text{intr}}$ is the so called intrinsic frequency, that would be seen by an observer moving with the background flow. Interestingly, for the incompressible case with $\rho_0 = \text{const.}$, in which the $1/4$ term vanishes, the formula in (40) is equal to the dispersion relation for internal gravity waves in a rotating fluid, see e.g. Gill (1982), but with the Coriolis parameter $f^2$ replaced by $\sigma \Theta_z^{(2)}$.

For the incompressible case, the dispersion relation can be written as a function of the angle $\alpha$ between the direction of the wavenumber vector $(k, m)$ of a wave and the horizontal $\omega_{\text{intr,incomp}} = \sqrt{(\cos^2(\alpha) + \sigma \sin^2(\alpha)) \theta_z^{(2)}}$. \hspace{1cm} (41)

b. Group velocity

Taking the derivative of (40) with respect to $k$ and $m$ yields the group velocity

$$c_g = (u_g, w_g) = (u^\infty, 0) \pm \frac{(1 - \sigma) \sqrt{\theta_z^{(2)}}}{(k^2 + m^2 + \frac{1}{4})^\frac{3}{2} (k^2 + \sigma(m^2 + \frac{1}{4}))^\frac{1}{2}} \left( k(m^2 + \frac{1}{4}), -mk^2 \right). \hspace{1cm} (42)$$
The group velocity is the travelling velocity of packets of waves with close-by wave numbers. It is identified with the transport of energy. In a dry ($\sigma = 0$), incompressible ($\mu = 0$, so no $\frac{1}{4}$ term) atmosphere, $c_g$ simplifies to the well-known expression for the group velocity of internal waves in a stratified fluid, see, e.g., Lighthill (1978),

$$c_{g,\text{dry,inc}} = (u^\infty, 0) \pm \frac{m\sqrt{\theta^{(2)}_z}}{(k^2 + m^2)^{\frac{3}{2}}} (m, -k).$$

(43)

One essential feature of these waves is $c_{g,\text{dry,inc}} \perp (k, m)$, i.e., the direction in which these waves transport energy is perpendicular to their phase direction. Because of the $1/4$ term, this no longer holds for (42), but waves with upward directed phase, i.e., either positive $m$ and positive branch in (40) and (42) or negative $m$ and negative branch in (40) and (42), still have a downward directed group velocity and vice versa.

With increasing $\sigma$, the coefficient in (42) decreases and eventually, for $\sigma = 1$, vanishes. Thus moisture reduces the transport of energy by waves and in completely saturated large-scale regions there is no energy transport by waves at all, only advection of energy by the background flow.

The ratio of the vertical and horizontal component of the group velocity determines the slope at which a wave packet propagates

$$\Delta_g = \frac{w_g}{u_g}.$$

(44)

Figure 1 shows the angle between a line with slope $\Delta_g$ and the horizontal depending on $\sigma$ for a flow with $\sqrt{\theta^{(2)}_z} = 1$ and $u^\infty = 0.1$ or, dimensionally, $N = 0.01 \text{ s}^{-1}$ and $u_\infty = 10 \text{ m s}^{-1}$. For all modes, moisture decreases the angle of the group velocity, so we expect the angle between the propagation direction of wave packets and the horizontal to decrease with increasing $\sigma$. This is demonstrated in the stationary solutions shown in section 4.a.2.
c. Taylor-Goldstein equation

A simplified but very elucidating class of solutions are those with height-dependent profile but plane wave structure in \( x \) and \( \tau \). Apply an ansatz

\[
\phi(x, z, \tau) = \phi^{(k)}(z) \exp(\mu z) \exp(ik(x - c\tau))
\]  

(45)

with \( c = \omega/k \) being the horizontal phase speed observed at a fixed height \( z \) and \( \phi \in \{ \bar{u}, \bar{w}, \bar{\theta}, \pi, \bar{w}', \bar{\theta}' \} \). The additional factor with parameter \( \mu \), as in the derivation of the dispersion relation, describes the amplitude growth caused by the decreasing density in the anelastic model. Inserting (45) into (1) and (2) and eliminating all \( \phi^{(k)} \) except for \( \bar{w}^{(k)} \) yields

\[
\left[ \frac{\theta_z^{(2)} - k^2(u^\infty - c)^2}{k^2(u^\infty - c)^2 - \sigma \theta_z^{(2)}} \right] \bar{w}^{(k)} + \mu(\mu - 1)\bar{w}^{(k)} + (2\mu - 1)\bar{w}_z^{(k)} + \bar{w}_{zz}^{(k)} = 0.
\]  

(46)

Set \( \mu = 1/2 \) as in subsection a, so that the final equation reads

\[
\left[ \frac{\theta_z^{(2)} - k^2(u^\infty - c)^2}{k^2(u^\infty - c)^2 - \sigma \theta_z^{(2)}} \right] \bar{w}^{(k)} + \bar{w}_{zz}^{(k)} = 0.
\]  

(47)

This is a Taylor-Goldstein equation which, in the incompressible dry case (i.e., with \( \sigma = 0 \) and without the 1/4 term) becomes the well-known equation for dry internal gravity waves, see, e.g., Etling (1996). The coefficient in (47) is the square of the local vertical wavenumber

\[
m(z, k) = \pm \sqrt{\left[ \frac{\theta_z^{(2)} - k^2(u^\infty - c)^2}{k^2(u^\infty - c)^2 - \sigma \theta_z^{(2)}} \right] k^2 - \frac{1}{4}}.
\]  

(48)

Figure 2 shows how the steady state vertical wavelength \( \lambda(k) = 2\pi/m(k) \) depends on \( \sigma \) for \( k = 1, \ldots, 4 \), constant \( \theta_z^{(2)} = 1 \) and \( u^\infty = 0.1 \). Obviously, moisture reduces the vertical wavelength.
(iii) Critical layers

