A Simple Justification of the Singular Limit for Equatorial Shallow-Water Dynamics

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Abstract
The equatorial shallow-water equations at low Froude number form a symmetric hyperbolic system with large variable-coefficient terms. Although such systems are not covered by the classical Klainerman-Majda theory of singular limits, the first two authors recently proved that solutions exist uniformly and converge to the solutions of the long-wave equations as the height and Froude number tend to 0. Their proof exploits the special structure of the equations by expanding solutions in series of parabolic cylinder functions. A simpler proof of a slight generalization is presented here in the spirit of the classical theory. © 2000 Wiley Periodicals, Inc.

1 Introduction
The climate of equatorial regions is of central importance in meteorology. Indeed, because the Coriolis force degenerates at the equator, the tropics behave as a wave guide with extremely warm surface temperatures, which affects the weather on a global level through hurricanes, monsoons, el Niño, and global teleconnections with the mid-latitude atmosphere [22, 24]. These phenomena, however, are still poorly understood. They involve a complex nonlinear interaction of clouds, moisture, and convection [2, 9, 18, 24] on a large variety of scales in both time and space, ranging from cumulus clouds over a few kilometers to intraseasonal oscillations over planetary scales of order 40,000 km. Current numerical simulations still fail to capture this complex interaction on multiple scales (see [2, 18, 20] and references therein). Hence the atmosphere-ocean science community is very interested in theoretical studies of tropical wave interactions and in developing reduced dynamical models that can explain, at least qualitatively, some key features of equatorial phenomena [1, 2, 9, 17, 18, 20, 21]. A mathematical introduction to these topics may be found in [16, chap. 9].
The present paper is concerned with a simple multiscale system of PDEs for the equatorial region. The plain equatorial shallow-water equations are

\begin{align}
\vec{v}_t + \vec{v} \cdot \nabla \vec{v} + \varepsilon^{-1}(y\vec{v}^\perp + \nabla h) &= 0, \\
h_t + \vec{v} \cdot \nabla h + h \nabla \cdot \vec{v} + \varepsilon^{-1}\nabla \cdot \vec{v} &= 0,
\end{align}

where \( h = h(t,x,y) \) is the height, \( \vec{v} = (u,v)(t,x,y) \) is the horizontal velocity, and \( \vec{v}^\perp = (-v,u) \). The longitude \( x \) is naturally periodic along the equator and so will be taken to belong to \( \mathbb{T} \), while the distance \( y \) to the equator will be assumed to vary in \( \mathbb{R} \). These equations are written in nondimensional variables under the assumption that both the Froude number (typical fluid velocity ratio to the gravity wave speed) and the height fluctuations are of order \( \varepsilon \). The actual value of \( \varepsilon \) is around \( 10^{-1} \), but we regard it as a positive parameter tending to 0. We remark that the Coriolis term \( y\vec{v}^\perp \) depends linearly on the latitude (and vanishes at the equator, where \( y = 0 \)).

In order to transform (1.1)–(1.2) into a multiscale system, we consider solutions that contain not only long waves of wavelength \( O(1/\varepsilon) \) in the \( x \)-direction, but also waves on the scale \( O(1) \) in that direction [20]. We therefore allow \( \vec{v} \) and \( h \) to be functions of \( \varepsilon x = X \in \mathbb{T} \) as well as of \( x \in \mathbb{T} \). Equations (1.1)–(1.2) then become

\begin{align}
u_t + uu_x + \varepsilon uu_X + vu_y + h_x + \varepsilon^{-1}(-vy + h_y) &= 0, \\
v_t + uv_x + \varepsilon uv_X + vv_y + \varepsilon^{-1}(yu + h_y) &= 0, \\
h_t + uh_x + \varepsilon uh_X + vh_y + (1 + \varepsilon h)u_x \\
&\quad + h(u_x + v_y) + \varepsilon^{-1}(u_x + v_y) = 0,
\end{align}

and we supplement them with initial conditions

\begin{align}
u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \quad h|_{t=0} = h_0.
\end{align}

