Intro to the paper "Dynamic trading with predictable returns + transaction costs," N. Garleanu + L H Pedersen, J Finance 68 (2013) pp 2309-2340

Goal: use discrete-time dynamic programming to consider how transaction costs should influence investment decisions.

The paper discusses a model with many stocks, examining how portfolio should be changed as time evolves (a time-dependent, transaction-cost-sensitive version of the Markowitz portfolio optimization problem).

I'll discuss just the simplest special case (Example 1 in the paper), when there is just a single stock

x_t = investor's position in this stock at time t (known for $t=-1$, to be decided today [$t=0$], and eventually at future times 1, 2, 3, etc)

and, at each time, investor has an estimate of the stock's anticipated return. Actually, we will focus only on excess return $r_{t+1} = p_{t+1} - (1+r^f)p_t$ where p_t is stock price at time t; we suppose investor knows f_t at time t, where

$$r_{t+1} = f_t + u_{t+1}$$

u_{t+1} mean zero
independent at each t
variance σ^2

We assume mean reverting dynamics for f_t

$$f_{t+1} - f_t = -\varphi f_t + \varepsilon_{t+1}$$

ε_{t+1} mean zero
independent at each t
variance Ω

We assume quadratic transaction costs: a trade of size Δx incurs trans costs $\frac{1}{2}\Lambda(\Delta x)^2$ for some const $\Lambda > 0$. (Logic of this: it assumes the trade moves the market transiently by an amount that's linear in Δx).

Use discount factor $1-\rho$ in discounting future income.

Investor's problem: at time 0, starting with position x_{-1} of stock + knowing est f_0 for coming period's returns, choose x_0 to maximize

risk-adjusted rtn in 1st period less trans. cost plus expected result of future periods.

ie

$$\max_{x_0, x_1, \ldots} \mathbb{E}\left[\sum_{t=0}^{\infty} (1-\rho)^{t+1} \left(r_{t+1} x_t - \frac{\gamma}{2}\sigma^2 x_t^2 \right) - \frac{(1-\rho)^t}{2} (\Delta x_t)^2 \Lambda \right]$$

risk adjusted return
($\gamma > 0$ measures sensitivity to risk).

Call this $V(x_{-1}, f_0)$. Principle of dyn progr tells us (with $\Delta x_t = x_{t+1} - x_t$)

$$V(x_{-1}, f_0) = \max_{x_0} \left\{ -\tfrac{1}{2} (\Delta x_0)^2 \Lambda + (1-\rho)(x_0 f_0 - \tfrac{\gamma}{2}\sigma^2 x_0^2) \right.$$
$$\left. (1-\rho) E\left[V(x_0, f_1) \right] \right\}$$

This is a stochastic "LQR-type" problem, so it's natural to guess that V has the form

$$V(x_t, f_{t+1}) = -\tfrac{1}{2} A_{xx} x_t^2 + A_{xf} x_t f_{t+1} + \tfrac{1}{2} A_{ff} f_{t+1}^2 + A_0$$

for some constants $A_{xx}, A_{xf}, A_{ff}, A_0$.

To find these constants + the optimal investment policy, we substitute this hypothesis into the Prin of dyn programming:

$$\text{LHS} = -\tfrac{1}{2} A_{xx} x_{-1}^2 + A_{xf} x_{-1} f_0 + \tfrac{1}{2} A_{ff} f_0^2 + A_0$$

RHS = max wrt x_0 of

$$-\tfrac{1}{2}(x_0 - x_{-1})^2\Lambda + (1-\rho)\left(x_0 f_0 - \tfrac{\gamma}{2}\sigma^2 x_0^2\right)$$

$$+ (1-\rho)\left[-\tfrac{1}{2}A_{xx}x_0^2 + A_{xf}x_0 f_0(1-\varphi) + A_{ff}f_0^2(1-\varphi)^2 + A_{ff}\Omega + A_0\right]$$

We observe that RHS is quadratic poly in x_0

$$-\tfrac{1}{2}x_0^2\left(\Lambda + (1-\rho)\gamma\sigma^2 + (1-\rho)A_{xx}\right)$$

$$+ x_0\left(\Lambda x_{-1} + (1-\rho)f_0 + (1-\rho)(1-\varphi)A_{xf}f_0\right)$$

$$+ \left(-\tfrac{1}{2}x_{-1}^2\Lambda + (1-\rho)A_{ff}\Omega + (1-\rho)A_{ff}f_0^2(1-\rho)^2 + (1-\rho)A_0\right)$$

Writing this as $-\tfrac{1}{2}x_0^2 P + x_0 Q + R$ we see that $x_0 = P/Q$ and

$$\text{RHS} = \tfrac{1}{2}\frac{Q^2}{P} + R \quad = \text{quadratic form in } x_{-1} + f_0 \text{ plus constant}$$

and matching LHS to RHS determines the values of A_{xx}, A_{xf}, + A_{ff}. (Explicit formulas exist - see the article - but I won't try to give them here).

How to understand optimal position x_0?
Return to principle of dyn programming:

$$-\tfrac{1}{2}A_{xx}x_{-1}^2 + A_{xf}x_{-1}f_0 + \tfrac{1}{2}A_{ff}f_0^2 + A_0 = \max_{x_0}\left\{-\tfrac{1}{2}(\Delta x_0)^2\Lambda + \text{stuff indep of } x_{-1}\right\}$$

This holds for all x_{-1} + f_0, so we can differentiate wrt to x_{-1}. Best x_0 depends on x_1, but we can use chain rule on RHS $\frac{\partial}{\partial x_{-1}}\text{RHS}(x_{-1}, x_0(x_{-1})) = \frac{\partial \text{RHS}}{\partial x_{-1}}(x_{-1}, x_0(x_{-1}))$

since $x_0(x_{-1})$ optimizes RHS wrt x_0. Thus

$$(*) \qquad -A_{xx}x_{-1} + A_{xf}f_0 = -(x_{-1} - x_0)\Lambda$$

How to interpret this? Let

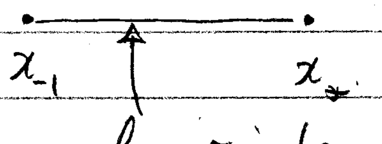
$$x_* \text{ minimize } -\tfrac{1}{2}A_{xx}x^2 + A_{xf}xf_0 + \tfrac{1}{2}A_{ff}f_0^2 + A_0$$

$$\text{ie} \qquad x_* = \frac{A_{xf}f_0}{A_{xx}}$$

At first you might expect $x_0 = x_*$; but that can't be right, since if x_{-1} is far from x_* you'd incur large trans costs to do that trade. Instead, eqn (*) says (after algebraic rearrangement)

$$x_0 = x_{-1}\left(1 - \frac{A_{xx}}{\Lambda}\right) + \frac{A_{xx}}{\Lambda}x_*$$

Thus: though the "target amount" is x_*, due to transaction costs you don't go all the way there — instead you go to a choice just *part* way between x_{-1} and x_*



linear interpolant, with weights $1 - \frac{A_{xx}}{\Lambda}$ and A_{xx}/Λ.

(Note: one can see, using the explicit formula for A_{xx}, that $0 < A_{xx}/\Lambda < 1$.)

Rmk: In our HW pbm on LQR, we had to consider an ODE in time. There is no such ODE here because we took the time horizon to be $+\infty$, so the value fn depends *only* on x_{-1} + f_0 (two real variables). This is, of course, what makes the problem amenable to by-hand analysis.