Note that if there is a height \( z_c \) for which

\[
\sigma(z_c) = \frac{(u^\infty - c)^2 k^2}{\theta_z^{(2)}} \tag{49}
\]

then \( m(z, k) \to \infty \) as \( z \to z_c \) indicating a critical layer, see, e.g., Bühler (2009). In the dry case without shear, this only happens if at some height the phase speed \( c \) is equal to the background velocity \( u^\infty \). In the moist case, critical layers also arise from the vertical profile of \( \sigma \) so that non-critical dry flows can develop critical layers if moisture is added. Also, the critical height \( z_c \) depends on the horizontal wavenumber \( k \) in that case. A detailed investigation of the local structure of solutions in the presence of critical layers will not be presented here but will be subject of future work.

d. Cut-off wavenumbers

For steady-state solutions, the intrinsic frequency \( \omega_{\text{intr}} \) is zero and the dispersion relation (39) can be rewritten to express the vertical wavenumber \( m \) as a function of the horizontal wavenumber \( k \) only

\[
m^2 = \frac{\theta_z^{(2)} - (u^\infty)^2 k^2}{(u^\infty)^2 k^2 - \sigma \theta_z^{(2)}} k^2 - \frac{1}{4} \tag{50}
\]

We neglect the \( 1/4 \) term as this simplifies the following derivation without qualitatively affecting the result. From (50) one can see, that for

\[
k^2 > \frac{\theta_z^{(2)}}{(u^\infty)^2} \tag{51}
\]

\( m \) becomes imaginary. Thus, there is an upper limit of the horizontal wavenumber up to which internal waves actually propagate. Different from the dry case, moisture also
introduces a lower cut-off, as for

\[ k^2 < \sigma \frac{\theta^{(2)}_z}{(u_\infty)^2} \]  

(52)

\( m \) also becomes imaginary. So only horizontal wavenumbers \( k \) with

\[ k_{\text{low}} := \sqrt{\sigma} \sqrt{\frac{\theta^{(2)}_z}{u_\infty}} \leq k \leq \sqrt{\frac{\theta^{(2)}_z}{u_\infty}} =: k_{\text{up}} \]  

(53)

are propagating while waves with horizontal wavenumbers outside this range are evanescent.

For increasing moisture, \( \sigma \) gets closer to unity and the range of propagating wavenumbers narrows. For \( \sigma = 1 \), the only propagating mode left is \( k = \sqrt{\theta^{(2)}_z}/u_\infty \).

A typical value for the stability frequency in dimensional terms is \( 0.01 \text{ s}^{-1} \) corresponding to \( \sqrt{\theta^{(2)}_z} = 1 \). Assume a background flow of \( 10 \text{ m s}^{-1} \), i.e., \( u_\infty = 0.1 \), and a not very moist atmosphere with \( \sigma = 0.1 \). Then the upper cut-off wavenumber is \( k_{\text{up}} = 10 \) and the lower is \( k_{\text{low}} = \sqrt{\sigma}10 \approx 3.162 \). Expressed in dimensional zonal wavelengths, this means that only wavelengths between roughly 6km and 20km propagate, while larger or smaller wavelengths are evanescent. The maximum wavelength decreases like \( 1/\sqrt{\sigma} \), so that small values of \( \sigma \) corresponding to small amounts of moisture can already filter a significant range of wavelengths: For \( \sigma = 0.2 \), the maximum wavelength is 14km and it is further reduced to about 8km for \( \sigma = 0.5 \). This low-wavenumber cut-off is especially interesting in the context of GWD parameterizations, as it primarily affects near-hydrostatic modes with long horizontal wavelengths, which are the most important ones in terms of GWD.

e. Release of condensate

To assess the amount of condensate released by condensation in a parcel of air, the vertical displacement from its initial position has to be computed. Denote by \( \xi(x, z, \tau) \) the
displacement of the parcel at \((x, z)\) at time \(\tau\). For a given vertical velocity field \(\bar{w}\) we have

\[
\frac{D\xi}{D\tau} = \xi_r(x, z, \tau) + u^\infty \xi_x(x, z, \tau) = \bar{w}(x, z, \tau)
\]  

(54)

so that \(\xi\) can be computed for a given \(\bar{w}\) by solving (54).

Consider a parcel at height \(z_0\) at time \(\tau = 0\). This parcel has a \(\eta\)-scale distribution of water vapor, given by \(q_v(\eta, x, z, 0)\). The air is saturated wherever \(q_v(\eta, x, z_0, \tau) \geq q_{vs}(z_0)\) and condensation will take place if the parcel is displaced upward, so the amount of water vapor in the parcel is reduced according to the decrease of saturation water vapor mixing ratio.

Denote by \(\delta q_v(\xi; x, z_0)\) the condensate released by a parcel, initially located at \((x, z_0)\), if it is displaced upward from \(z_0\) to \(z_0 + \xi\). For a parcel with \(q_v(\eta, x, z_0) \geq q_{vs}(z_0)\) for every \(\eta\), this amount can be approximated by

\[
\delta q_v(\xi; x, z_0) = q_{vs}(z_0 + \xi) - q_{vs}(z_0) \approx dq_{vs}(z_0) dz \xi.
\]

(55)

If the parcel is not saturated everywhere, according to (27), \(\sigma(x, z)\) is the horizontal area fraction of saturated small scale columns and the condensate release can be approximated as

\[
\delta q_v(\xi; x, z_0) = q_{vs}(z_0 + \xi) - q_{vs}(z_0) \approx \sigma(x, z) \frac{dq_{vs}(z_0)}{dz} \xi.
\]

(56)

This approximation fails to account for small-scale areas that are initially not saturated but reach saturation somewhere on the parcels ascend from \(z_0\) to \(z_0 + \xi\), so \(\delta q_v\) is more like a lower bound for the condensate release. However, as our linear model is only valid for small displacements anyhow, (56) will be a decent approximation for the actual condensate release except for peculiar distributions of \(q_v(\eta)\) with large non-saturated regions that are very close to saturation.
There is an interesting possible extension of the model emerging from this derivation: if we assume leading order saturation everywhere from the start, i.e., \( q_{vs}^{(0)} = q_v^{(0)} \), and define \( \sigma \) according to the first order water vapor distribution \( q_v^{(1)} \), \( \sigma \) is no longer passively advected by the background flow. Instead, the equation for \( \sigma \) then contains the vertical velocity \( \bar{w} \), making this modified model nonlinear. The discussion of this extension is subject of future work.