The first rigorous treatment of this singular limit appeared in [5], where it was proved that, under suitable assumptions on the initial data, solutions to (1.3)–(1.6) depending on \( X \) but not on \( x \) exist on a time interval independent of \( \varepsilon \) and converge as \( \varepsilon \to 0 \) to the solution \( (u,h) \) of the long-wave equations

\begin{align}
\partial_t u + \partial_x h - yV &= 0, \\
\partial_t h + \partial_x u + \partial_y V &= 0, \\
yu + \partial_x h &= 0,
\end{align}

for some Lagrange multiplier \( V \). We also have a similar result for solutions depending on \( x \) but not on \( X \) [6]. Instead of the long-wave equations, zonal jets then constitute the slow limiting dynamics. Both weak and strong convergence results for viscous equatorial shallow-water equations were later obtained by Gallagher and Saint-Raymond [10] by a very different approach following the solution strategy of Leray-Hopf at fixed viscosity and utilizing a detailed study of resonances.

For flows in mid-latitudes, theorems have been proved that justify the quasi-geostrophic dynamics with great generality [3, 4, 7, 8, 16, 19, 25]. The dependence on \( y \) of the fast term makes it difficult to do the same in the equatorial context: energy estimates on solutions to (1.3)–(1.6) in the usual Sobolev spaces blow up as \( \varepsilon \to 0 \), because differentiating the equations with respect to \( y \) leads to terms of
order $O(\varepsilon^{-1})$ from the commutators. In fact, few examples of singular limit of a symmetric hyperbolic system with fast variable coefficients are known (one may be found in [12, sec. 4]).

It is possible, however, to get uniform estimates in modified Sobolev spaces by exploiting the special structure hidden in (1.3)–(1.5) [5, 6]. This special structure involves utilizing Riemann invariant coordinates so that (1.3)–(1.5) becomes a symmetric hyperbolic system with a fast-wave operator involving a matrix of the raising and lowering operators for the harmonic oscillator. The modified Sobolev space $\tilde{W}_n$ is defined as the Banach space of functions having finite norm

\begin{equation}
\sum_{j+k+l+m \leq n} \| y^j \partial^k_x \partial^l_y \partial^m w \|^2_{L^2}.
\end{equation}

We prove in this paper the following theorem:

**Theorem 1.1.** Suppose that $(u_0, v_0, h_0) \in \tilde{W}_{2n}$ for some $n \geq 3$. Suppose in addition that the initial data are well prepared in the sense that

\begin{equation}
(-y v_0 + \partial_x h_0, y u_0 + \partial_y h_0, \partial_x u_0 + \partial_y v_0) = O(\varepsilon)
\end{equation}

in $\tilde{W}_{(n-1)}$. Then the solutions

\begin{equation}
(u^\varepsilon, v^\varepsilon, h^\varepsilon) \in C([0, T]; \tilde{W}_{2n}) \cap C^1([0, T]; \tilde{W}_{2(n-1)})
\end{equation}

to the initial-value problems (1.3)–(1.6) exist for a time $T > 0$ independent of $\varepsilon$ and converge in $C([0, T]; \tilde{W}_{2(n-1)})$ to $(u^0, 0, h^0)$, where $u^0$ and $h^0$ depend only on $t$, $X$, and $y$ and satisfy the system (LW) along with the initial conditions

\begin{equation}
u^0|_{t=0} = u_0, \quad h^0|_{t=0} = h_0.
\end{equation}

The estimates in $\tilde{W}_n$ were obtained in our previous papers by expanding the solutions in a basis of special functions involving Hermite polynomials. In contrast, the estimates are obtained below by using direct algebraic properties of the fast operator. This allows for a much simpler proof, and to preserve this simplicity, we restrict ourselves to even values of $n$ and also assume that the initial data are well prepared. Although the filtering technique [11, 23] implies that the bounds obtained here suffice to obtain convergence to a limiting profile even when the initial data are not well prepared, expansions in basis functions would be needed to calculate the resonant interactions occurring in that limit, as was done in [6, 10].

