(1) Let $u$ solve $u_t - \Delta u = f(u)$ in a bounded domain $\Omega$, with $u = 0$ at $\partial \Omega$ (and with enough smoothness to apply the maximum principle). Suppose $f(0) = 0$, and let $m \leq 0$ and $M \geq 0$ have the property that the interval $[m, M]$ is invariant for the ODE $\frac{da}{dt} = f(a)$ (in the sense that if $a(0) \in [m, M]$ then $a(t) \in [m, M]$ for all $t > 0$). Show that this interval is invariant for the PDE as well (in the sense that if $u(x, 0) \in [m, M]$ for all $x$ then $u(x, t) \in [m, M]$ for all $x$ and all $t > 0$). [Hint: to show that $u \geq m$, start by showing that $\phi = u - m$ satisfies a relation of the form $\phi_t - \Delta \phi + c(x, t) \phi \geq 0$, for a suitable function $c(x, t)$.

(2) In Lecture 10 we made repeated use of the following Lemma: Suppose a nonnegative real-valued function $a(t)$ satisfies a differential inequality $\frac{da}{dt} \leq f(a(t))$ with initial condition $a(0) = a_0$, and some $C^1$ function $f$ that’s strictly positive and increasing on $[a_0, \infty)$. Then $a(t) \leq \alpha(t)$ for all $t > 0$, where $\alpha$ solves the ODE $\frac{d\alpha}{dt} = f(\alpha(t))$ with the same initial data $\alpha(t) = a_0$. Prove it.

(3) This problem guides you through a semigroup-based proof that when $f : \mathbb{R} \to \mathbb{R}$ is $C^1$ with $f(0) = 0$, the 1D nonlinear heat equation

$$u_t - u_{xx} = f(u) \quad \text{for } t > 0, \text{ with } u = u_0(x) \text{ at } t = 0$$

has a unique local-in-time solution in $C([0, T], H^1)$ for any $u_0 \in H^1$. (Here and throughout this problem, I write $H^1$ for the space $H^1(\mathbb{R})$.)

(a) Show that when $\Delta u = u_{xx}$ is the 1D Laplacian, $e^{t\Delta}$ is a bounded linear map from $H^1$ to itself, with operator norm at most 1 (in other words, $\|e^{t\Delta} u\|_{H^1} \leq \|u\|_{H^1}$).

(b) Show that if $u \in C([0, T], H^1)$ then

$$\int_0^t e^{(t-s)\Delta} f(u(s)) \, ds \in C([0, T], H^1).$$

(c) Now consider the iteration

$$u^{n+1}(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} f(u^n(s)) \, ds,$$

with $u^0(x) = 0$. Show that if $T > 0$ is small enough the iteration converges in $C([0, T], H^1)$. 

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**PDE, Spring 2020, HW5.** Distributed Thursday 4/9/2020, due Friday 4/24/2020 (two full weeks from distribution). Upload your solution using the Assignments tool in NYU Classes; if possible, please provide a single pdf. Corrections and additions 4/19: I added a hint for problem 1; in Problem 2, I added the hypotheses that $f$ is $C^1$ and increasing; in Problem 4(b), in the sentence starting “Your task is to show . . .”, I corrected a typo in the integral form of the PDE $u_t - \Delta u = u^3$; in Problem 5, I corrected the characterization of the Galerkin approximation by inserting the variable coefficient $a(x)$ where it belongs; and in Problem 6, I changed the PDE at the beginning of the problem to $u_t - \Delta u = u^3$, since that is what I had mind when writing the rest of the problem.
(d) Conclude that our initial value problem has a unique solution in $C([0,T],H^1)$.

(Note: the strategy outlined here amounts to an application of the contraction mapping fixed point theorem. The overall outline of the argument should be familiar from your study of ODE.)

(4) I argued in Lecture 10 (using the scale-invariance of the equation) that for the initial-value problem $u_t - \Delta u = u^3$ in $\mathbb{R}^n$, a well-posedness result in $L^p(\mathbb{R}^n)$ should need $p > n$. This problem shows that there is indeed such a well-posedness result when $p > n$.

(a) Let $\Delta$ be the Laplacian in $\mathbb{R}^n$. Show that if $u \in L^p(\mathbb{R}^n)$ then $e^{t\Delta}u \in L^q(\mathbb{R}^n)$ for $t > 0$, and

$$\|e^{t\Delta}u\|_{L^q} \leq C \frac{1}{t^{\frac{n}{2}}(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p},$$

where $C$ is independent of $t$ and $u$. (Note: this amounts to an estimate of the operator norm $\|e^{t\Delta}\|_{L^p \to L^q}$.) [Hint: use the inequality from Real Variables: $\|f * g\|_{L^m} \leq \|f\|_{L^k}\|g\|_{L^l}$ when $\frac{1}{k} + \frac{1}{l} = \frac{1}{m} + 1$.]

