PDE for Finance, Spring 2015 – Homework 3. Distributed 3/12/15, due 3/30/15.

1. We showed, in the Section 2 notes, that the solution of

$$w_t = w_{xx} \quad \text{for } t > 0 \text{ and } x > 0, \text{ with } w = 0 \text{ at } t = 0 \text{ and } w = \phi \text{ at } x = 0$$

   is

$$w(x,t) = \int_0^t \frac{\partial G}{\partial y}(x, 0, t-s)\phi(s)\, ds \tag{1}$$

   where $G(x, y, s)$ is the probability that a random walker, starting at x at time 0, reaches y at time s without first hitting the barrier at 0. (Here the random walker solves $dy = \sqrt{2}dw$, i.e. it executes the scaled Brownian whose backward Kolmogorov equation is $u_t + u_{xx} = 0$.) Let's give an alternative demonstration of this fact, following the line of reasoning at the end of the Section 1 notes.

   (a) Express, in terms of G, the probability that the random walker (starting at x at time 0) hits the barrier before time t. Differentiate in t to get the probability that it hits the barrier at time t. (This is known as the *first passage time* density).

   (b) Use the forward Kolmogorov equation and integration by parts to show that the first passage time density is $\frac{\partial G}{\partial y}(x, 0, t)$.

   (c) Deduce the formula (1).

2. As noted in HW2 problem 5, questions about Brownian motion with drift can often be answered using the Cameron-Martin-Girsanov theorem. But we can also study this process directly. Let's do so now, for the process $dz = \mu dt + dw$ with an absorbing barrier at $z = 0$.

   (a) Suppose the process starts at $z_0 > 0$ at time 0. Let $G(z_0, z, t)$ be the probability that the random walker is at position z at time t (and has not yet hit the barrier). Show that

$$G(z_0, z, t) = \frac{1}{\sqrt{2\pi t}} e^{-|z - z_0 - \mu t|^2/2t} - \frac{1}{\sqrt{2\pi t}} e^{-2\mu z_0} e^{-|z + z_0 - \mu t|^2/2t}.$$

   (Hint: just check that this G solves the relevant forward Kolmogorov equation, with the appropriate boundary and initial conditions.)

   (b) Show that the first passage time density is

$$\frac{1}{2}\frac{\partial G}{\partial z}(z_0, 0, t) = \frac{z_0}{t\sqrt{2\pi t}} e^{-|z_0 + \mu t|^2/2t}.$$

3. Consider the linear heat equation $u_t - u_{xx} = 0$ on the interval $0 < x < 1$, with boundary condition $u = 0$ at $x = 0, 1$ and initial condition $u = 1$.

   (a) Interpret u as the value of a suitable double-barrier option.

(b) Express $u(t,x)$ as a Fourier sine series, as explained in Section 3.

(c) At time $t = 1/100$, how many terms of the series are required to give $u(t,x)$ within one percent accuracy?

4. Consider the SDE $dy = f(y)dt + g(y)dw$. Let $G(x,y,t)$ be the fundamental solution of the forward Kolmogorov PDE, i.e. the probability that a walker starting at x at time 0 is at y at time t. Show that if the infinitesimal generator is self-adjoint, i.e.

$$-(fu)_x + \tfrac{1}{2}(g^2u)_{xx} = fu_x + \tfrac{1}{2}g^2u_{xx},$$

then the fundamental solution is symmetric, i.e. $G(x,y,t) = G(y,x,t)$.

5. Consider the stochastic differential equation $dy = f(y,s)ds + g(y,s)dw$, and the associated backward and forward Kolmogorov equations

$$u_t + f(x,t)u_x + \tfrac{1}{2}g^2(x,t)u_{xx} = 0 \quad \text{for } t < T\text{, with } u = \Phi \text{ at } t = T$$

and

$$\rho_s + (f(z,s)\rho)_z - \tfrac{1}{2}(g^2(z,s)\rho)_{zz} = 0 \quad \text{for } s > 0\text{, with } \rho(z) = \rho_0(z) \text{ at } s = 0.$$

Recall that $u(x,t)$ is the expected value (starting from x at time t) of payoff $\Phi(y(T))$, whereas $\rho(z,s)$ is the probability distribution of the diffusing state $y(s)$ (if the initial distribution is ρ_0).