### 2 Operators and Identities

We begin by adapting the analysis of [5], which treated the case when the solution does not depend on the original $x$-variable. In order to symmetrize the equations, substitute

\[ h = \frac{(1 + \varepsilon h/2)^2 - 1}{\varepsilon} \]
into (1.3)–(1.5) and divide the resulting equation for $\tilde{h}$ by $1 + \varepsilon \tilde{h}/2$. After dropping the tilde, this yields

\[
\begin{align*}
u_t + uu_x + \varepsilon uu_x + vv_y + \frac{1}{2} \varepsilon h_x + (1 + \frac{1}{2} \varepsilon \tilde{h}) h_x + \varepsilon^{-1} (-yv + h) & = 0, \\
v_t + uv_x + \varepsilon uv_x + vv_y + \frac{1}{2} h_y + \varepsilon^{-1} (yv + h) & = 0, \\
h_t + uh_x + \varepsilon uh_x + vh_y + \frac{1}{2} h(u_x + v_y) + (1 + \frac{1}{2} \varepsilon \tilde{h}) u_x + \varepsilon^{-1} (u_x + v_y) & = 0.
\end{align*}
\]

Making the substitutions

\[
(2.1) u = \frac{r - l}{\sqrt{2}} \quad \text{and} \quad h = \frac{r + l}{\sqrt{2}}
\]

then leads to

\[
(2.2) r_t + \frac{3r - l}{2\sqrt{2}} (r_x + \varepsilon r_x) + r_x + vr_y + \frac{r + l}{4} v_y + \varepsilon^{-1} \left( r_x + \frac{v_y - yv}{\sqrt{2}} \right) = 0,
\]

\[
(2.3) l_t + \frac{r - 3l}{2\sqrt{2}} (l_x + \varepsilon l_x) - l_x + vl_y + \frac{r + l}{4} v_y + \varepsilon^{-1} (-l_x + \frac{v_y + yv}{\sqrt{2}}) = 0,
\]

\[
(2.4) v_t + \frac{r - l}{\sqrt{2}} (v_x + \varepsilon v_x) + vv_y + \frac{l + r}{4} (l_y + r_y) + \varepsilon^{-1} \left( r_y + yv \frac{l_y - yl}{\sqrt{2}} \right) = 0.
\]

In the terms of order $\varepsilon^{-1}$ in (2.2)–(2.4), $y$-derivatives and multiplication by $y$ always occur in the combinations

\[
L_\pm := \frac{1}{\sqrt{2}} (\partial_y \mp y).
\]

The operators $L_\pm$ are the well-known “raising” and “lowering” operators of the harmonic oscillator Hamiltonian

\[
(2.5) H := L_- L_+ + L_+ L_- = \partial_y^2 - y^2.
\]

As their names suggest, the raising or lowering operator transforms each eigenfunction of $H$ into a multiple of the eigenfunction having the next higher or lower eigenvalue, respectively. The analyses in both [5] and [10] were performed by expanding solutions in terms of those eigenfunctions. The main novelty here is that
by using the key identity (2.14) below, we obtain uniform estimates without resorting to such expansions. Besides being simpler, the resulting theory is also more in the spirit of the classical analysis [13, 14, 15] of singular limits.

Using the operators $L_{\pm}$, the equations (2.2)–(2.4) may be written as

$$
\begin{align*}
\dot{r}_t + \frac{3r - l}{2\sqrt{2}}(rx + \varepsilon rx_X) + r_X + vr_y + \frac{r + l}{4}vy \\
+ \varepsilon^{-1}(r_x + L_+ v) &= 0,
\end{align*}
$$

$$
\begin{align*}
\dot{l}_t + \frac{r - 3l}{2\sqrt{2}}(lx + \varepsilon lx_X) - l_X + vl_y + \frac{r + l}{4}vy \\
+ \varepsilon^{-1}(-l_x + L_- v) &= 0,
\end{align*}
$$

$$
\begin{align*}
\dot{v}_t + \frac{r - l}{\sqrt{2}}(vx + \varepsilon vX) + vv_y + \frac{l + r}{4}(l_y + r_y) \\
+ \varepsilon^{-1}(L_- r + L_+ l) &= 0.
\end{align*}
$$

These equations have the form

$$
\begin{align*}
\vec{U}_t + A_1(\vec{U})U_x + A_2(\vec{U})U_X + A_3(\vec{U})U_y + \varepsilon^{-1}M\vec{U} &= 0,
\end{align*}
$$

where the matrices $A_j$ are symmetric and depend smoothly on $\vec{U}$, and $M$ is the matrix operator

$$
M = \begin{pmatrix}
\partial_x & 0 & L_+ \\
0 & -\partial_x & L_- \\
L_- & L_+ & 0
\end{pmatrix}.
$$

The identities

$$
L_+^* = -L_-^*
$$

imply that $M$ is antisymmetric.

