These problems are concerned with linear elasticity. Problem 1 asks you to explain why an isotropic Hooke’s law is described by just two constants. Problems 2 and 3 explore Korn’s inequality. Problem 4 gives an example of an elasticity problem with an explicit separation-of-variables solution. Problems 5-8 examine some important special solutions and reductions of linear elasticity.

1. Elastic symmetries. A linearly elastic material is symmetric under a rotation $R$ if its Hookes’ law satisfies $\alpha(R^T e R) = R^T \alpha(e) R$. Show, by a direct argument, that if this holds for any $R \in SO(3)$ then $\alpha e = 2\mu e + \lambda (\text{tr } e) I$ for some constants $\lambda, \mu$. (Hint: start by showing that $\sigma = \alpha e$ must be simultaneously diagonal with $e$.) What about the case of “cubic symmetry”, when $\alpha$ is only symmetric under 90 deg rotations (i.e. under any $R$ which permutes the coordinate axes)?

2. Korn’s inequality for periodic deformations. Korn’s inequality for periodic deformations says
\[ \int_Q |\nabla u|^2 \, dx \leq C \int_Q |e(u)|^2 \, dx \]
when $u : \mathbb{R}^n \to \mathbb{R}^n$ is periodic in each variable with period 1 and $Q = [0,1]^n$ is the unit cell. Give a proof using the Fourier representation of $u$. What is the best possible value of the constant $C$? Why is there no condition about $\int \nabla u$ being symmetric?

3. Korn’s inequality for beams. Let $\Omega_h \subset \mathbb{R}^2$ be the long, thin domain $\{0 < x < 1, -h/2 < y < h/2\}$ where $h \ll 1$. Korn’s second inequality for this domain says
\[ \int_{\Omega_h} |\nabla u|^2 \, dx \leq C(h) \int_{\Omega_h} |e(u)|^2 \, dx \quad \text{provided } \int_{\Omega_h} \nabla u \text{ is symmetric.} \]
(a) Show that $C(h)$ must be at least of order $h^{-2}$, by considering deformations of the form $u(x,y) = (-y \phi_x, \phi)$ where $\phi = \phi(x)$.
(b) Show that the inequality is true with $C_h \sim h^{-2}$. You may assume (for simplicity, this is not really necessary) that $1/h$ is an integer. Hint: divide $\Omega_h$ into $1/h$ squares of side $h$. Korn’s inequality (for squares) controls $\nabla u - \begin{pmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{pmatrix}$ on the jth square in terms of the strain on that square, for some $\omega_j \in R$. Use Korn’s inequality again (this time for rectangles of eccentricity 2) to control $\omega_j - \omega_{j-1}$ in terms of the strain on the (j-1)st and jth squares. Then apply a discrete version of Poincare’s inequality in one space dimension to control the variation of $\omega_j$ with $j$.
(c) How do you think these results would extend to a thin plate-like domain $\{0 < x < 1, 0 < y < 1, -h/2 < z < h/2\}$ in $\mathbb{R}^3$? (Just discuss how the 3D problem is similar or different; I’m not asking for a complete solution.)

4. Separation of variables. Let $\Omega$ be a “ball with a hole removed”:
\[ \Omega = \{ x : \rho^2 < |x|^2 < 1 \} . \]
Suppose it is filled with an isotropic, homogeneous, linearly elastic material with Lamé moduli \( \lambda \) and \( \mu \), and constant pressure \( P \) is applied at the outer boundary \( |x| = 1 \). The inner boundary \( |x| = \rho \) is traction-free. Find the displacement \( u(x) \) and the associated stress \( \sigma(x) \) using separation of variables.

5. **The torsion problem.** Let \( D \) be a domain in the \( x - y \) plane, and consider a long cylinder with cross-section \( D \). Imagine twisting the cylinder at its ends. The lateral boundaries are traction-free, and gravity is ignored. The linearized version of such a deformation is achieved by

\[
 u(x, y, z) = \tau(-yz, xz, \phi(x, y))
\]

for \( \tau \in \mathbb{R} \) and \( \phi : D \to \mathbb{R} \).

(a) Find the associated stress and strain, assuming an isotropic and homogeneous Hooke’s law. Show that \( u \) solves the equations of elastostatics with traction-free boundary condition \( \sigma \cdot n = 0 \) at the lateral boundaries (and a suitable displacement boundary condition at the ends) if and only if \( \Delta \phi = 0 \) in \( D \) and \( \partial \phi / \partial n = (y, -x) \cdot n \) at \( \partial D \).

(b) Verify that the consistency condition \( \int_{\partial D} (y, -x) \cdot n = 0 \) is satisfied [thus \( \phi \) exists and is unique up to an additive constant].

(c) Show that the elastic energy per unit length is \( \tau^2 T \) where \( T = \mu \int_D (\phi_x - y)^2 + (\phi_y + x)^2 \, dx \, dy \). This \( T \) is called the torsional rigidity of the cylinder.

[Comment: This example is more than just a special solution: “Saint Venant’s principle” says that no matter how you twist the ends of a cylinder, far from the ends the deformation will approach the special solution described above.]

6. **Antiplane shear.** Consider once again a cylinder with cross-section \( D \), but consider a uniform body load in the \( z \) direction (gravity), and suppose the lateral boundaries are clamped. Show that these conditions are consistent with the displacement \( u = (0, 0, \phi(x, y)) \) with \( \Delta \phi = 1 \) in \( D \) and \( \phi = 0 \) at \( \partial D \).

7. **Bending of a thin plate.** Consider now a thin, constant-thickness plate whose midplane occupies a region \( D \) in the \( x - y \) plane. The upper and lower surfaces are \( z = \pm h/2 \), so the thickness is \( h \). Consider a deformation of the form \( u = (-z\phi_x, -z\phi_y, \phi + \frac{\alpha}{2} z^2 \Delta \phi) \). Find the associated strain and stress, keeping only terms of order \( h \). Show that for the faces to be traction-free (to this order) we need \( \alpha = \lambda / (\lambda + 2\mu) \). Do the \( z \)-integrations in the basic variational principle, to obtain a new variational principle for \( \phi(x, y) \). Notice that it involves second derivatives of \( \phi \), so the associated PDE is a fourth-order equation!

8. **Plane stress.** Consider the same thin plate, but rather than bending it we suppose it is loaded within its plane. The top and bottom are traction-free, so \( \sigma_{ij} = 0 \) there. If the plate is thin enough we may expect that \( \sigma \) is independent of \( z \). This does not imply that \( u_z \) are independent of \( z \), but we can nevertheless consider \( \bar{u}(x, y) = \text{the average of } u_z \text{ with respect to } z \). Show that \( \bar{u}_1, \bar{u}_2 \) solve the system of “2D elasticity” with a suitable choice of elastic constants.