

PDE for Finance Notes – Section 9

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The martingale method for dynamic portfolio optimization. Sections 5-7 were devoted to stochastic control. We discussed the *value function* and the *principle of dynamic programming*. In the discrete-time setting dynamic programming gives an iterative scheme for finding the value function; in the continuous-time setting it leads to the Hamilton-Jacobi-Bellman PDE. Stochastic control is a powerful technique for optimal decision-making in the presence of uncertainty. In particular it places few restrictions on the sources of randomness, and it does not require special hypotheses such as market completeness.

In the continuous-time setting, a key application (due to Merton, around 1970) is *dynamic portfolio optimization*. We examined two versions of this problem: one optimizing the utility of consumption (Section 5), the other optimizing the utility of final-time wealth (Homework 5).

This section introduces an alternative approach to dynamic portfolio optimization. It is much more recent – the main papers were by Cox & Huang; Karatzas, Lehoczky, & Shreve; and Pliska, all in the mid-80's. A very clear, rather elementary account is given in R. Korn and E. Korn, *Option Pricing and Portfolio Optimization: Modern Methods of Financial Mathematics* (American Mathematical Society, 2001). My discussion is a simplified (i.e. watered-down) version of the one in Korn & Korn.

This alternative approach is called the “martingale method,” for reasons that will become clear presently. It is closely linked to the modern understanding of option pricing via the discounted risk-neutral expected value, which is why we reviewed that theory in Section 8. (It will also be briefly reviewed in this section, to make our discussion more or less self-contained.) The martingale method is much less general than stochastic control; in particular, it requires that the market be complete. When it applies, however, it provides an entirely fresh viewpoint, quite different from Merton's. To capture the main idea with a minimum of complexity, I shall (as in Section 5 and HW 5) consider just the case of a single risky asset. Moreover I shall focus on the case (like Problem 1 of HW5) where there is no consumption. So the goal is this: consider an investor who starts with wealth x at time 0. He can invest in a risk-free asset (“bond”) which offers constant interest r , or a risky asset (“stock”) whose price satisfies

$$dS = \mu S dt + \sigma S dw. \quad (1)$$

He expects to mix these, putting fraction $\theta(t)$ of his wealth in stock and the rest in the bond; the resulting wealth process satisfies

$$dX = [(1 - \theta)r + \theta\mu]X dt + \theta\sigma X dw \quad (2)$$

with initial condition $X(0) = x$. His goal is to maximize his expected utility of final-time wealth:

$$\max_{\theta(t)} E [h(X(T))]$$

where h is his utility function and T is a specified time. Of course his investment decisions must be non-anticipating: $\theta(t)$ can depend only on knowledge of S up to time t .

Review of risk-neutral option pricing. Recall that the time-0 value of an option with payoff $f(S_T)$ is its discounted risk-neutral expected payoff:

$$\text{option value} = e^{-rT} E_{\text{RN}}[f(S_T)].$$

Moreover the risk-neutral process differs from (1) by having a different drift: it solves $dS = rSdt + \sigma Sdw$. By Girsanov's theorem we can write the risk-neutral expected payoff using the subjective probability as

$$E_{\text{RN}}[f(S_T)] = E[e^{-z(T)} f(S_T)] \quad (3)$$

where

$$z(t) = \int_0^t \frac{\mu - r}{\sigma} dw + \frac{1}{2} \int_0^t \left(\frac{\mu - r}{\sigma} \right)^2 ds. \quad (4)$$

Clearly (3) can be written as

$$\text{option value} = E[H(T)f(S_T)]$$

with $H(t) = e^{-rt}e^{-z(t)}$. One verifies using Ito's formula and (4) that

$$dH = -rHdt - \frac{\mu - r}{\sigma} Hdw \quad (5)$$

with initial condition $H(0) = 1$.

This option pricing formula doesn't come from thin air. It comes from the absence of arbitrage, together with the fact that the option payoff $f(S_T)$ can be replicated by a *hedge portfolio*, consisting of stock and bond in suitable weights. The option value is the value of the hedge portfolio at time 0, i.e. the initial capital needed to establish this (time-varying but self-financing) replicating portfolio.

The connection with portfolio optimization. What does this have to do with portfolio optimization? Plenty. The hedge portfolio represents a specific choice of weights $\theta(t)$. The option pricing formula tells us the condition under which a random final-time wealth of the form $X(T) = f(S(T))$ is achievable by a suitable trading strategy: the only restriction is that f satisfy $x = E[H(T)f(S_T)]$.