The commutation relations

$$
[H, L_{\pm}] = \pm 2L_{\pm}
$$

imply by induction that for any polynomial $P$,

$$
P(H \pm 2)L_{\pm} = L_{\pm}P(H).
$$

Replacing $P(\cdot)$ in (2.11) by $P(\cdot \mp 2)$ yields the similar identity

$$
P(H)L_{\pm} = L_{\pm}P(H \mp 2).
$$

Now define

$$
D := \begin{pmatrix}
H + 2 & 0 & 0 \\
0 & H - 2 & 0 \\
0 & 0 & H
\end{pmatrix}.
$$

Identities (2.11)–(2.12) with $P(s) = s$ imply that $D$ commutes with $M$:

$$
DM = MD.
$$
This identity will serve as our substitute for the commutation of scalar derivative operators with constant-coefficient derivative operators in the classical theory [13] of singular limits. Note that although $D$ is not a scalar operator, the highest-order part of $D$ is scalar, which will suffice for obtaining energy estimates.

3 Function Spaces Adapted to $H$

We will need to estimate derivatives of functions and powers of $y$ times functions in terms of $H$ applied to those functions. In particular, we will show that when $Hw$ belongs to $L^2$, then each of the terms $\partial^2_y w$ and $y^2 w$ making up that expression belongs to $L^2$ separately.

Define $\tilde{W}_{2n}$ to be the space of functions having finite norm

$$\|w\|_{\tilde{W}_{2n}}^2 := \sum_{k+l+2p \leq 2n} \|\partial^k_x \partial^l_y H^p w\|_{L^2}^2;$$

the modified Sobolev spaces $\tilde{W}_{2n}$ are Hilbert spaces whose inner products are obtained by replacing the squares of norms on the right side of (2.3) by the corresponding inner products. The norm (2.3) is equivalent to the variant in (1.7):

Lemma 3.1. For every $w$ in $\tilde{W}_{2n}$,

$$\sum_{j+k+l+m \leq 2n} \|y^j \partial^k_x \partial^l_y \partial^m_y w\|_{L^2}^2 \leq c_n \|w\|_{\tilde{W}_{2n}}^2.$$

Proof. Define the $y$-weight of a term $y^j \partial^k_x \partial^l_y \partial^m_y w$ to be $j + m$. By using integration by parts and moving powers of $y$ from one factor to the other in the integrals occurring on the left side of (3.2) and then applying the Cauchy-Schwarz inequality, we can estimate that left side by a multiple of the sum of those terms having even $y$-weight. Thanks to the density of functions of the form $w_1(x, X)w_2(y)$, it then suffices to prove the estimate for functions of $y$ alone, i.e., to prove that

$$\sum_{j+m \leq 2n} \|y^j \partial^m_y w\|_{L^2}^2 \leq c_n \sum_{p=0}^n \|H^p w\|_{L^2}^2.$$

Integration by parts yields the basic identities

$$\|yw\|_{L^2}^2 + \|w\|_{L^2}^2 = - \int \langle w \rangle Hw \leq \|w\|_{L^2} \|H w\|_{L^2} \leq \|H w\|_{L^2}^2 + \|w\|_{L^2}^2$$

and

$$\|w\|_{L^2}^2 + 2\|yw\|_{L^2}^2 = \|H w\|_{L^2}^2 + 2\|w\|_{L^2}^2,$$

which show that (3.3) holds for $n = 1$. That case allows us to estimate any term of the form $|y^j \partial^k_x H^m w|$ by similar terms having two fewer occurrences of $y$ and/or $\partial_y$ but with an extra factor of $H$ on the left. The commutation relations

$$[H, y] = 2\partial_y, \quad [H, \partial_y] = 2y, \quad [\partial_y, y] = 1,$$
allow us to move that factor of $H$ back to the right and then move all factors of $y$ to the left of all factors of $\partial_y$ modulo lower-order terms. By induction, this yields (3.3).