4. Stationary and non-stationary solutions

A projection method is used to solve the full system (1), (2) numerically. It consists of a predictor step, advancing the equations in time ignoring the divergence constraint and the pressure gradient. In a second step, the predicted velocity field is projected onto the space of vector fields satisfying the anelastic constrain by applying the “correct” pressure gradient, obtained by solving a Poisson problem in each time step. The predictor step uses a third order Adams-Bashforth scheme in time together with a fourth-order central difference scheme for the advective terms. The application of this scheme to advection problems was investigated in Durran (1991) and found to be a viable alternative to the commonly used leapfrog scheme. To solve the Poisson problem occurring in the projection step, we use the discretization described in Vater (2005), Vater and Klein (2009) with slight modifications to account for the density stratification \( \rho_0(z) \).

In subsection a, visualizations of analytical stationary solutions for the case with constant coefficients are shown. The effect of the lower cut-off wavenumber is demonstrated as well as the damping of wave amplitudes by moisture and the reduced angle of propagation. In
the subsections b and c, non-stationary, numerically computed, approximate solutions are shown. First, the effects of a cloud envelope being advected through a mountain wave pattern are discussed, then waves initiated by a perturbation in potential temperature and travelling through a pair of clouds are investigated. A reduction of momentum flux by moisture is demonstrated for stationary and non-stationary mountain waves.

The code solves the non-dimensionalized equations, but for a more descriptive presentation, all quantities have been converted back into dimensional units in the figures.

a. Steady state solutions for a uniform atmosphere

If $\theta_z^{(2)}$, $\sigma$ and $u^\infty$ are constant and periodic boundary conditions are assumed in the horizontal direction, analytical solutions of the form

$$\bar{w}(x, z) = \exp\left(\frac{1}{2}z\right) \sum_{n=-N_x}^{N_x} \hat{w}^{(n)}(z) \exp(i k_n x)$$  \hspace{1cm} (57)

with $k_n = \frac{2\pi n}{L}$, $L$ equal to the length of the domain, can be derived, whereas every $w^{(n)}$ is a solution of (47) with $k = k_n$. For constant coefficients, these solutions simply read

$$\hat{w}^{(n)} = A^{(n)} \exp(im(k_n)z) + B^{(n)} \exp(-im(k_n)z).$$  \hspace{1cm} (58)

To avoid energy propagating downward, i.e., a negative vertical component of $c_g$, we choose the negative branch in (48) and set $B^{(n)} = 0$. The coefficient $A^{(n)}$ is determined according to the lower boundary condition

$$\bar{w}(x, z = 0, t) = u^\infty h_x(x)$$  \hspace{1cm} (59)

where $h$ describes the topography.
1) Sine shaped topography

We illustrate the change of the vertical wavenumber and the lower cut-off for a simple, sine shaped topography

\[ h(x) = H \sin(2x) \]  

(60)

on a domain \([0, 2\pi] \times [0, 1]\) or \([0, 62.8] \text{ km} \times [0, 10] \text{ km}\), where only a single mode with horizontal wavenumber \(k = 2\) is excited. Values for \(\sigma\) are \(\sigma = 0\), \(\sigma = 0.02\) and \(\sigma = 0.05\). The stratification is \(\sqrt{\theta_2^{(2)}} = 1\) or 0.01 s\(^{-1}\) and the background flow \(u^\infty = 0.1\) or 10 m s\(^{-1}\). The lower cut-off wavenumbers are \(k_{\text{low}} = \sqrt{0.02 \cdot (1/0.1)} \approx 1.41\) and \(k_{\text{low}} = \sqrt{0.05 \cdot (1/0.1)} \approx 2.24\) respectively. The height of the topography is \(H = 0.04\) or 400 m. Figure 3 shows contour lines of the vertical velocity \(\bar{w}\) for the three different values of \(\sigma\). The first figure shows the dry solution. The second shows the solution for \(\sigma = 0.02\) where still \(k_{\text{low}} < k\) and the excited wave is propagating. Compared to the dry case, the direction of propagation is slightly tilted to the vertical, corresponding to the increase of the vertical wavenumber \(m(k)\) with increasing \(\sigma\). In the last figure with \(\sigma = 0.05\), the solution has completely changed. Now \(k < k_{\text{low}}\) so the excited wave no longer propagates but is evanescent and its amplitude decays exponentially with height.

2) Witch of Agnesi

A more complex case is the Witch of Agnesi topography, exciting modes of all wavenumbers

\[ h(x) = \frac{HL^2}{L^2 + (x - x_c)^2}. \]  

(61)
$H$ is the height of the hill, $L$ a measure of its length and $x_c$ the center of the domain, so that the top of the hill is in the middle of it. Figure 4 shows solutions with $N_x = 201$ for $\sigma = 0$, $\sigma = 0.1$ and $\sigma = 0.5$ for $\sqrt{\theta_z^{(2)}} = 1$, $u^\infty = 0.1$ or 0.01 s$^{-1}$ and 10 m s$^{-1}$. The computational domain is $[0,8] \times [0,1]$ or $[0,80] \text{ km} \times [0,10] \text{ km}$ but the solution is plotted only on $[2,6] \times [0,1]$. Set $H = 0.04$ or 400 m and $L = 0.1$ or 1000 m.