(a) The solution of the backward equation has the following property: if $m = \min_z \Phi(z)$ and $M = \max_z \Phi(z)$ then $m \le u(x,t) \le M$ for all $t < T$. Give two distinct justifications:

(a1) Explain why this is an easy consequence of the probabilistic interpretation of u.

(a2) Explain why this amounts to a "maximum principle" for solutions of $u_t + f(x,t)u_x + \frac{1}{2}g^2(x,t)u_{xx} = 0$. Then show, by a PDE argument (similar to what we did for $u_t - u_{xx} = 0$ in all space, but easier), that such a max principle is valid provided $|f|$ is uniformly bounded and we know in advance that u is uniformly bounded. (Hint: let ψ be a smooth function such that $\psi(x) = |x|$ for $|x| \ge 1$. Consider $u_{\epsilon,\delta} = u(x,t) \pm \epsilon t \pm \delta\psi$. Apply the maximum principle to $u_{\epsilon,\delta}$ then consider a suitable limit in which ϵ and δ tend to 0.)

(b) The solution of the forward equation does *not* in general have the same property; in particular, $\max_z \rho(z,s)$ can be larger than the maximum of ρ_0. Explain why not, by considering the example $dy = -yds$. (Intuition: $y(s)$ moves toward the origin; in fact, $y(s) = e^{-s}y_0$. Viewing $y(s)$ as the position of a moving particle, we see that particles tend to collect at the origin no matter where they start. So $\rho(z,s)$ should be increasingly concentrated at $z = 0$.) Show that the solution in this case is $\rho(z,s) = e^s\rho_0(e^s z)$. This counterexample has $g = 0$; can you also give a counterexample using $dy = -yds + \epsilon dw$?

6. The solution of the forward Kolmogorov equation is a probability density, so we expect it to be nonnegative (assuming the initial condition $\rho_0(z)$ is everywhere nonnegative). In light of Problem 2b it's natural to worry whether the PDE has this property. Let's show that it does.

   (a) Consider the initial-boundary-value problem

   $$w_t = a(x,t)w_{xx} + b(x,t)w_x + c(x,t)w$$

   with x in the interval $(0,1)$ and $0 < t < T$. We assume as usual that $a(x,t) > 0$. Suppose furthermore that $c < 0$ for all x and t. Show that if $0 \le w \le M$ at the initial time and the spatial boundary then $0 \le w \le M$ for all x and t. (Hint: a positive maximum cannot be achieved in the interior or at the final boundary. Neither can a negative minimum.)

   (b) Now consider the same PDE but with $\max_{x,t} c(x,t)$ positive. Suppose the initial and boundary data are nonnegative. Show that the solution w is nonnegative for all x and t. (Hint: apply part (a) not to w but rather to $\bar{w} = e^{-Ct}w$ with a suitable choice of C.)

   (c) Consider the solution of the forward Kolmogorov equation in the interval, with $\rho = 0$ at the boundary. (It represents the probability of arriving at z at time s without hitting the boundary first.) Show using part (b) that $\rho(z,s) \ge 0$ for all s and z.

   [Comment: statements analogous to (a)-(c) are valid for the initial-value problem as well, when we solve for all $x \in R$ rather than for x in a bounded domain. The justification takes a little extra work however, and it requires some hypothesis on the growth of the solution at ∞.]

7. Consider the solution of

   $$u_t + au_{xx} = 0 \quad \text{for } t < T, \text{ with } u = \Phi \text{ at } t = T$$

   where a is a positive constant. Recall that in the stochastic interpretation, a is $\frac{1}{2}g^2$ where g represents volatility. Let's use the maximum principle to understand qualitatively how the solution depends on volatility.

   (a) Show that if $\Phi_{xx} \ge 0$ for all x then $u_{xx} \ge 0$ for all x and t. (Hint: differentiate the PDE.)

   (b) Suppose $\bar{u}$ solves the analogous equation with a replaced by $\bar{a} > a$, using the same final-time data Φ. We continue to assume that $\Phi_{xx} \ge 0$. Show that $\bar{u} \ge u$ for all x and t. (Hint: $w = \bar{u} - u$ solves $w_t + \bar{a}w_{xx} = f$ with $f = (a - \bar{a})u_{xx} \le 0$.)