**Remark 3.2.** Thanks to the factors of $y$ included in estimate (3.2) and the fact that $x$ and $X$ belong to the compact space $\mathbb{T}$, an easy variant of Rellich’s lemma shows that bounded sets of $\tilde{W}_2n$ are precompact in $\tilde{W}_{2(n+1)}$ without the need to localize by multiplying by a $C^\infty_0$ function.

### 4 Uniform Estimates

**Lemma 4.1.** Let $n$ be an integer greater than or equal to 2. Assume that the initial data $(r_0, l_0, v_{0, e})$ for system (2.6)–(2.8) are bounded uniformly in $\tilde{W}_2n$ for $0 < \epsilon \leq \epsilon_0$. Then for some positive $T$ there exists a unique solution to that system for such $\epsilon$ on the fixed time interval $[0, T]$, during which the solution remains uniformly bounded in $\tilde{W}_2n$.

**Proof.** Since (2.6)–(2.8) is a symmetric hyperbolic system, by standard results it suffices to prove a uniform estimate for the solution. Differentiating (2.9) or applying the operator $D$ from (2.13) yields equations of the form

\begin{equation}
\hat{V}_t + A_1(\hat{U})V_x + A_2(\hat{U})\hat{V}_x + A_3(\hat{U})\hat{V}_y + \epsilon^{-1}MV = \tilde{F}.
\end{equation}

Since the $A_j$ are symmetric and the operator $M$ is antisymmetric, multiplying (4.1) by $\hat{V}$, integrating over the spatial variables, and integrating by parts yields the uniform $L^2$ estimate

\begin{equation}
\frac{d}{dt} \|\hat{V}\|^2_{L^2} \leq \left( \sum_j \|A_j(\hat{U})\|_{C^1} \right) \|\hat{V}\|^2_{L^2} + \|\hat{F}\|^2_{L^2} + \|\hat{V}\|^2_{L^2} + \|\hat{F}\|^2_{L^2}.
\end{equation}

In particular, since solutions $\hat{U}$ of (2.9) satisfy (4.1) with $\tilde{F} = 0$,

\begin{equation}
\frac{d}{dt} \|\hat{U}\|^2_{L^2} \leq \left( \sum_j \|A_j(\hat{U})\|_{C^1} \right) \|\hat{U}\|^2_{L^2}.
\end{equation}

Now apply $\partial_x^j \partial_y^k D^p$ to (2.9). Using (2.14) we obtain that $\hat{V} = \tilde{V}_{k,l,p} := \partial_x^j \partial_y^k D^p \hat{U}$ satisfies (4.1) with

\begin{equation}
\tilde{F} = \tilde{F}_{k,l,p} := \left[ \partial_x^j \partial_y^k D^p, A_1(\hat{U}) \right] \hat{U}_x + \left[ \partial_x^j \partial_y^k D^p, A_2(\hat{U}) \right] \hat{U}_x + \left[ \partial_x^j \partial_y^k D^p, A_3(\hat{U}) \right] \hat{U}_y.
\end{equation}

We will now show that

\begin{equation}
\|\tilde{F}_{k,l,p}\|^2_{L^2} \leq G(\|\hat{U}\|^2_{\tilde{W}_2n})
\end{equation}