The dashed lines visualize the average over the slopes (44) of all propagating modes, where each mode is weighted by its amplitude. With increasing $\sigma$, the slope of the wave pattern is reduced, according the the reduced slope of the group velocity pointed out in 3b. This is compatible with the following consideration: By decreasing the stability $N$ of the atmosphere, moisture increases the baroclinic Froude number $\text{Fr}_{\text{baroclinic}}$, cf. (11), which is small for waves close to hydrostatic balance. As hydrostatic waves propagate only in the vertical and the horizontal component increases as waves become more non-hydrostatic, it seems reasonable that an increase of $\text{Fr}_{\text{baroclinic}}$ by moisture leads to a reduction of the propagation angle.

An important mechanism caused by gravity waves is the vertical transfer of horizontal momentum. While for propagating steady-state modes the horizontally averaged vertical flux of momentum\(^2\)

$$\rho^{(0)} \overline{uw}$$

is constant, for evanescent modes it decays exponentially with height. Thus by turning propagating modes into evanescent, moisture inhibits the momentum transfer by gravity waves. Table 1 lists the values of the horizontally averaged vertical flux of momentum at the top of the domain for different values of $\sigma$.

\(^2\)For once here, the overbar over $\overline{uw}$ denotes the horizontal average in $x$ and not in $\eta$.\[^{28}\]
b. Mountain waves disturbed by a moving cloud envelope

In this subsection, we demonstrate the effect of a cloud packet being advected through an established mountain wave pattern, so the fully time-dependent problem (1), (2) is solved numerically. The domain is $[-3, 3] \times [0, 1.5]$ or $[-30, 30] \times [0, 15]$ km in dimensional terms. To realize a transparent upper boundary condition, a Rayleigh damping layer as described in Klemp and Lilly (1978), reaching from $z = 1$ to $z = 1.5$, is used. Subsequent figures show the solution on the subset $[-2, 2] \times [0, 1]$. The topography is a Witch of Agnesi hill with $H = 0.04$ or 400 m and $L = 0.1$ or 1000 m and the maximum located at $x = 0$. The background flow is linearly increased from $\tau = 0$ to $\tau = 0.25$ up to its maximum value of $u^\infty = 0.1$ or 10 m s$^{-1}$.

The simulation is run until $\tau_\sigma = 3$ with $\sigma \equiv 0$. Then, a cloud packet described by a Gaussian-distributed $\sigma$, defining its local intensity and envelope, is introduced and advected with velocity $u^\infty$, crossing the domain and finally exiting at the right boundary. For $\tau \geq \tau_\sigma$, set

$$
\sigma(x, z, \tau) = \sigma_{\max} \exp \left(-\frac{1}{2} \left[ \frac{(x - x_c - u^\infty [\tau - \tau_\sigma])^2}{s_x^2} + \frac{(z - z_c)^2}{s_z^2} \right] \right)
$$

(63)

with $s_x = 0.25$, $s_z = 0.25$ and $z_c = 0.4$. To avoid a sudden introduction of moisture at $\tau = \tau_\sigma$, set

$$
x_c = x_{\text{left}} - 2 \cdot s_x \quad \text{with} \quad x_{\text{left}} = -4
$$

(64)

so that at $\tau_\sigma$ the maximum of $\sigma$ is outside the actual domain of computation at $x = -4$. The cloud is then advected with velocity $u^\infty = 0.1$ into the domain so that its maximum enters at $\tau = \tau_\sigma + \frac{2s_x}{u^\infty} = 8$ at the left boundary and would leave the domain at $\tau = \tau_\sigma + \frac{2s_x}{u^\infty} + \frac{6}{u^\infty} = 68$. The maximum is always located at a height of 4 km.
The simulation uses 300 nodes in the horizontal direction and 75 in the vertical, resulting in a resolution of $\Delta x = \Delta z = 0.02$ or 200 m. The time step is $\Delta \tau = 0.05$ or 5 s.

The first four figures in 5 show contour lines of the vertical velocity $\vec{w}$ at different times for a cloud envelope with $\sigma_{\text{max}} = 0.5$. In the first figure, the cloud pattern has entered the domain from the left. Figures two, three, and four show how the cloud envelope travels through the mountain waves, while the last figure shows a dry reference solution at $\tau = 50$ for comparison. Considering the third and fourth figures and comparing them with the dry solution in figure five, we see how the cloud packet has damped the waves in the upper region. Also in figure four a decreased propagation angle is observed.

Figure 6 shows the net vertical flux of horizontal momentum (62) over time for the dry case, the case with $\sigma_{\text{max}} = 0.5$ and a third simulation where only the value of $\sigma_{\text{max}}$ has been changed to 0.2. The negative momentum flux increases towards zero as the cloud envelope travels through the wave pattern, i.e., its magnitude is reduced as in the analysis of the stationary solutions.

c. Waves travelling through clouds

The domain is $[-5, 5] \times [0, 1.25]$ or $[-50, 50]$ km $\times$ [0, 12.5] km and there is no background flow here, i.e. $u^\infty = 0$. The stability frequency is $\sqrt{\theta_z^{(2)}} = 1$ or 0.01 s$^{-1}$. Between $-30$ km to $-10$ km and 10 km to 30 km, two clouds packets are located.

