Recall from Lecture 6: for linear elasticity, eqn (7) is

$$\text{div}(\sigma(u)) + f = 0 \quad \text{in} \quad \Omega$$

where $\sigma$ is the Hooke's law (a parabolic, linear, zero, from symmetric matrices to symmetric matrices).

Typically, it is

$$u = u_0 \quad \text{on} \quad \partial \Omega \quad \text{("Displacement Boundary")}$$

or

$$\begin{cases} u = u_0 & \text{on part of} \ \partial \Omega \\ \sigma(u) \cdot \mathbf{n} = g & \text{at} \ \partial \Omega \\ \text{("Traction boundary")} \end{cases}$$

or may be for $u = u_0$ on part of $\partial \Omega$ and fix $\sigma(u) \cdot \mathbf{n}$ on the rest of $\partial \Omega$.

I'll discuss:

a) uniqueness

b) existence

c) convergence of a basic Galerkin approach

Uniqueness. In nonlinear setting, uniqueness is false. This is clear, for example, in bending of a beam or column, where buckling can occur. Even when de forms $z(x)$ on the entire body, uniqueness can fail, as
This classic thought experiment due to E. John shows: consider a 2D annulus, with boundary conditions \( x(X) = X \) at \( \partial \Omega \). Clearly, \( x(X) \equiv X \) is a soln. But we also expect equilibrium, in which \( \nabla \Phi = 0 \) = cast wind only fixed at \( \partial \Omega \) (see figure).

By contrast, in linear setting solutions are usually unique (the sole exception being the case of traction be on the entire boundary when soln is unique "up to rigid motion").

Fundamental reason: in linear elasticity, the assoc. var. principle is convex (in fact quadratic + linear).

Actually, a very elementary pf of uniqueness is possible, taking advantage of the quadratic character of elastic energy. As warm up, recall this simple pf of uniqueness for Laplace is \( \nabla^2 u + f = 0 \) in \( \mathcal{D} \), \( u = u_0 \) at \( \partial \mathcal{D} \); if \( u_1, u_2 \) are two solutions then difference solves \( \nabla^2 w = 0 \) in \( \mathcal{D} \) and \( w = 0 \) at \( \partial \mathcal{D} \). Multiply by \( w \) and integrate by parts \( \Rightarrow \int_{\mathcal{D}} w \Delta w = \int_{\partial \mathcal{D}} w \partial_w w = 0 \).
So \( w = \text{const} \), but \( w = 0 \) at \( \partial Q \) so \( w = 0 \).

A similar argument works for elasticity: consider \( u \) such that

\[
-\text{div}(\sigma e(u)) = \mathbf{f} \quad \text{in} \; Q \\
\sigma e(u) \quad \text{on} \; \Gamma_1 \\
\sigma e(u) \cdot \mathbf{n} = \mathbf{g} \quad \text{on} \; \Gamma_2
\]

where \( \partial Q = \Gamma_1 \cup \Gamma_2 \); \( \sigma \) is unique.

If we need only show \( \mathbf{f} = \mathbf{g} = 0 \), \( u_0 = 0 \) \( \Rightarrow \) \( u \equiv 0 \). Argue as for Laplace: with \( \sigma = \sigma e(u) \),

\[
0 = \int_Q \left< \nabla u, \nabla \sigma \right> = -\int_Q \left< \nabla u, \sigma \right> + b e \quad \text{by Green's}
\]

\[
= -\int_{\Gamma_2} \left< \sigma e(u), \sigma \right> \quad \text{since } \sigma \text{ is symmetric}
\]

\[
= -\int_{\Gamma_2} \left< \sigma e(u), e(u) \right> \quad \text{since } \sigma \text{ is symmetric}
\]

\[
\Rightarrow e(u) \equiv 0 \quad \text{since } \sigma \text{ is positive}
\]

Now need Lemma: if \( e(u) \equiv 0 \) in connected region \( \Omega \) then \( u \) is an "infinitesimal rotation" i.e.

\[
u(x) = \sum w_i x_i + \mathbf{d}_i
\]

for some (constant) skew-symmetric \( w_i \) and some constant \( \mathbf{d}_i \).
(Note: This is linear analogue of statement that $F^{-1}F = I \Rightarrow \chi(X) \text{ is locally a rigid motion.}

