PDE for Finance, 4/27/2015 (Section 8)

These notes discuss why "risk neutral discounted expected payoff" can be used to price (even path-dependent) options.

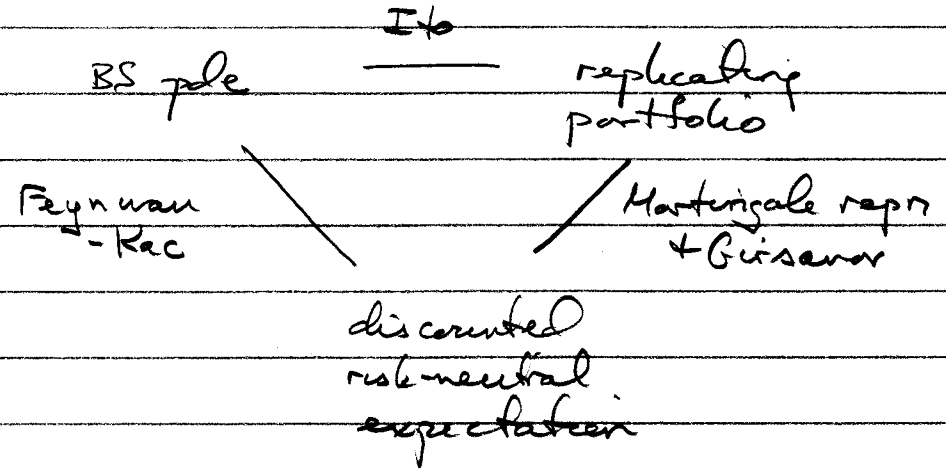
To keep things concrete I'll focus on our usual classes of (simple) asset price models: lognormal with constant interest rate r, or else $dS = \mu(S,t) S\,dt + \sigma(S,t) S\,dw$ (still Markovian!). However the discn extends far beyond that (even to $dS = \mu_t S\,dt + \sigma_t S\,dw$ where μ_t, σ_t are just $\mathcal{F}_t$ measurable, and even to stochastic interest rates).

Besides its intrinsic interest, we need this to explain the "Martingale approach" to portfolio optimization (next week's topic).

Places to read about this:

* the book by Baxter + Rennie
* the book by Karr + Karr

Big picture: we have thus far understood only the "Feynman-Kac" leg of the following triangle:



First, relatively easy goal: explain the "Ito" leg. This applies only to options whose payoff has the form $\phi(S(T))$ (no path dependence), since the Black-Scholes PDE is restr to that setting.

<u>Claim</u>: if $V(S,t)$ solves BS pde with $V=\phi$ at $t=T$, then starting with initial capital $V(S_0,0)$ at time 0, option's payoff can be <u>replicated</u> by a <u>self-financing trading strategy</u>. [So (by absence of arbitrage) option's value at time 0 must be $V(S_0,0)$.]

<u>Pf of claim</u>: consider trading strategy:

- at time t hold $\phi_t = \frac{\partial V}{\partial S}(S_t,t)$ units of stock and $\psi_t = (V(S_t,t) - \phi_t S_t)/B_t$ units of bond

where (since we take the interest rate to be constant) $B_t = e^{rt}$ is the value of a risk-free bond at time t.

Clearly its total value is $V(S_t, t)$. To show it is self-financing we must show that

$$\underbrace{dV}_{\text{change in value}} = \underbrace{\varphi_t\, dS + \psi_t\, dB}_{\text{profit or loss on stock + bond holdings}}$$

LHS is $dV = V_t\, dt + V_S\, dS + \frac{1}{2} V_{SS}\, dS\, dS$ by Ito,

RHS is $\frac{\partial V}{\partial S} dS + \left(V - \frac{\partial V}{\partial S} S\right) r$ since $dB = rB\, dt$

The fact that these are equal is precisely the BS pde.

Of course we know that BS pde is assoc to Feynman-Kac applied to $dS = rS\,dt + \sigma S\, dW$. So we recover this way that option prices are assoc to discounted expected payoffs wrto a suitable stock process (the "risk-neutral process"), different from the subjective one.