All initial values are zero, except for a concentrated Gaussian peak of negative $\vec{\theta}$, placed at the center of the domain with its maximum at $(x, z) = (0, 0.5)$. Figure 7 visualizes the distribution of $\sigma$ as well as the initial $\vec{\theta}$. The simulation uses 400 nodes in the horizontal
direction, 40 nodes up to $z = 10 \text{ km}$ and 10 more nodes to realize the sponge layer between $z = 10 \text{ km}$ and $z = 12.5 \text{ km}$. The resulting resolution is $\Delta x = \Delta z = 250 \text{ m}$. The time step is $\Delta \tau = 0.1$ or 10 s. For comparison, a reference solution is computed with identical parameters but $\sigma \equiv 0$.

The initial potential temperature perturbation starts to excite waves, which form a typical X-shaped pattern (not shown, see for example Clark and Farley (1984)). Figure 8 shows a cross section through the vertical velocity at 5 km of the cloudy case (continuous line) as well as the non-cloudy reference simulation (dashed line). In the first figure, waves have formed and started travelling outwards. Inside the cloud, the updraft from the entering wave is amplified by latent heat release. Because the wave is also slowed down inside the cloud, there is some steepening before the cloud. In the next figure the steepening before and the amplification inside the cloud are even more pronounced. In the last figure one can see the newly formed extrema and a generally strongly distorted distribution of vertical velocity inside the cloud. In the region behind the cloud, the amplitude of the wave is noticeably reduced compared to the non-cloudy case.

5. Conclusions

The paper presents the derivation and analysis of a model for modulation of internal waves by deep convection. In the analysis, the dispersion relation, group velocity, and Taylor-Goldstein equation of the extended model are computed. Moisture, represented by the saturated area fraction $\sigma$, introduces multiple effects compared to the dry dynamics: by altering the group velocity, it inhibits wave propagation and changes the propagation
direction of wave packets. It introduces a lower cut-off horizontal wavenumber below which modes turn from propagating into evanescent. This is in contrast to the dispersion properties of waves in a non-rotating dry atmosphere for which only an upper cut-off wavenumber exists.

The lower cut-off leads to a moisture related reduction of the vertical flux of horizontal momentum. As gravity wave drag (GWD) is closely related to momentum flux, including this effect in parameterizations of GWD could improve simulations, because near-hydrostatic modes with small horizontal wavenumber, which are primarily affected by the cut-off, significantly contribute to wave-drag. We also note that moisture can cause critical layers for flows that would be non-critical under dry conditions.

Examples of stationary solutions obtained analytically demonstrate the cut-off, the reduced angle of propagation of wave packets, and the reduced momentum flux. The examples include stationary solutions for mountain waves excited from sine shaped and Witch of Agnesi topographies. The non-stationary results show how a cloud packet, represented as a Gaussian-distributed $\sigma$, is advected through Witch of Agnesi mountain waves. A significant damping of the waves by the cloud pattern is observed, and the reduction of momentum flux is documented. The second example begins with a small perturbation of potential temperature between two clouds, so that the excited waves travel through them. Inside the clouds, we observe an amplification of the amplitudes of wave-induced up- and downdrafts, while beyond the clouds, wave-amplitudes are damped.

The derivation starts from the results in Klein and Majda (2006) and yields a model for the interaction of non-hydrostatic internal gravity waves with a timescale of 100 s and a lengthscale of 10 km and convective hot towers with horizontal variations on a 1 km scale. An analytically computed closure of the model is achieved without requiring additional
approximations by applying weighted averages over the small 1 km horizontal scale. In the final model, moisture is present only as a parameter $\sigma$ which describes the area fraction of saturated regions on the tower-scale.

The resulting model involves anelastic, moist, large-scale dynamics described by an extension of the linearized dry anelastic equations. The equations include a source term for the potential temperature determined by two additional equations for the averaged dynamics on the small tower-scale. The averaged equations for the small scales, in turn, include terms from the large-scale dynamics so that there is bi-directional coupling between large- and tower-scale flow components in the model.

The presented model provides some interesting possibilities for future research. The similarity of the saturated area fraction $\sigma$ with the cloud cover fraction in GCMs might make the model a good starting point for the development of GWD parameterizations that include moist effects. Also of interest with respect to GWD parameterizations would be an attempt to validate the hypothesized lower horizontal cut-off wavenumber in a model employing a full bulk micro-physics scheme. An extension of the model to the case of weak under-saturation, in which $\sigma$ will turn into a prognostic variable and the model will become nonlinear, is work in progress.

Acknowledgments.

This research is partially funded by Deutsche Forschungsgemeinschaft, Project KL 611/14, and by the DFG Priority Research Programs “PQP” (SSP 1167) and “MetStroem” (SPP 1276). We thank Oliver Bühler, Peter Spichtinger and Stefan Vater for helpful discussions.
and suggestions. We also thank the three anonymous reviewers for their very helpful comments on the initial version of the manuscript.
Key steps in the derivation of the model

The non-dimensional conservation laws for mass, momentum, energy (expressed as potential temperature) in Klein and Majda (2006) read

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{u}) + (\rho w)_z &= 0 \\
\mathbf{u}_t + \mathbf{u} \cdot \nabla\mathbf{u} + w\mathbf{u}_z + \epsilon f (\Omega \times \mathbf{v})_{\parallel} + \epsilon^{-4} \rho^{-1} \nabla p &= 0 \\
w_t + \mathbf{u} \cdot \nabla w + w w_z + \epsilon f (\Omega \times \mathbf{v})_{\perp} + \epsilon^{-4} \rho^{-1} p_z &= -\epsilon^{-4} \\
\theta_t + \mathbf{u} \cdot \nabla \theta + w \theta_z &= \epsilon^2 (\tilde{S}_\theta^{\epsilon} + S_{q,\epsilon}^\theta) \tag{A1}
\end{align*}
\]

where

\[
S_{q,\epsilon}^{\theta} = \Gamma^{**} L^{**} q_{vs}^{**} \frac{\theta}{\rho} \left( \epsilon^{-n} \hat{C}_d - \hat{C}_{ev} \right) \tag{A2}
\]

is the source term related to evaporation and condensation, while \(\tilde{S}_\theta^{\epsilon}\) is a given external source of energy like, for example, radiation. Inserting (9), (15) plus the expansions of \(\rho, w, p\) in Klein and Majda (2006) yields the following leading order equations.
(i) **Horizontal momentum**

\[
\mathcal{O}(\varepsilon^2) : \quad p_\eta^{(3)} = 0
\]

\[
\mathcal{O}(\varepsilon^3) : \quad \rho^{(0)}u_\tau^{(0)} + \rho^{(0)}u_\tau^{(1)} + \rho^{(0)}u^\infty u_x^{(0)} + \rho^{(0)}u^\infty u_\eta^{(1)} + p_\tau^{(3)} + p_\eta^{(4)} = 0.
\]

