PDE for Finance, Spring 2015 – Homework 2
Distributed 2/23/15, due 3/09/15.

(1) This problem uses a PDE to value a zero-coupon bond when the short-term interest rate is described by the Vasicek model. Suppose $r(t)$ solves $dr = (\theta - ar)\,dt + \sigma\,dw$, where θ, a, and σ are positive constants. If today is time t_0 and the short-term rate today is $r(t_0) = r_0$, the value of a zero-coupon bond with maturity T and face value of one dollar is

$$E_{r(t_0)=r_0}\left[e^{-\int_{t_0}^{T} r(s)\,ds}\right].$$

(a) Explain why this is equal to $V(t_0, r(t_0))$, where $V(t,r)$ solves the PDE

$$V_t + (\theta - ar)V_r + \tfrac{1}{2}\sigma^2 V_{rr} - rV = 0$$

for $t < T$, with the final-time condition $V(T,r) = 1$ for all r.

(b) Look for a solution of the form $V(t,r) = A(t,T)e^{-B(t,T)r}$. Show that A and B should satisfy

$$A_t - \theta AB + \tfrac{1}{2}\sigma^2 AB^2 = 0 \quad \text{and} \quad B_t - aB + 1 = 0$$

with final-time conditions

$$A(T,T) = 1 \quad \text{and} \quad B(T,T) = 0.$$

(c) Solving for B first, then A, show that the solution is

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)})$$

and

$$A(t,T) = \exp\left[\left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right)(B(t,T) - T + t) - \frac{\sigma^2}{4a}B^2(t,T)\right].$$

(2) Consider the linear heat equation $u_t - u_{xx} = 0$ in one space dimension, with discontinuous initial data

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

(a) Show by evaluating the solution formula that

$$u(x,t) = N\left(\frac{x}{\sqrt{2t}}\right) \tag{1}$$

where N is the cumulative normal distribution

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-s^2/2}\,ds.$$

(b) Explore the solution by answering the following: what is $\max_x u_x(x,t)$ as a function of time? Where is it achieved? What is $\min_x u_x(x,t)$? For which x is $u_x > (1/10)\max_x u_x$? Sketch the graph of u_x as a function of x at a given time $t > 0$.

(c) Show that $v(x,t) = \int_{-\infty}^{x} u(z,t)\,dz$ solves $v_t - v_{xx} = 0$ with $v(x,0) = \max\{x, 0\}$. Deduce the qualitative behavior of $v(x,t)$ as a function of x for given t: how rapidly does v tend to 0 as $x \to -\infty$? What is the behavior of v as $x \to \infty$? What is the value of $v(0,t)$? Sketch the graph of $v(x,t)$ as a function of x for given $t > 0$.

(3) This problem obtains convenient representations for the solutions of some particular initial-boundary-value problems for the linear heat equation on the half-line:

$$w_t - w_{xx} = 0 \quad \text{for } t > 0 \text{ and } x > 0.$$

(a) Let w_1 be the solution with $w_1 = 0$ at $x = 0$ and $w_1 = 1$ at $t = 0$. Express it in terms of the function $u(x,t)$ defined in Problem 2.

(b) Let w_2 be the solution with $w_2 = 0$ at $x = 0$ and $w_2 = (x-K)_+$ at $t = 0$. Assume that $K > 0$. Express w_2 in terms of the function $v(x,t)$ defined in Problem 2(c).

(c) Let w_3 be the solution with $w_3 = 0$ at $x = 0$ and $w_3 = (x-K)_+$ at $t = 0$, when $K < 0$. Find a convenient representation of w_3 analogous to those you gave for w_1 and w_2.

(d) Let w_4 be the solution with $w_4 = 1$ at $x = 0$ and $w_4 = 0$ at $t = 0$. Find a convenient representation, analogous to those you gave for the other w_i. (Hint: what boundary value problem does $w_4 - 1$ solve?)

(e) Interpret each w_i as the expected payoff of a suitable barrier-type instrument, whose underlying executes the scaled Brownian motion $dy = \sqrt{2}dw$ with initial condition $y(0) = x$ and an absorbing barrier at 0. (Example: $w_1(x,T)$ is the expected payoff of an instrument which pays 1 at time T if the underlying has not yet hit the barrier and 0 otherwise.)

NOTE: One can, of course, use the general representation formula for solutions of the half-space problem to get a "formula" for each w_i. But I'm not asking you to do this. Rather, I'm asking you to find (using the functions introduced in Problem 2) a solution of the PDE with the correct initial and boundary conditions. This is *much* easier.

(4) Let's look more closely at the function w_1 introduced in Problem 3(a).

(a) Show that for fixed $x > 0$, $w_1(x,t) \to 0$ as $t \to \infty$.

(b) How fast does it decay? (Suggestion: show that as $t \to \infty$, $w_1(x,t) \sim Ct^{-\alpha}$. What is the best possible value of α?)

(5) The Section 2 notes reduce the Black-Scholes PDE to the heat equation by brute-force algebraic substitution. This problem achieves the same reduction by a probabilistic route. Our starting point is the fact that

$$V(s,t) = e^{-r(T-t)} E_{y(t)=s}\left[\Phi(y(T)\right] \tag{2}$$

where $dy = rydt + \sigma ydw$.

(a) Consider $z = \frac{1}{\sigma}\log y$. By Ito's formula it satisfies $dz = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)dt + dw$. Express the right hand side of (2) as a discounted expected value with respect to z process.

(b) The z process is Brownian motion with drift $\mu = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$. The Cameron-Martin-Girsanov theorem tells how to write an expected value relative to z as a weighted expected value relative to the standard Brownian motion w. Specifically:

$$E_{z(t)=\frac{1}{\sigma}\log s}\left[\Phi(e^{\sigma z(T)})\right] = E_{w(t)=\frac{1}{\sigma}\log s}\left[e^{\mu(w(T)-w(t))-\frac{1}{2}\mu^2(T-t)}\Phi(e^{\sigma w(T)})\right] \quad (3)$$

where the left side is an expectation using the path-space measure associated with z, and the right hand side is an expectation using the path-space measure associated with Brownian motion. Apply this to get an expression for $V(s,t)$ whose right hand side involves an expected value relative to Brownian motion.

(c) An expected payoff relative to Brownian motion is described by the heat equation (more precisely by an equation of the form $u_t + \frac{1}{2}u_{xx} = 0$). Thus (b) expresses the solution of the Black-Scholes PDE in terms of a solution of the heat equation. Verify that this representation is the same as the one given in the Section 2 notes.