**PDE for Finance, Spring 2015 – Homework 1**
*Distributed 2/2/2015, due 2/23/2015.* Typo corrected in 4($a$) on 2/9.

1) Consider the lognormal random walk

$$dy = \mu y dt + \sigma y dw$$

starting at $y(0) = x$. Assume $\mu \neq \frac{1}{2}\sigma^2$. The Section 1 notes examine the mean exit time from an interval $[a, b]$ where $0 < a < x < b$. There we used the PDE for the mean exit time

$$\mu x u_x + \tfrac{1}{2}\sigma^2 x^2 u_{xx} = -1 \quad \text{for } a < x < b \tag{1}$$

with boundary conditions $u(a) = u(b) = 0$ to derive an explicit formula for $u$.

(a) Show that the general solution of (1), without taking any boundary conditions into account, is

$$u = \frac{1}{\frac{1}{2}\sigma^2 - \mu} \log x + c_1 + c_2 x^\gamma$$

with $\gamma = 1 - 2\mu/\sigma^2$. Here $c_1$ and $c_2$ are arbitrary constants. [The formula given in the notes for the mean exit time is easy to deduce from this fact, by using the boundary conditions to solve for $c_1$ and $c_2$; however you need not do this calculation as part of your homework.]

(b) Argue as in the notes to show that the mean exit time from the interval $(a, b)$ is finite. (Hint: mimic the argument used to answer Question 3, using $\phi(y) = \log y$.)

(c) Let $p_a$ be the probability that the process exits at $a$, and $p_b = 1 - p_a$ the probability that it exits at $b$. Give an equation for $p_a$ in terms of the barriers $a, b$ and the initial value $x$. (Hint: mimic the argument used in the answer to Question 4, using $\phi(y) = y^\gamma$.) How does $p_a$ behave in the limit $a \to 0$?

2) Examine the analogues of Problem 1(a)–(c) when $\mu = \frac{1}{2}\sigma^2$. (Hint: notice that $x u_x + x^2 u_{xx} = u_{zz}$ with $z = \log x$.)

3) Consider a diffusion $dy = f(y)ds + g(y)dw$ starting at $x$ at time 0, with $a < x < b$. Let $\tau$ be its exit time from the interval $[a, b]$, and assume $E[\tau] < \infty$.

(a) Let $u_a(x)$ be the probability it exits at $a$. Show that $u_a$ solves $f u_x + \frac{1}{2}g^2 u_{xx} = 0$ with boundary conditions $u_a(a) = 1, u_a(b) = 0$.

(b) Apply this method to Problem 1(c). Is this approach fundamentally different from the one indicated by the hint above?

4) Consider once again a diffusion $dy = f(y)ds + g(y)dw$ starting at $x \in (a, b)$ at time 0, and assume the mean arrival time to the boundary is finite. We know from the Section 1 notes that the mean arrival time $v(x) = E_{y(0)=x}[\tau]$ satisfies $f v_x + \frac{1}{2}g^2 v_{xx} = -1$ on $(a, b)$,

with $v = 0$ at $x = a, b$. In this problem we consider the *second moment* of the arrival time, $h(x) = E_{y(0)=x}[\tau^2]$. It's clear that $h = 0$ at $x = a, b$. I'll sketch (and you'll complete) two different proofs that $h$ solves the differential equation

$$fh_x + \tfrac{1}{2}g^2 h_{xx} = -2v(x). \tag{2}$$

(This equation, combined with the boundary condition, completely determines the function $h$). For efficiency of notation, we let $\mathcal{L}$ be the infinitesimal generator of the diffusion, i.e. we define $\mathcal{L}u = fu_x + \frac{1}{2}g^2 u_{xx}$ for any function $u$.

(a) Consider the solution of

$$\mathcal{L}u_\varepsilon - \varepsilon u_\varepsilon = 0 \quad \text{for } a < x < b \tag{3}$$

with boundary conditions $u_\varepsilon(a) = u_\varepsilon(b) = 1$. (Assume that $u_\varepsilon(x)$ exists and depends smoothly on both $\varepsilon$ and $x$ – these assertions are true, though they lie beyond the scope of this class.) Show that

$$u_\varepsilon(x) = E_{y(0)=x}\left[e^{-\varepsilon\tau}\right].$$

Differentiate this formula to identify $\frac{\partial u_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}$ and $\frac{\partial^2 u_\varepsilon}{\partial \varepsilon^2}|_{\varepsilon=0}$. Then differentiate (3) with respect to $\varepsilon$ and deduce the validity of (2).

(b) We know (from the Section 1 notes) a characterization of the solution to (2):

$$h(x) = E_{y(0)=x}\left[\int_0^\tau 2v(y(s))\, ds\right].$$

Our task is thus to show that the function $h$ defined this way also satisfies $h(x) = E_{y(0)=x}[\tau^2]$. Do this, using Ito's formula in the form $d[sv(y(s))] = s\, dv(y(s)) + v(y(s))\, ds$ and $dv(y(s)) = \mathcal{L}v(y(s))\, ds + \text{stuff}\, dw$ (and, of course, making appropriate use of Dynkin's formula).

5) Let $w(t)$ be standard Brownian motion, starting at 0. Let $\tau_n$ be the first time $w$ exits from the interval $[-n, 1]$, and let $\tau_\infty$ the the first time it reaches $w = 1$.

(a) Find the expected value of $\tau_n$, and the probability that the path exits $[-n, 1]$ at $-n$.

(b) Verify by direct evaluation that $w(\tau_n)$ has mean value 0. (This must of course be true, since $E\left[\int_0^{\tau_n} dw = 0\right]$ by Dynkin's theorem.)

(c) Conclude from (a) that $E[\tau_\infty] = \infty$.

(d) Show that $\tau_\infty$ is almost-surely finite.