The second equation can be rewritten as

\[
\rho^{(0)}u_\tau^{(0)} + \rho^{(0)}u^\infty u_x^{(0)} + p_x^{(3)} = -\left[\rho^{(0)}u_\tau^{(1)} + \rho^{(0)}u^\infty u_\eta^{(1)} + p_\eta^{(4)}\right].
\]

By integrating this equation along a characteristic \(\tau' + u^\infty \eta = \text{const.}\) and employing sublinear growth condition for the higher order quantities \(u^{(1)}\) and \(p^{(4)}\), we conclude that the right hand side must be zero and the equation simplifies to

\[
\rho^{(0)}u_\tau^{(0)} + \rho^{(0)}u^\infty u_x^{(0)} + p_x^{(3)} = 0.
\]

Note that as \(u^\infty\) is assumed to be constant, there is no term \(w^{(0)}u_z^\infty\).

(ii) **Vertical momentum**

\[
\mathcal{O}(1) : \quad p_z^{(0)} = -\rho^{(0)}
\]

\[
\mathcal{O}(\varepsilon) : \quad p_z^{(1)} = -\rho^{(1)}
\]

\[
\mathcal{O}(\varepsilon^2) : \quad \rho^{(0)}w_\tau^{(0)} + \rho^{(0)}u^\infty w_\eta^{(0)} + p_z^{(2)} = -\rho^{(2)}
\]

\[
\mathcal{O}(\varepsilon^3) : \quad \rho^{(0)}w_\tau^{(1)} + \rho^{(0)}u^\infty w_\eta^{(1)} + \rho^{(1)}w_\tau^{(0)} + \rho^{(1)}u^\infty w_\eta^{(0)} + \rho^{(0)}u_\tau^{(0)} + \rho^{(0)}u^\infty u_x^{(0)} + \rho^{(0)}u_\tau^{(1)} + \rho^{(0)}u^\infty u_\eta^{(1)} + p_z^{(3)} = -\rho^{(3)}
\]
We assume $\rho^{(1)} = 0$ here and employ again the sublinear growth condition. The last equation then becomes

$$
\rho^{(0)} w^{(0)} + \rho^{(0)} u^{\infty} w^{(0)} + \rho^{(0)} u^{(0)} w^{(0)} + p^{(3)} = -\rho^{(3)}.
$$
\[(A7)\]

Assuming that the specific heats $c_v$ and $c_p$ are constants and employing the Newtonian limit, expanding the equation of state yields

$$
-\frac{\rho^{(3)}}{\rho^{(0)}} = \theta^{(3)} - \frac{p^{(3)}}{p^{(0)}}.
$$
\[(A8)\]

Using this and the hydrostatic balance for the leading order density and pressure, one obtains

$$
w^{(0)} + u^{\infty} w^{(0)} + u^{(0)} w^{(0)} \pi^{(3)} = \theta^{(3)}
$$
\[(A9)\]

with $\pi^{(3)} := p^{(3)}/\rho^{(0)}$.

\((iii)\) Mass

$$
\rho^{(0)} u^{(1)}_{\eta} + \rho^{(0)} u^{(0)}_{x} + (\rho^{(0)} u^{(0)})_{z} = 0
$$
\[(A10)\]

Sublinear growth yields

$$
\rho^{(0)} u^{(0)}_{x} + (\rho^{(0)} w^{(0)})_{z} = 0.
$$
\[(A11)\]
(iv) Potential temperature

\[ \mathcal{O}(\epsilon^3) : \quad \theta_{\tau'}^{(3)} + u^\infty \theta_\eta^{(3)} = 0 \]

\[ \mathcal{O}(\epsilon^4) : \quad \theta_{\tau}^{(3)} + \theta_{\tau'}^{(4)} + u^\infty \theta_x^{(3)} + u^\infty \theta_\eta^{(4)} + u^{(0)} \theta_\eta^{(3)} + w^{(0)} \theta_z^{(2)} \]

\[ = \frac{L^{**} \Gamma^{**} q_{\text{vs}}^{**}}{p_0} \left( H_{\text{qv}} C_d^{(0)} + (H_{\text{qv}} - 1) C_{e\text{v}}^{(0)} \right) \]  \hspace{1cm} (A12)

Assume that there are no external sources of heat, i.e. \( \tilde{S}_\theta^\epsilon = 0 \). Again, the advective derivative of \( \theta^{(4)} \) along \( \tau' - \eta \)-characteristics vanishes by sublinear growth condition

\[ \theta_{\tau}^{(3)} + u^\infty \theta_x^{(3)} + u^{(0)} \theta_\eta^{(3)} + w^{(0)} \theta_z^{(2)} = \frac{L^{**} \Gamma^{**} q_{\text{vs}}^{**}}{p_0} \left( H_{\text{qv}} C_d^{(0)} + (H_{\text{qv}} - 1) C_{e\text{v}}^{(0)} \right). \] \hspace{1cm} (A13)