In portfolio optimization we are not mainly interested final-time wealths of the form $f(S_T)$. Rather, we are interested in those achievable by a suitable trading strategy (i.e. a non-anticipating choice of $\theta(t)$ for $0 \leq t \leq T$). The resulting random final-time wealth will, in general, be path-dependent; however it is certainly \mathcal{F}_T measurable, i.e. it is determined by knowledge of the entire Brownian process $\{w(t)\}_{0 \leq t \leq T}$.

Our discussion of option pricing extends, however, to more general (\mathcal{F}_T -measurable) final-time wealths. The crucial question is: which (random, \mathcal{F}_T -measurable) final-time wealths

B are associated with nonanticipating trading strategies $\theta(t)$ using initial capital x ? The answer is simple: B has this property exactly if

$$x = E[H(T)B] \quad (6)$$

and in that case the associated wealth process $X(t)$ satisfies

$$H(t)X(t) = E[H(T)B | \mathcal{F}_t]. \quad (7)$$

We'll prove just the easy direction: that if $B = X(T)$ for some trading strategy $\theta(t)$ then (6) and (7) hold. (See Korn & Korn for the converse.) Let's apply Ito's formula in the form

$$d(HX) = HdX + XdH + dHdX$$

(there is no factor of 1/2 in front of the last term, because $f(h, x) = xh$ has $\partial^2 f / \partial h \partial x = \partial^2 f / \partial x \partial h = 1$). Substituting (2) and (5) gives

$$d(HX) = -HX(rdt + \frac{\mu - r}{\sigma}dw) + HX(\{(1 - \theta)r + \theta\mu\}dt + \theta\sigma dw) - HX\frac{\mu - r}{\sigma}\theta\sigma dt.$$

The dt terms cancel, leaving only dw terms. So

$$H(T)X(T) - H(t)X(t) = \int_t^T [\text{stuff}] dw.$$

Setting $t = 0$ and taking the expectation of both sides gives

$$E[H(T)X(T)] = H(0)X(0) = x;$$

similarly, for any t we take the expectation of both sides conditioned on time- t data to get

$$E[H(T)X(T) | \mathcal{F}_t] = H(t)X(t).$$

These are the desired assertions. (See e.g. Korn & Korn for the converse.)

The martingale approach to portfolio optimization. Relation (6) changes the task of dynamic portfolio optimization to a *static* optimization problem: the optimal final-time wealth B solves

$$\max_{E[H(T)B]=x} E[h(B)].$$

The solution is easy. To avoid getting confused let's pretend the list of possible final-time states was discrete, with state α having probability p_α , $1 \leq \alpha \leq N$. Then the random variable B would be characterized by its list of values (B_1, \dots, B_N) and our task would be to solve

$$\max_{\sum H(T)_\alpha B_\alpha p_\alpha = x} \sum h(B_\alpha) p_\alpha$$

for the optimal $B = (B_1, \dots, B_N)$. This is easily achieved using the method of Lagrange multipliers. If λ is the Lagrange multiplier for the constraint then the solution satisfies $\partial L / \partial B_\alpha = 0$ and $\partial L / \partial \lambda = 0$ where

$$L(B, \lambda) = \sum h(B_\alpha) p_\alpha + \lambda \left(x - \sum H(T)_\alpha B_\alpha p_\alpha \right).$$

The derivative in λ recovers the constraint

$$\sum H(T)_\alpha B_\alpha p_\alpha = x$$

and the derivative with respect to B_α gives

$$h'(B_\alpha) = \lambda H(T)_\alpha$$

for each α . These $N + 1$ equations determine the values of the $N + 1$ unknowns B_α and λ .