for some smooth function $G$. Recall that the $y$-weight of a term $y^j \partial_x^k \partial_y^l D^m \hat{U}$ equals $j + m$. In addition, define the total weight of such a term to be $j + k + l + m$, and the
total weight of a product to be the sum of the total weights of the factors. Unlike
the case of scalar operators, the order of the commutator of two matrix operators is
not in general lower than the sum of the orders of those operators. Here, however,
the term \((\partial_y^2 - y^2)/I\) in \(D\) having highest \(y\)-weight is a scalar operator, and hence so
is the term \((\partial_y^2 - y^2)/I\) having highest \(y\)-weight in \(D^p\). The total weight of every
term in \(F_{k,l,p}\) is therefore at most \(2n\). Since \(n \geq 2\), at most one factor in each term of
\(F_{k,l,p}\) has total weight greater than \(2n - 2\). Sobolev’s theorem for dimension 3 plus
(3.2) ensure that any factors having total weight at most \(2n - 2\) can be pulled out in
the sup norm from the \(L^2\) norm of \(F_{k,l,p}\) and then estimated in terms of
\(\|U\|_{\tilde{W}^{2n}}\). We
leave one factor inside the integral, having total weight at most \(2n\), which by (3.2)
can also be estimated in terms of
\(\|U\|_{\tilde{W}^{2n}}\). This yields (4.3).

Adding (4.2) with \(V = V_{k,l,p}\) for \(0 \leq k + l + 2p \leq 2n\) and using the fact that
(4.4)
\[\|A(\bar{U})\|_{C^1} \leq G_1(\|\bar{U}\|_{H^1}) \leq G_2(\|\bar{U}\|_{\tilde{W}^{2n}})\]
then yields the closed uniform energy estimate
\[\frac{d}{dt}\|\bar{U}\|^2_{\tilde{W}^{2n}} \leq G_3(\|\bar{U}\|^2_{\tilde{W}^{2n}}),\]
which implies a uniform bound for \(\|\bar{U}\|_{\tilde{W}^{2n}}\) up through some \(T > 0\). □

5 Convergence

Lemma 5.1. In addition to the assumptions of Lemma 4.1, assume that
\[\|\partial_x r_{0,\epsilon} + L_+ v_{0,\epsilon}\|_{\tilde{W}^{2n-1}} + \|\partial_x l_{0,\epsilon} + L_- v_{0,\epsilon}\|_{\tilde{W}^{2n-1}} + \|L_- r_{0,\epsilon} + L_+ l_{0,\epsilon}\|_{\tilde{W}^{2n-1}} \leq c\epsilon\]
and that
\[r_{0,\epsilon} \to r_{0,0} \quad \text{and} \quad l_{0,\epsilon} \to l_{0,0} \quad \text{in} \quad \tilde{W}_n \quad \text{as} \quad \epsilon \to 0.\]
Then as \(\epsilon \to 0,
\[r \to r^0, \quad l \to l^0, \quad v \to 0 \quad \text{in} \quad C^0([0, T]; \tilde{W}_{2(n-1)}) \cap C^0([0, T]; C^1_{loc})\]
and
\[\epsilon^{-1}v \to v^1,\]
where \(r^0, l^0, \) and \(v^1\) are independent of \(x\) and comprise the unique solution of
\[r^0_t + r^0_x + L_+ v^1 = 0, \tag{5.5}\]
\[l^0_t - l^0_x + L_- v^1 = 0, \tag{5.6}\]
\[L_- r^0 + L_+ l^0 = 0, \tag{5.7}\]

having the initial data \(r_{0,0}\) and \(l_{0,0}\), and \(T\) is the time of uniform existence obtained
in Lemma 4.1.
Proof. Estimates similar to those used to bound the terms $F_{k,l,m}$ in the proof of Lemma 4.1 show that the $W_{2(n-1)}$ norm of $\bar{U}_t(0, x, X, y)$ is bounded by some function of $\|U(0, x, X, y)\|_{W_2}$ plus $c\varepsilon^{-1}$ times the left side of (5.1). Hence the assumption that (5.1) holds implies that

$$\|\bar{U}_t(0, x, X, y)\|_{W_{2(n-1)}} < \infty.$$  

Applying $\partial^n_k \partial^j_l D^p \partial$ with $j+k+2p \leq 2(n-1)$ to (2.9) and estimating as in (4.2)–(4.4) then yields the inequality

$$\frac{d}{dt}\|\bar{U}_t\|^2_{W_{2(n-1)}} \leq G_4(\|\bar{U}\|^2_{W_2}) \|\bar{U}_t\|^2_{W_{2(n-1)}},$$

which implies a uniform estimate for $\|\bar{U}_t\|_{W_{2(n-1)}}$ as well. In view of Remark 3.2, standard compactness methods [15] then show that along some sequence $\varepsilon_j \to 0$, the solutions $\bar{U}^\varepsilon$ converge in $C^0([0, T]; W_{2(n-1)}) \cap C^0([0, T]; H^{2n-\delta})$ to a limit $\bar{U} = (r^0, \rho^0, \nu^0)$.