Proof is easy in $\mathbb{R}^2$: $u_{1,1} = 0$, $u_{2,2} = 0 \Rightarrow u_1 = f(x_2), u_2 = f(x_1)$

$u_{1,2} + u_{2,1} = 0 \Rightarrow f'(x_2) + g'(x_1) = 0$

$\Rightarrow f = \omega x_2 + \text{const} \quad g = -\omega x_1 + \text{const}$

Proof in $\mathbb{R}^3$ ($\omega \mathbb{R}^n, n \geq 3$) can be done similarly, by induction on dimension. Or, there's another (less intuitive) way: observe that

$$\partial_k u_i = \partial_j e_{ik} + \partial_k e_{ij} - \partial_i e_{jk}$$

Therefore $e(u) = 0 \Rightarrow \nabla \nabla u = 0 \Rightarrow u$ is linear

in $x$. Since $e(u) = 0$, $Du$ is skew-symmetric.

Wrap up proof of uniqueness: we were assuming $d
\sigma = 0$ in $\Omega$, $u = 0$ at $\Gamma_1$, $\sigma \cdot n = 0$ at $\Gamma_2$.
We concluded $u = u_{\text{rigid}}$.

If $\Gamma_1 \neq \emptyset$ this forces $u \equiv 0$.

What about pure traction plan? Situation
is like Neumann plan in Laplace eqn. Recall: for \( \Delta u = f \) in \( \Omega \), \( \frac{\partial u}{\partial n} = g \) at \( \partial \Omega \), we have a consistency condition: \[ \int_{\Omega} f = \int_{\partial \Omega} g \] when consistency holds, \( u \) is unique up to a constant.

Similar situation for linear elasticity: traction plan
\[-\text{div} \sigma = f, \quad \sigma = x \varepsilon (u) \quad \text{in} \quad \Omega \]
\[ \sigma \cdot n = g \quad \text{at} \quad \partial \Omega \]

can have only if
\[ \int_{\partial \Omega} < g, \hat{\mathbf{n}} > ds + \int_{\Omega} < f, \hat{\mathbf{u}} > dx = 0 \]

\( \hat{\mathbf{u}} \) is an unstable rigid motion.

If it does have a value, that value is unique up to addition of an unstable rigid motion.

If consistency: if \( \varepsilon (\hat{\mathbf{u}}) = 0 \) then
\[ \int_{\Omega} < \hat{\mathbf{u}}, f > = - \int_{\partial \Omega} < \hat{\mathbf{u}}, \sigma \cdot n > + \int_{\Omega} < \varepsilon (\hat{\mathbf{u}}), \sigma > \]
\[ = - \int_{\partial \Omega} < \hat{\mathbf{u}}, \sigma \cdot n > \]
Pf of uniqueness: Same as before except row $\gamma = \phi$ so uniqueness is only true up to unl. rigid motion.

What about existence? Again, with a lot like scalar Laplace eqn, or dir. harm scalar eqn $\nabla \cdot (\sigma \nabla u) = f$. Main techniques:

1. Variational principles 
   \( \nabla \cdot (\sigma \nabla u) = f \) very closely
2. Lax–Milgram Lemma connected
3. Boundary integral techniques (different)

Boundary integral methods are basically restricted to constant-coeff setting (won't discuss them here).

Variational + Lax–Milgram are simple & general; also form basis of most numerical schemes (e.g., finite elements), we'll focus on former.

Again, use scalar Laplace as guide. Solve

\[-\Delta u = f \quad \text{in } \Omega, \quad u = u_0 \quad \text{on part of } \partial \Omega \quad (17)\]
\[\frac{\partial u}{\partial n} = g \quad \text{on rest of } \partial \Omega \quad (17')\]

can be found using variational prin
(8) \[ \min_{u=0} \int_{\partial \Omega} -\nabla u \cdot \mathbf{n} \, dS - \int_{\Omega} f \, dx \]

Note: if \( \Gamma = \emptyset \) and data are inconsistent, then functional is unbounded below; we can drive it to \(-\infty\) by taking \( u = \text{constant} \).