The continuous case is no different: the optimal B satisfies

$$h'(B) = \lambda H(T) \tag{8}$$

as random variables. Since h is concave, h' is invertible, so we can solve (8) for B :

$$B = \{h'\}^{-1}(\lambda H(T))$$

where $\{h'\}^{-1}$ is the inverse function of h' . The value of λ is uniquely determined by the condition that this B satisfy

$$E[H(T)B] = x.$$

An example: the Merton problem with logarithmic utility. So far we have not assumed anything about the drift and volatility: they can be functions of time and stock price ($\mu = \mu(t, S)$ and $\sigma = \sigma(t, S)$). But to bring things down to earth let's consider a familiar example: the constant-drift, constant-volatility case, with utility $h(x) = \log x$. Notice that for this utility $h'(x) = 1/x$, so $\{h'\}^{-1}(y) = 1/y$. Also, since the drift and volatility are constant

$$H(t) = e^{-rt - z(t)}$$

with

$$z(t) = \frac{\mu - r}{\sigma} w(t) + \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 t$$

while

$$S(t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma w(t)}.$$

Thus both $H(t)$ and $S(t)$ are determined by knowledge of t and $w(t)$. Put differently: in this case $H(T)$ is a function of $S(T)$.

Specializing (8) to our example gives $\frac{1}{B} = \lambda H(T)$, so

$$B = \frac{1}{\lambda H(T)}.$$

The value of λ is fixed by (6), which gives $x = E[H(T)B] = \frac{1}{\lambda}$ thus

$$\lambda = \frac{1}{x}.$$

The wealth at time T is determined by (7); it gives

$$H(t)X(t) = E[H(T)B | \mathcal{F}_t] = E[\lambda^{-1} | \mathcal{F}_t] = 1/\lambda$$

whence

$$X(t) = H^{-1}(t)x.$$

To implement this solution practically, what we really need is the weight $\theta(t)$ associated with the optimal policy. To find it, observe that by Ito's formula applied to (5),

$$d[H^{-1}x] = -H^{-2}x dH + H^{-3}x dH dH = H^{-1}x \left[r dt + \left(\frac{\mu - r}{\sigma} \right)^2 dt + \frac{\mu - r}{\sigma} dw \right].$$

This can be written as a wealth-process SDE, i.e. it has the form

$$dX = [(1 - \theta)r + \theta\mu]X dt + \theta\sigma X dw$$

with $X(t) = H^{-1}x$ and

$$\theta = \frac{\mu - r}{\sigma^2}.$$

Thus the optimal policy is to keep fraction $\theta = (\mu - r)/\sigma^2$ of your wealth in the stock, and fraction $1 - \theta$ in the risk-free bond. This agrees, as it should, with the solution to Problem 1c of HW5.

The optimal portfolio is easy to describe but difficult to maintain, since it requires continuous-time trading. Indeed, each time $S(t)$ changes, some stock must be bought or sold so the value of the stock portfolio remains a constant proportion of the total wealth. This is of course impractical – one gets killed by transaction costs. In the constant-drift, constant-volatility case a convenient alternative presents itself. We saw that the optimal B is a function of $S(T)$ alone. So it can be expressed as the payoff of a suitable option, i.e. there is a function f (depending on T) such that $B = f(S(T))$. (It is not hard to find a formula for f , using the information given above.) If an option with payoff f were available in the marketplace, you could simply buy this option at time 0 and do no trading. (The optimal portfolio is identical to this option's hedge portfolio, which is in turn functionally equivalent to the option itself.) In practice the option is not available in the marketplace; even so, it can be approximated by a suitable combination of calls. This combination of options, held statically, reproduces (approximately) the optimal Merton payoff without incurring any transaction costs. For discussion of this observation and its consequences see M. Haugh and A. Lo, *Asset allocation and derivatives*, Quantitative Finance 1 (2001) 45-72.

In considering portfolio optimization problems we have mainly assumed lognormal stock dynamics. However the problem can also be considered with a different price process (for example one of the form $dS = \mu(t, S)S dt + \sigma(t, S)S dw$ where the drift $\mu(t, S)$ and the volatility $\sigma(t, S)$ are known, deterministic functions of stock price and time.) There is no difficulty formulating the HJB equation (however it must typically be solved numerically). There is also no difficulty identifying the solution using the martingale method – however the analogue of the Girsanov factor H in the discussion above is typically path dependent, i.e. $H(t)$ is usually not a function of t and $S(t)$ alone.

We observed that in the lognormal setting, the optimal final-time wealth can be achieved by buying a well-chosen European option at time 0 and not trading at all. For which market models is this true? Answer: it holds if and only if the analogue of H is not path dependent. Stock price evolution laws with this property are rare, but the lognormal case is not the only example. For an exploration of this topic, including numerous other examples of evolution laws with this property, see my paper with Oana Papazoglu-Statescu, *On the equivalence of the static and dynamic asset allocation problems*, Quantitative Finance 6(2), 2006, 173-183.