But: this applies only to path-independent options. I'd like a viewpt that applies also to path-dependent options (ie any $\mathcal{F}_T$-measurable payoff).

Key fact: (Girsanov's Thm) changing the drift of an SDE amounts to changing the measure we use on path space (and the Radon-Nikodym derivative can be made explicit). In fact, if

$$dS = \mu S\,dt + \sigma S\,dw$$ under the original measure, call it P

then there's a different measure, call it Q, st

(*) $$dS = rS\,dt + \sigma S\,d\tilde{w}$$ where $\tilde{w}$ is a Q-Brownian motion

(so: $d\tilde{w} = \frac{\mu - r}{\sigma}dt + dw$) and for every $\mathcal{F}_T$-meas random variable X we have (at time 0)

$$\mathbb{E}_Q[X] = \mathbb{E}_P[M_T X]$$

with

$$M_t = \exp\left(-\int_0^t \left(\frac{\mu-r}{\sigma}\right)dw - \frac{1}{2}\int_0^t \left(\frac{\mu-r}{\sigma}\right)^2 ds\right).$$

Essence of (*): under the measure Q, S_t/B_t is a martingale. Easy consequence of (*) via Ito, since

$$d(S/B) = d(e^{-rt}S) = (dS - rS\,dt)e^{-rt} = \sigma(S/B)\,d\tilde{w}$$

Equivalent stmt: $S_t/B_t = \mathbb{E}_Q[S_T/B_T \mid \mathcal{F}_t]$

i.e. $M_t (S_t / B_t) = \mathbb{E}_P [M_T S_T / B_T \mid \tilde{\mathcal{F}}_t]$

Easy to see also from form of M_t (indeed, this is why M_t has its form): need to show $M_t (S_t / B_t)$ is a martingale wrt P

$$M_t = e^{-Z_t} \qquad dZ = \frac{\mu - r}{\sigma} dW + \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 dt$$

$$\Rightarrow dM = -e^{-Z} dZ + \frac{1}{2} e^{-Z} dZ\, dZ$$
$$= -M \left(\frac{\mu - r}{\sigma} dW + \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2 dt \right) + \frac{1}{2} M \left(\frac{\mu - r}{\sigma} \right)^2 dt$$
$$= -M \left(\frac{\mu - r}{\sigma} \right) dW$$

$$d(S/B) = (\mu - r)(S/B)\, dt + \sigma (S/B)\, dW$$

$$d\left(M \cdot \frac{S}{B}\right) = M\, d\left(\frac{S}{B}\right) + \frac{S}{B} dM + dM\, d\left(\frac{S}{B}\right)$$
$$= M \left(\frac{S}{B}\right) \left[(\mu - r)\, dt + \sigma\, dW \right]$$
$$- M \left(\frac{S}{B}\right) \left[\frac{\mu - r}{\sigma} \right] dW$$
$$- M \left(\frac{\mu - r}{\sigma} \right) \cdot \sigma \left(\frac{S}{B}\right) dt$$
$$= \text{stuff } dW$$ since the "dt" terms cancel

Why is this useful? We can use it to show (by an arg't similar to that given above for European options) that the value V_t at time t of a path-dependent option worth X (an $\mathcal{F}_T$ measurable random variable) at time T satisfies

$$(**) \quad V_t / B_t = E_Q \left[X/B_T \mid \mathcal{F}_t \right]$$

in which RHS is conditional expectation wrt info avail at time t.

To demonstrate (**), we must show existence of a replicating portfolio that's self-financing, whose value at time t is the V_t given by (**).

Key tool: "martingale repn thm". Recall that soln of $dy = \gamma \, d\tilde{W}$ is a Q martingale. Repn thm says every Q martingale has this form. So (using $d(S/B) = \sigma (S/B) \, d\tilde{W}$)

$$(**) \Rightarrow d\left(V_t / B_t\right) = \varphi_t \, d\left(S_t / B_t\right)$$

for some ($\mathcal{F}_t$-measurable) φ_t.