Existence via variational prin arises convexity of functional, plus lemma that it's odd below. Key to latter is a pair of Poincaré-type reps:

**Easy Poincaré rep**: \[ \int_{\Omega} |\nabla u|^2 \leq C \int_{\partial \Omega} |u|^2 \]

provided \( u = 0 \) at \( \partial \Omega \)

**Hard Poincaré rep**: \[ \int_{\Omega} |\nabla (u - \bar{u})|^2 \leq C \int_{\partial \Omega} |\nabla \bar{u}|^2 \]

where \( \bar{u} = \text{avg of } u \) on \( \partial \Omega \).

They assure us that although "energy" controls only \( \int |\nabla u|^2 \) directly, it also controls \( \int |\nabla u|^2 \) indirectly.

Situation for electricity is just the same. Variational is...
\[
\min_{u = u_0 \text{ on } \Gamma_1} \left\{ \int_{\Omega} \frac{1}{2} \langle \mathbf{e}(u), \mathbf{e}(u) \rangle + \langle f, u \rangle \, dx - \int_{\Gamma_2} \langle \mathbf{e}(u), \mathbf{g} \rangle \, dS \right\}
\]

Analogue of Poincaré inequality is consequence of Korn's inequality. Two versions:

1. \( \int_{\Omega} |\mathbf{u}|^2 \leq C \int_{\Omega} |\mathbf{e}(\mathbf{u})|^2 \)

provided \( \mathbf{u} = 0 \) at \( \partial \Omega \).

2. \( \int_{\Omega} |\mathbf{u}|^2 \leq C \int_{\Omega} |\mathbf{e}(\mathbf{u})|^2 \)

provided \( \int_{\Omega} \mathbf{u} \) is symmetric matrix.

They ensure that although "energy" controls only \( \int |\mathbf{e}(\mathbf{u})|^2 \) directly, it controls \( \int |\mathbf{u}|^2 \) indirectly (and therefore also \( \int |\mathbf{u}|^2 \) indirectly).

Same intuition on (hard) Korn inequality; it clearly implies:

\[
\int_{\Omega} |\mathbf{u} - \mathbf{\bar{u}}|^2 \leq C \int_{\Omega} |\mathbf{e}(\mathbf{u})|^2 \quad \text{for some weak rigid motion}
\]

which is linear analogue of estimate (true, but much harder). That \([\text{small volume} \implies \text{close to rigid rotation}] \implies [\text{close to rigid rotation}] \) constant depends on domain, of course,
and long thin domains $\Rightarrow$ very large constants

locally close to a rigid structure.

But not globally!

"Easy Korn way" can be proved by an elementary integral by parts, or by an easy Fourier-transform-based argument. Here is the former:

For $u \in C_0^\infty(\Omega)$ (with $\Omega$ bounded),

$$\int_{\Omega} |\nabla u|^2 = \int_{\Omega} \sum_i \left( \frac{\partial_i u^i + \partial_i u^i}{2} \right)^2$$

$$= \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \int_{\Omega} \frac{1}{2} \sum_i \partial_i u^i \partial_i u^i$$

But since $u = 0$ near $\partial \Omega$,

$$\int_{\Omega} \partial_i u^i \partial_i u^i = \int_{\Omega} \partial_i u^i \partial_i u^i$$

so

$$\int_{\Omega} |\nabla u|^2 = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\Omega} |\nabla u|^2$$
Thus when \( u=0 \) at \( \partial \Omega \), Korn's inequality states that

\[
\frac{1}{52} \int_{\Omega} u^2 \leq C \left( \frac{1}{52} \int_{\Omega} \left| \nabla u \right|^2 + \frac{1}{2} (\det \mathbf{F})^2 \right)
\]

Evidently true with \( C = 2 \).

"Hard Korn inequality" has interesting history:

- 1st "proof" by Korn, about 1910

  - Friedrichs wrote a paper about 1947, pointing out importance of this inequality, giving a new (still not simple) proof of 1st "modern" proof of existence of solutions (by Hilbert-space methods)

  - Many proofs since Friedrichs. Some are very efficient but not so elementary (see e.g. Duvaut + Lions book). Also much study of other "coerciveness inequalities" - when does control of selected combinations of \( \| \mathbf{u} \| \) yield control of all derivatives individually (K.T. Smith, Amazigo, others - 60's, 70's).
using pseudo-differential operators.