Multiplying the equations by $\varepsilon$ and taking the limit yields

$$r_x^0 + L_x \nu^0 = 0, \quad -l_x^0 + L_x \nu^0 = 0,$$

and (5.7). Taking the $x$-derivative of (5.7) and substituting in for $r_x^0$ and $l_x^0$ from (5.8) yields

$$0 = [L_+L_-] \nu^0 = \nu^0,$$

which shows that $\nu^0 = 0$ as claimed.

By (5.8), (5.9) implies that $r^0$ and $l^0$ are independent of $x$. For any function $\nu$, let $\nu$ denote its average with respect to $x$. Averaging (2.6)–(2.7) yields

$$\tilde{r}_t + \frac{3r - l}{2\sqrt{2}} (r_x + \varepsilon \nu_x) + \tilde{r}_x + \tilde{\nu} + \frac{r + l}{4} \nu_y + \varepsilon^{-1} L_+ \nu = 0,$$

$$\tilde{l}_t - \frac{r - 3l}{2\sqrt{2}} (l_x + \varepsilon \nu_x) - \tilde{l}_x + \tilde{\nu} + \frac{r + l}{4} \nu_y + \varepsilon^{-1} L_- \nu = 0.$$

Since $\nu$ converges to 0, and $r$ and $l$ converge to limits that are independent of $x$, we can write (5.10)–(5.11) in the shorter form

$$\bar{r}_t + \bar{r}_x + \varepsilon^{-1} L_+ \bar{\nu} = o(1),$$

$$\bar{l}_t - \bar{l}_x + \varepsilon^{-1} L_- \bar{\nu} = o(1).$$

Subtracting $L_+$ applied to (5.13) from $L_-$ applied to (5.12) and using the commutator relation

$$[L_+, L_-] = I$$

yields

$$\partial_t (L_- \tau - L_+ \bar{T}) + (L_- \tau + L_+ \bar{T})_x - \varepsilon^{-1} \nu = o(1),$$

where $\tau$ and $\bar{T}$ are defined as

$$\tau = \frac{r - l}{2\sqrt{2}} (r_x + \nu_x) + \nu + \frac{r + l}{4} \nu_y + \varepsilon^{-1} L_+ \nu,$$

$$\bar{T} = \frac{r - l}{2\sqrt{2}} (l_x + \nu_x) - \nu + \frac{r + l}{4} \nu_y + \varepsilon^{-1} L_- \nu.$$
which shows that $\varepsilon^{-1}v$ is bounded and converges weakly as $\varepsilon_j \to 0$ to some limit $v^1$. Since $r^0$ and $l^0$ are independent of $x$, so is $v^1$; in fact, by (5.7), taking the limit of (5.15) yields

\begin{equation}
(5.16) 
v^1 = 2L - r^0 = -2L + l^0.
\end{equation}

Taking the limit of (5.12)–(5.13) without using (5.16) therefore yields (5.5)–(5.6), which by (5.16), (2.5), and (5.14) can also be written as

\begin{align*}
(2 + H)r^0_t + r^0_X &= 0, \\
(2 - H)l^0_t - l^0_X &= 0.
\end{align*}

By (5.2) the initial data for the linear system (5.5)–(5.7) is $(r_0, 0, l_0, 0)$. In view of (5.7) and (2.10), an $L^2$ energy estimate for (5.5)–(5.6) shows that the solution is unique, and hence that convergence occurs without restricting $\varepsilon$ to a sequence. \qed

Upon inverting the transformation (2.1), (5.5)–(5.7) yields the long-wave equations (LW). Combining Lemmas 4.1 and 5.1 therefore yields the proof of Theorem 1.1.

Acknowledgment. Andrew Majda acknowledges generous support from the National Science Foundation under NSF Grant DMS-0456713 and the Office of Naval Research under ONR Grant N00014-05-1-0164.

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Received Month 200X.