Using this, we identify the trading strategy of the replicating portfolio: it holds φ_t units of stock + $(V_t - \varphi_t S_t)/B_t$ units of bond at time t.

Value of proposed portfolio is certainly V_t.
<u>Need to show it is self-financing.</u> In fact,

$$V = \frac{V}{B} B \;\Rightarrow\; dV = d\left(\frac{V}{B}\right) B + \frac{V}{B}\, dB$$

(I'm using Ito in form $d(XY) = X\,dY + Y\,dX + dX\,dY$, and fact that $dX\,dB = 0$ since $dB = rB\,dt$ has no "dW" term.)

Also

$$S = \frac{S}{B} B \;\Rightarrow\; dS = d(S/B)\, B + \frac{S}{B}\, dB$$

So, writing $\psi_t = \dfrac{V_t - \phi_t S_t}{B_t}$, we have

$$\begin{aligned} \phi\, dS + \psi\, dB &= \phi B\, d(S/B) + \cancel{\phi \frac{S}{B}\, dB} \\ &\qquad \left(\frac{V}{B} - \cancel{\phi \frac{S}{B}}\right) dB . \end{aligned}$$

So the choice of ϕ st $\phi\, d(S/B) = d\left(\frac{V}{B}\right)$ gives

$$\phi\, dS + \psi\, dB = dV$$

Thus: changes in portfolio's value are entirely attributable to market gains/losses + bond interest, as desired.

Rmk: since we know how to turn Q-expectations into P-expectations (using Girsanov) we can also write down the value of an option using the P-expectation.

Some examples, to bring this down to earth

① stock with const dividend yield at rate q. The tradeable in this case is the stock with dividends reinvested.

Claim: if stock price process (under P) is

$$dS = \mu S\,dt + \sigma S\,dw$$

then RN process (assoc Q) is

$$(**) \qquad dS = (r-q)S\,dt + \sigma S\,d\tilde{w}.$$

The reason is that if you start at $t=0$ with 1 share then your holding at time t is e^{qt} shares, and its value is $S_t e^{qt}$. Eqn $(**)$ is the condn that

$$S_t e^{qt}/B_t = S_t e^{(q-r)t} \text{ is a Q-martingale.}$$

② Option on a foreign exchange rate

Suppose US dollar risk-free rate $= r$
British pound risk-free rate $= q$
exchange rate (dollars/pound) is lognormal,
$dC = \mu C\,dt + \sigma C\,dW$.

To dollar investor, a pound looks like "stock with cont's div yield q", so from ex ① the dollar-investor's risk-neutral process is Q, where

$$dC = (r-q)\,C\,dt + \sigma C\,d\tilde{W}$$ & $d\tilde{W}$ is a Q-Brownian motion.

What about the pound investor. His exchange rate is $1/C$. By Ito, under the P measure,

$$d(1/C) = -C^{-2}dC + C^{-3}\,dC\,dC$$
$$= (-\mu+\sigma^2)\frac{1}{C}\,dt - \sigma\cdot\frac{1}{C}\,dW$$

What is the Pound investor's risk-neutral measure? Certainly not Q! By analogy to what we did for the dollar investor, pound investor's RN meas $\bar{Q}$ is st

$$d(1/C) = (q-r)(1/C)\,dt - \sigma(1/C)\,d\bar{W}$$

where $\bar{W}$ is a $\bar{Q}$ Brownian motion. Evidently

$$(-\mu+\sigma^2)\,dt - \sigma\,dW = (q-r)\,dt - \sigma\,d\bar{W}$$

$$\text{whence} \quad d\bar{W} = \left(\frac{\mu+q-r}{\sigma} - \sigma\right)dt + dW$$

whereas for the dollar investor

$$\mu ¢\,dt + \sigma ¢\,dW = (r-q)\,¢\,dt + \sigma ¢\,d\tilde{W}$$

$$\Rightarrow d\tilde{W} = \left(\frac{\mu+q-r}{\sigma}\right)dt + dW$$

Is this strange? Well, no. The "RN measure" is simply the one asserting that (value of tradable $/B_t$) is a martingale. Different investors in this case see different tradables, so they have different RN measures.