A really simple, elementary proof was finally given by Oleinik & Kondratiev about 1989 (CRAS Paris Ser I, 1989, 483-487; also Rev. Mat. Appl. 7(10), 1990, vol 3, 641-666). I'll distribute a handout on this.

I promised a discussion of existence via variational principle. Actually let's do a little less (I won't actually prove existence - though those who know the variational proof of existence for Laplace's eqn well see what to do) and a little more (I'll discuss convergence of a typical finite element method calculation). Focus on:

Linear elasticity in \( \mathbb{R}^2 \) (no body load)

\[ u = 0 \text{ on part of } \partial \Omega \text{ (call it } \Gamma) \]

\[ \sigma \cdot n = g \text{ on rest of } \partial \Omega \text{ (call it } \Gamma_g) \]

For which \( u \) will this be

\[ \min \int_{\Omega} \frac{1}{2} \langle \nabla u(x), \nabla u(x) \rangle - \int_{\Omega} u(x) g(x) \]

\[ u = 0 \text{ on } \Gamma \]

\[ \Gamma_g \]
Typical numerical method: minimize exactly the same functional in a finite dened space of functions \( V \) (closed so that \( u \) = 0 whenever \( u \in V \)). For example:

\[ V \]

could consist of functions that are piecewise linear + cubic on a fixed triangulation of \( \Omega \).

Let

\[
E[u] = \int_{\Omega} \frac{1}{2} \langle \epsilon(eu), \epsilon(eu) \rangle - \int_{\Omega} \langle u, f \rangle
d\Omega
\]

where \( u_\ast = \) minimizer (i.e., actual solution of elasticity problem). Since \( E \) is quadratic + linear, Taylor expand around \( u_\ast \) keeping only quadratic terms in exact. So for any \( v \),

\[
E(v) = E(u_\ast) + \frac{1}{2} \int_{\Omega} \langle \delta \epsilon(eu_\ast), \epsilon(v-eu_\ast) \rangle
d\Omega
\]

\[
+ \int_{\Omega} \langle v-u_\ast, f \rangle
d\Omega
\]

\[
+ \frac{1}{2} \int_{\Omega} \langle \delta (e(u)-e(u_\ast)), \epsilon(v-eu_\ast) \rangle
d\Omega
\]

The square of elasticity (the EL energy for \( E \) at \( u_\ast \)) assure that boxed terms (the \( 1^{st} \) term) vanish, so
\[ \frac{1}{2} \int_{52} < e(v-u_k), e(v-u_k) > = E[v] - E[u_k] \]

If \( u_k \) can be approx well in the subspace \( V \) then RHS will be small at the best \( v \in V \)

Claim: if RHS is small, then \( v \) is close to \( u_k \) in \( H \). This follows immediately from (*) using positivity of \( A \) and the Korn inequality.

\[ \frac{1}{52} \int_{52} |v|^2 \leq C \frac{1}{52} \int_{52} |e(w)|^2 \quad \text{if} \quad w = 0 \text{ on } \Gamma. \]

Explanation of (*): well, by the "hard" Korn inequality

\[ \frac{1}{52} \int_{52} |v|^2 \leq C \frac{1}{52} \int_{52} |e(w)|^2 \]

for some skew-symmetric, constant matrix \( \mathbf{v} \). A Poincaré-type inequality Then gives

\[ \frac{1}{52} \int_{52} |v|^2 \leq C \frac{1}{52} \int_{52} |e(w)|^2 \]

and a standard "trace" theorem for \( H(\mathbf{v}) \) gives

\[ \int_{\Gamma} |v - (\mathbf{v} \cdot \mathbf{x} + d)|^2 ds \leq C \frac{1}{52} \int_{52} |e(w)|^2 \]

But \( w = 0 \) on \( \Gamma \), and the map
\( (s, 0) \to \left( \int \frac{18 \cdot x + \theta^2}{\sqrt{x}} \, ds \right)^{1/2} \)

is a norm on the finite-dimensional space of all \( \phi' + \phi'' \). So,

\[ \|s\| + \|t\| \leq \frac{C''}{\sqrt{2}} \|e(t)\| \]

Thus finally,

\[ \frac{\|w\|}{\sqrt{2}} \leq C'' \frac{\|e(t)\|}{\sqrt{2}} \]

by triangle inequality.