(b) Show that the strategy of Problem 3 applies also here, for initial data $u_0 \in L^p(\mathbb{R}^n)$ with $p > n$. (Your task is to show that for sufficiently small $T > 0$, there is a unique $u \in C([0,T],L^p)$ such that $u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}u^3(s)\,ds$ for $0 \leq t \leq T$.)

(5) Let us examine the accuracy of a specific Galerkin scheme for the initial-value problem

$$u_t - \nabla \cdot (a(x)\nabla u) = 0 \text{ in } \Omega, \text{ with } u = 0 \text{ at } \partial \Omega \text{ and } u = u_0 \text{ at } t = 0.$$

We assume $\Omega$ is a bounded domain in $\mathbb{R}^n$ (with nice enough boundary), and take as the Galerkin space $V_N$ the span of the first $N$ eigenfunctions of the constant-coefficient Dirichlet Laplacian. (More carefully: let $\{\phi_j\}$ be an orthonormal basis for $L^2$ satisfying $-\Delta \phi_j = \lambda_j \phi_j$ in $\Omega$ and $\phi_j = 0$ at $\partial \Omega$, ordered so that $\lambda_j \leq \lambda_{j+1}$; then $V_N$ is the span of $\{\phi_j\}_{j=1}^N$.) As a reminder: the Galerkin approximation $u_N$ is characterized by the properties that $u_N(t) \in V_N$,

$$\int_\Omega (\partial_t u_N) v \,dx + \int_\Omega \langle a(x)\nabla u_N, \nabla v \rangle \,dx = 0 \quad \text{for all } v \in V_N,$$

and

$$u_N(0) = \pi_N(u_0) = \text{orthogonal projection of } u_0 \text{ to } V_N \text{ using the } L^2 \text{ inner product.}$$

(a) Show that $w_N = u_N - \pi_N(u)$ satisfies an estimate of the form

$$\frac{d}{dt} \int_\Omega |w_N|^2 \,dx + C_1 \int_\Omega |w_N|^2 \,dx \leq C_2 \int_\Omega |
abla u - \nabla \pi_N(u)|^2 \,dx,$$

where $C_1$ and $C_2$ are positive constants.
(b) Show that any function \( u \in H^1_0(\Omega) \cap H^2(\Omega) \) is well-approximated in \( H^1 \) by its \( L^2 \) projection to \( V_N \), in the sense that
\[
\int_\Omega |\nabla u - \nabla \pi_N(u)|^2 \, dx \leq \frac{1}{\lambda_N} \int_\Omega |\nabla \nabla u|_2^2 \, dx.
\]

(c) When \( \partial \Omega \) is nice enough, it is known that \( \lambda_N \sim C \Omega N^{2/n} \). (This is known as Weyl’s law. You can find a formula for \( C \Omega \) in Section 6.5 of Evans. A proof can be found in volume 1 of Courant & Hilbert’s Methods of Mathematical Physics, which is available online through Bobcat.) Also: for initial data \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \), the PDE solution \( u \) remains uniformly bounded for all time in this space. (This is part of the basic existence theory; see e.g. Theorem 5 of Evans’ Section 7.1.) Using these facts together with (a) and (b), prove that
\[
\|u_N(t) - u(t)\|_{L^2(\Omega)} \leq CN^{-1/n}
\]
with a constant \( C \) that’s independent of time.

(d) Now suppose \( u \) is smoother, specifically that \( \int_\Omega |\Delta^k u|^2 \, dx \) is uniformly bounded in time for some integer \( k \geq 2 \). Can you adjust the preceding arguments to get a better estimate for \( \|u_N(t) - u(t)\|_{L^2(\Omega)} \) than the one stated in part (c)?

