MECHANICS – Problem Set 3, distributed 3/1/19, due 3/27/19. I’m giving you several weeks, since this problem set is long and 3/20 is spring break.

These problems provide practice with basic concepts of 3D nonlinear elasticity, and explore various reductions including balloons, elastic membranes, and compressible flow. Problem 1 is perhaps the richest (so don’t leave it to the last minute). Problems 1-4 use only material we have already covered in class; problem 5 concerns incompressible hyperelastic materials, a topic we’ll cover on 3/6 (it is discussed at the end of the Lecture 5 notes).

(1) Consider a spherical rubber balloon (such as you might buy in a toy store). To a reasonable approximation we may:

- consider the reference domain to be a thin spherical annulus \( \Omega = \{ x : r_0 - \epsilon < |X| < r_0 + \epsilon \} \);
- consider the air pressure in the balloon to be a constant \( p \);
- ignore the atmospheric pressure outside the balloon;
- consider experiments that are volume-controlled (fixing the volume of the interior of the balloon) or pressure-controlled (fixing the air pressure in the balloon).

From common experience, it is difficult to start blowing up a balloon, but then it gets easier, though eventually as the balloon gets large the blowing gets hard again (unless it bursts). This suggests a pressure-volume relation of the type shown in figure 1 below.

(a) Assume the rubber is hyperelastic. Show that variational principle associated with a pressure-controlled experiment involves the energy \( E = \int_{\Omega} W(F) \, dX - p(\text{volume inside balloon}) \). (In other words, check that this gives the correct equilibrium and boundary conditions.) What variational principle is associated with a volume-controlled experiment?

(b) Consider the limit \( \epsilon \to 0 \) and assume the deformation is uniform expansion (i.e. the sphere \( X = r_0 \) is mapped by \( x(X) = \lambda X \) to a sphere of radius \( \lambda r_0 \)). Suppose the rubber is isotropic and incompressible, so \( W \) has the form \( \Phi(\lambda_1, \lambda_2, \lambda_3) \) where \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are the principal stretches (eigenvalues of \( (F^T F)^{1/2} \)), which must satisfy \( \lambda_1 \lambda_2 \lambda_3 = 1 \). Show that when restricted to the case of “uniform expansion” the pressure-controlled variational principle takes the form \( E(\lambda) = c_1 F(\lambda) - c_2 p \lambda^3 \) with

\[
F(\lambda) = \Phi(\lambda, \lambda, \lambda^{-2}).
\]

What are the constants \( c_1 \) and \( c_2 \)?

(c) Two commonly-used constitutive laws for rubber are the neo-Hookean energy

\[
\Phi(\lambda_1, \lambda_2, \lambda_3) = a(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3)
\]

with \( a > 0 \), and the Mooney-Rivlin energy

\[
\Phi(\lambda_1, \lambda_2, \lambda_3) = a(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + (a/K)(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3)
\]

with \( a > 0 \) and \( K > 0 \) (typically \( 4 < K < 8 \)). Are these laws consistent with the nonmonotone pressure-volume relation shown in figure 1?
(d) Let’s think about the 1D energy $E(\lambda)$, using the non-monotonicity of the pressure-volume relation (as shown in Figure 1) but not using any special formula for $F$ (such as those in part c). Evidently, certain values of the pressure $p$ are consistent with 3 different volumes rather than just one. For such $p$, $E$ must have “double-well” structure, as shown in Figure 2. Show that the two wells have exactly the same depth precisely when $p = p_0$ satisfies the “equal area rule” sketched in Figure 3.

(e) In real pressure-controlled experiments, as $p$ crosses the value $p_0$, the balloon size changes (relatively suddenly) so that the volume occupies the deeper well (the energetically preferred state). How can this be reconciled with our 1D model?

(2) A homogeneous elastic fluid is a hyperelastic material with an energy function $W(F) = h(\det F)$. Show that the Cauchy stress is then $\tau = -p(\rho)I$, where $p(\rho) = -h'(\rho_R/\rho)$. [Here $\rho_R$ is the density in Lagrangian, assumed constant, and $\rho$ is the density in Eulerian variables.] Show that in this case the equations of elastodynamics are precisely the compressible Euler equations

$$\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p(\rho) + \rho f$$

$$\frac{\partial p}{\partial t} + \sum \frac{\partial}{\partial x_i}(p v_i) = 0 .$$

[Note: to calculate $\partial W/\partial F_{\alpha} \alpha$ when $W(F) = h(\det F)$ you’ll to use Cramer’s Rule, which says that $\frac{\partial(\det F)}{\partial F} = (\det F)(F^T)^{-1}$.]