From the water-vapor transport equation in Klein and Majda (2006) we get

\[ \mathcal{O}(\epsilon^{-n}) : \quad C_d^{(-n)} \sim C_d^{**} \delta q_{\text{vs}}^{(0)} H_{\theta} q_c^{(0)} = 0 \]

\[ \mathcal{O}(\epsilon^{-n+1}) : \quad C_d^{(-n+1)} \sim C_d^{**} \delta q_{\text{vs}}^{(1)} H_{\theta} q_c^{(0)} = 0 \]

\[ \mathcal{O}(\epsilon^{-n+2}) : \quad C_d^{(-n+2)} \sim C_d^{**} \delta q_{\text{vs}}^{(2)} H_{\theta} q_c^{(0)} = 0 \] \hspace{1cm} (A14)

with \( \delta q := q_{\text{vs}} - q_v \). So we can distinguish the regime of saturated air, where the saturation deficit \( \delta q_{\text{vs}} \) is nonzero only at higher orders, and the non-saturated regime with \( \delta q_{\text{vs}}^{(0)} = 0 \) and \( q_c^{(0)} = 0 \), i.e. zero leading order cloud-water mixing ratio.
(v) Saturated air

\[
q^{(2)}_{vs,\tau'} + u^\infty q^{(2)}_{vs,\eta} + w^{(0)} q^{(0)}_{vs,x} = -C^{(0)}_d
\]

\[
q^{(1)}_{c,\tau'} + u^\infty q^{(1)}_{c,\eta} = 0
\]

\[
q^{(2)}_{c,\tau'} + u^\infty q^{(2)}_{c,\eta} + q^{(1)}_{c,\tau} + u^\infty q^{(1)}_{c,x} + u^{(0)} q^{(1)}_{c,\eta} = C^{(0)}_d - C^{(0)}_{cr}
\] (A15)

\[
q^{(0)}_{r,\tau'} + u^\infty q^{(0)}_{r,\eta} = 0
\]

\[
q^{(1)}_{r,\tau'} + u^\infty q^{(1)}_{r,\eta} + q^{(0)}_{r,\tau} + u^\infty q^{(0)}_{r,x} + u^{(0)} q^{(0)}_{r,\eta} = 0
\]

Again, by using sublinear growth condition, the equations simplify to

\[
-w^{(0)} q^{(0)}_{vs,x} = C^{(0)}_d
\]

\[
q^{(1)}_{c,\tau} + u^\infty q^{(1)}_{c,x} + u^{(0)} q^{(1)}_{c,\eta} = C^{(0)}_d - C^{(0)}_{cr}
\] (A16)

\[
q^{(0)}_{r,\tau} + u^\infty q^{(0)}_{r,x} + u^{(0)} q^{(0)}_{r,\eta} = 0.
\]

(vi) Non-saturated air

\[
q^{(0)}_{v,\tau'} + u^\infty q^{(0)}_{v,\eta} = 0
\]

\[
q^{(1)}_{v,\tau'} + u^\infty q^{(1)}_{v,\eta} + q^{(0)}_{v,\tau} + u^\infty q^{(0)}_{v,x} + u^{(0)} q^{(0)}_{v,\eta} = 0
\]

\[
q^{(0)}_{c,\tau'} + u^\infty q^{(1)}_{c,\eta} = 0
\]

\[
q^{(1)}_{c,\tau} + u^\infty q^{(1)}_{c,\eta} + q^{(0)}_{c,\tau} + u^\infty q^{(0)}_{c,x} + u^{(0)} q^{(0)}_{c,\eta} = -C^{(0)}_{cr}
\] (A17)

\[
q^{(0)}_{r,\tau'} + u^\infty q^{(0)}_{r,\eta} = 0
\]

\[
q^{(1)}_{r,\tau'} + u^\infty q^{(1)}_{r,\eta} + q^{(0)}_{r,\tau} + u^\infty q^{(0)}_{r,x} + u^{(0)} q^{(0)}_{r,\eta} = 0
\]
Sublinear growth yields

\begin{align*}
q^{(0)}_{v,\tau} + u^\infty q^{(0)}_{v,x} + u^{(0)} q^{(0)}_{v,\eta} &= 0 \\
q^{(1)}_{c,\tau} + u^\infty q^{(1)}_{c,\eta} + u^{(0)} q^{(1)}_{c,\eta} &= -C^{(0)}_{ct} \\
q^{(0)}_{t,\tau} + u^\infty q^{(0)}_{t,x} + u^{(0)} q^{(0)}_{t,\eta} &= 0.
\end{align*}

(A18)
REFERENCES


Davies, T., A. Staniforth, N. Wood, and J. Thuburn, 2003: Validity of anelastic and other


List of Tables

1. Vertical flux of horizontal momentum for different values of $\sigma$ in the constant coefficient, steady-state solution. 46
Table 1. Vertical flux of horizontal momentum for different values of $\sigma$ in the constant coefficient, steady-state solution.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>Momentum flux in N m$^{-2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-0.71$</td>
</tr>
<tr>
<td>0.1</td>
<td>$-0.60$</td>
</tr>
<tr>
<td>0.5</td>
<td>$-0.22$</td>
</tr>
</tbody>
</table>
List of Figures

1 Angle between the direction of group velocity and the horizontal for wavenumbers $k = 1, \ldots, 4$ depending on $\sigma$ in a steady-state flow with $\sqrt{\theta_z^{(2)}} = 1$ or $0.01 \text{ s}^{-1}$ and $u^\infty = 0.1$ or $10 \text{ m s}^{-1}$

2 Vertical wavelength $\lambda$ for modes $k = 1, \ldots, 4$ depending on $\sigma$ for $\sqrt{\theta_z^{(2)}} = 1$ or $0.01 \text{ s}^{-1}$ and $u^\infty = 0.1$ or $10 \text{ m s}^{-1}$. The dimensional, horizontal wavelengths corresponding to $k = 1, \ldots, 4$ are approximately 63 km, 31 km, 21 km and 16 km. Values of $\sigma$ where $\lambda(k, \sigma) = 0$ indicate the lower-cutoff.