(6) Now let’s consider the analogue of Problem 5 for the semilinear heat equation
\[
u_t - \Delta u = u^3 \text{ in } \Omega, \text{ with } u = 0 \text{ at } \partial \Omega \text{ and } u = u_0 \text{ at } t = 0,
\]
when \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) (with sufficiently nice boundary). We assume that
\[
\sup_{0 \leq t \leq T} \int_\Omega |\nabla \nabla u|^2 \, dx \leq M
\]
for some constant \( M \). (This amounts to taking \( u_0 \in H^2(\Omega) \) and assuming the solution has not blown up by time \( T \).) As in Problem 5, we denote by \( u_N \) the solution of the Galerkin approximation obtained using the first \( N \) eigenfunctions of the Dirichlet Laplacian; it is determined by the conditions that \( u_N(t) \in V_N \) for all \( t \) and
\[
\int_\Omega (\partial_t u_N) v \, dx + \int_\Omega (\nabla u_N, \nabla v) \, dx = \int_\Omega u_N^3 v \, dx \quad \text{for all } v \in V_N,
\]
together with the initial condition
\[
u_N(t) = \pi_N(u_0)
\]
(as before, \( \pi_N \) denotes orthogonal projection from \( L^2(\Omega) \) to \( V_N \) using the \( L^2 \) inner product).

(a) Let \( w_N = u_N - \pi_N(u) \). In Problem 5 we relied on an energy estimate involving \( \frac{d}{dt} \int_\Omega |w_N|^2 \, dx \), and this problem can be done that way too. However when I sketched the local-in-time existence theory in Lecture 10, I relied mainly on an
energy estimate that involves \( \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx \), so it is natural in this setting look for a related estimate involving \( \frac{d}{dt} \int_{\Omega} |\nabla w_N|^2 \, dx \). Show that in fact
\[
\frac{d}{dt} \int_{\Omega} |\nabla w_N|^2 \, dx + 2 \int_{\Omega} |\Delta w_N|^2 \, dx = -2 \int_{\Omega} (u_N^3 - u^3) \Delta w_N \, dx.
\]
It is convenient to rewrite the integral on the RHS as
\[
\int_{\Omega} (u_N^3 - [\pi_N(u)]^3) \Delta w_N \, dx + \int_{\Omega} ([\pi_N(u)]^3 - u^3) \Delta w_N \, dx = I + II.
\]
(b) Show that
\[
I \leq C \left( \int_{\Omega} |\Delta w_N|^2 \, dx \right)^{1/2} \left( \int_{\Omega} w_N^6 \, dx \right)^{1/6} \left( \int_{\Omega} |u_N|^6 + |\pi_N(u)|^6 \, dx \right)^{1/3}.
\]
As a start toward estimating the last of the three terms in this product, explain why
\[
\left( \int_{\Omega} |\pi_N(u)|^6 \, dx \right)^{1/3} \leq C \int_{\Omega} |\nabla \pi_N(u)|^2 \, dx \leq C \int_{\Omega} |\nabla u|^2 \, dx,
\]
and why in combination with (1) this gives
\[
\sup_{0 \leq t \leq T} \int_{\Omega} |\pi_N(u)|^6 \, dx \leq C_1 M.
\]
(c) Since we expect to show that \( u_N \) is close to \( \pi_N(u) \), in light of part (b) it is natural to expect that
\[
\sup_{0 \leq t \leq T} \int_{\Omega} |u_N|^6 \, dx \leq 2C_1 M. \tag{2}
\]
(Estimates proved using this assumption are valid up to the first time when (2) fails. We’ll see in part (d) that if \( N \) is large enough then it never fails for \( t \in [0, T] \).) Argue using (2) and part (b) that for any \( \varepsilon > 0 \),
\[
I \leq \varepsilon \int_{\Omega} |\Delta w_N|^2 \, dx + C_{\varepsilon, M} \int_{\Omega} |\nabla w_N|^2 \, dx
\]
and
\[
II \leq \varepsilon \int_{\Omega} |\Delta w_N|^2 \, dx + C_{\varepsilon, M} \int_{\Omega} |\nabla \pi_N(u) - \nabla u|^2 \, dx.
\]
Conclude (by arguing as in Problem 5 and using that the spatial dimension is \( n = 3 \)) that
\[
\frac{d}{dt} \int_{\Omega} |\nabla w_N|^2 \, dx \leq C_2 \int_{\Omega} |\nabla w_N|^2 \, dx + C_3 N^{-2/3},
\]
and show using this an estimate of the form
\[
\|u(t) - u_N(t)\|_{H^1_0(\Omega)}^2 \leq C N^{-2/3}
\]
up to the first time that (2) fails. (The constant \( C \) can depend on \( M \) and \( T \), but is otherwise independent of \( u \).)
(d) Show finally that (2) holds for all \( t \in [0, T] \) if \( N \) is sufficiently large, so that
\[
\sup_{0 \leq t \leq T} \|u(t) - u_N(t)\|_{H^1_0(\Omega)}^2 \leq C N^{-2/3}.
\]