(3) Consider a hyperelastic material, whose Piola-Kirchhoff stress tensor is given by $P_{\alpha} = \partial W/\partial F^{\alpha}_{\alpha}$. Show that if $W$ is frame-indifferent (i.e. if $W(F) = W(RF)$ for all orientation-preserving rotations $R$) then the associated Cauchy stress $\tau$ satisfies $\tau(RF) = R\tau(F)R^T$. 

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(4) Consider a homogeneous, isotropic, hyperelastic material with energy function $W(F) = \psi(I_1, I_2, I_3)$, where $I_1, I_2, I_3$ are the elementary symmetric functions of $B = FF^T$ ($I_1 = \text{tr} B$, $I_2 = \frac{1}{2}[(\text{tr} B)^2 - \text{tr}(B^2)]$, $I_3 = \det B$). Show that the associated Cauchy stress has the form $\tau = \phi_0 I + \phi_1 B + \phi_2 B^2$ with

\[
\phi_0 = 2 \frac{\partial \psi}{\partial I_3} \det F,
\phi_1 = 2 \frac{\partial \psi}{\partial I_1} (\det F)^{-1} + 2 \frac{\partial \psi}{\partial I_2} (\text{tr} B)(\det F)^{-1},
\phi_3 = -2 \frac{\partial \psi}{\partial I_2} (\det F)^{-1}.
\]

(5) Rubber is typically modelled as a homogeneous, isotropic, incompressible hyperelastic material. The energy function for such a material has the form $W(F) = \psi(I_1, I_2)$, since all deformations must satisfy the constraint $\det F = 1$. Its Cauchy stress has the form $\tau = -pI + \phi_1 B + \phi_2 B^2$, where $\phi_1, \phi_2$ have the form derived in Problem 4. Let’s explore how $W$ can be determined experimentally, using relatively simple experiments on thin membranes.

Consider a sheet (in reference coordinates) of length $2A$, width $2B$, and thickness $2h$, with $A, B \gg h$. Consider deformations of the form

\[x_i = \lambda_i X_i, \quad i = 1, 2, 3,\]

which can be maintained by edge tractions alone (i.e. for which the the faces $X_3 = \pm h$ are traction-free). Show that

\[
I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2 \lambda_2^2},
I_2 = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \lambda_1^2 \lambda_2^2
\]

and that the Cauchy stress is

\[
\tau_{11} = 2(\lambda_1^2 - \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{\partial \psi}{\partial I_1} + \lambda_2^2 \frac{\partial \psi}{\partial I_2}),
\tau_{22} = 2(\lambda_2^2 - \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{\psi}{\partial I_1} + \lambda_1^2 \frac{\psi}{\partial I_2}),
\tau_{33} = 0,
\tau_{ij} = 0 \quad i \neq j.
\]

Conclude that $\frac{\partial \psi}{\partial I_1}$ and $\frac{\partial \psi}{\partial I_2}$ satisfy

\[
\frac{\partial \psi}{\partial I_1} = \frac{1}{2(\lambda_1^2 - \lambda_2^2)} \left( \frac{\lambda_2^2 \tau_{11}}{\lambda_1^2 - 1/\lambda_1^2 \lambda_2^2} - \frac{\lambda_1^2 \tau_{22}}{\lambda_2^2 - 1/\lambda_1^2 \lambda_2^2} \right),
\frac{\partial \psi}{\partial I_2} = \frac{-1}{2(\lambda_1^2 - \lambda_2^2)} \left( \frac{\tau_{11}}{\lambda_1^2 - 1/\lambda_1^2 \lambda_2^2} - \frac{\tau_{22}}{\lambda_2^2 - 1/\lambda_1^2 \lambda_2^2} \right).
\]

Thus by measuring the dependence of $\tau_{11}$ and $\tau_{22}$ on $\lambda_1$ and $\lambda_2$ one can determine the function $\psi$.  

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