3 Contour lines of the steady-state vertical velocity for a sine shaped topography with horizontal wavenumber $k = 2$ for $\sigma = 0$, $\sigma = 0.02$ and $\sigma = 0.05$. The interval between contours is 0.2 m s$^{-1}$. Dotted contours represent negative values.

4 Contour lines of steady-state vertical velocity for a Witch of Agnesi topography with $H = 400 \text{ m}$ and $L = 1000 \text{ m}$ for $\sigma = 0$, $\sigma = 0.1$ and $\sigma = 0.5$. The interval between contours is 0.25 m s$^{-1}$. Dotted contours represent negative values. The dashed lines visualize the averaged slope of the group velocity of all propagating modes.

5 Contour lines of vertical velocity at different times for a cloud packet advected through the waves excited by a Witch of Agnesi. Interval between contours is 0.25 m s$^{-1}$ in dimensional terms. Dotted contours represent negative values. The two thin circles are the $\sigma = 0.05$ and $\sigma = 0.25$ isoline. The last figure shows a reference solution without any moisture at $\tau = 50$ for comparison.
Net vertical flux of horizontal momentum $\rho^{(0)} \overline{uw}$ across the height $z = 10$km over time. The solid line is the dry reference simulation, the dashed line is a moving cloud envelope with $\max(\sigma) = 0.2$ while the dash-dotted line represents a cloud envelope with $\max(\sigma) = 0.5$.

$\sigma$ modelling two clouds. The dashed line is the cross-section along which the vertical velocity is plotted in figure 8. The continuous lines are isolines of $\sigma$ with a difference between isolines of 0.1 and the outer isoline corresponding to $\sigma = 0.1$. The dotted line shows the initial distribution of $\tilde{\theta}$, the difference between the isolines is $-0.025$, the outer line corresponding to $\tilde{\theta} = -0.025$.

Cross-section of the vertical velocity $\bar{w}$ at different times for the cloudy and non-cloudy case. The dotted line is the cross-section through $\sigma$ at the same height, but multiplied by a factor 0.05 so that the shape is reasonably visible in the given scaling of the $y$-axis.
Angle of group velocity depending on $\sigma$ for $N=0.01\ s^{-1}$, $U=10\ m\ s^{-1}$

**Fig. 1.** Angle between the direction of group velocity and the horizontal for wavenumbers $k = 1, \ldots, 4$ depending on $\sigma$ in a steady-state flow with $\sqrt{\theta_z^{(2)}} = 1$ or $0.01\ s^{-1}$ and $u^\infty = 0.1$ or $10\ m\ s^{-1}$
Vertical wavelength depending on $\sigma$ for $N=0.01 \text{ s}^{-1}$, $U=10 \text{ m s}^{-1}$

![Graph of vertical wavelength $\lambda(k, \sigma)$ vs $\sigma$ for $k=1, \ldots, 4$ depending on $\sigma$ for $\sqrt{\theta_z^{(2)}} = 1$ or $0.01 \text{ s}^{-1}$ and $u^{\infty} = 0.1$ or $10 \text{ m s}^{-1}$. The dimensional, horizontal wavelengths corresponding to $k=1, \ldots, 4$ are approximately 63 km, 31 km, 21 km and 16 km. Values of $\sigma$ where $\lambda(k, \sigma) = 0$ indicate the lower-cutoff.]
Fig. 3. Contour lines of the steady-state vertical velocity for a sine shaped topography with horizontal wavenumber $k = 2$ for $\sigma = 0$, $\sigma = 0.02$ and $\sigma = 0.05$. The interval between contours is 0.2 m s$^{-1}$. Dotted contours represent negative values.
Fig. 4. Contour lines of steady-state vertical velocity for a Witch of Agnesi topography with $H = 400$ m and $L = 1000$ m for $\sigma = 0$, $\sigma = 0.1$ and $\sigma = 0.5$. The interval between contours is $0.25$ m s$^{-1}$. Dotted contours represent negative values. The dashed lines visualize the averaged slope of the group velocity of all propagating modes.
Fig. 5. Contour lines of vertical velocity at different times for a cloud packet advected through the waves excited by a Witch of Agnesi. Interval between contours is 0.25 m s$^{-1}$ in dimensional terms. Dotted contours represent negative values. The two thin circles are the $\sigma = 0.05$ and $\sigma = 0.25$ isoline. The last figure shows a reference solution without any moisture at $\tau = 50$ for comparison.
Fig. 6. Net vertical flux of horizontal momentum $\rho^{(0)} \overline{uw}$ across the height $z = 10$km over time. The solid line is the dry reference simulation, the dashed line is a moving cloud envelope with $\text{max}(\sigma) = 0.2$ while the dash-dotted line represents a cloud envelope with $\text{max}(\sigma) = 0.5$. 
Fig. 7. $\sigma$ modelling two clouds. The dashed line is the cross-section along which the vertical velocity is plotted in figure 8. The continuous lines are isolines of $\sigma$ with a difference between isolines of 0.1 and the outer isoline corresponding to $\sigma = 0.1$. The dotted line shows the initial distribution of $\bar{\theta}$, the difference between the isolines is $-0.025$, the outer line corresponding to $\bar{\theta} = -0.025$. 
Fig. 8. Cross-section of the vertical velocity \( \bar{w} \) at different times for the cloudy and non-cloudy case. The dotted line is the cross-section through \( \sigma \) at the same height, but multiplied by a factor 0.05 so that the shape is reasonably visible in the given scaling of the \( y \)-axis.