Today's topic (spelling, probably, into next week): a 1D bending theory (Euler's "elasticity"), both as 1st exposure to "torque" and "bending", and as an example of bifurcation.

Antman's chapter 4 has a good discussion of beams + rods + his chapter 5 has a very nice introduction to bifurcation in this context (of course, we'll do much less than he does).

Big picture:

- in 2D, there is resistance to bending even for a sheet that's inextensible (e.g., the "xerox paper problem" to be discussed later in these notes + explored in HW2).

Essential mechanism: if a thin strip is mapped to an annulus with midline mapping isometrically, thin lines parallel to midline are stretched or shrunk due to effect of curvature.
for a rod in 3D (or a thin ribbon - which is just a rod with rectangular cross-section) picture is similar except that rod can bend (in two possible planes) + twist (example: a Mobius band).

To keep things simple we'll focus on an inextensible beam (initially straight + uniform, e.g. a piece of paper or a ruler, deformed by bending consistent with a 1D model).

To get started, we must discuss:

a) kinematics (description of deformation)

b) statics (forces + bending moments, and assoc balance laws)

c) constitutive laws (in this case: relation between curvature + bending moment)

Kinematics: for a 1D inextensible rod this is easy: use $\Delta \mathbf{L}$ as reference body, and $\mathbf{F}(\mathbf{x})$ as deformed position. We want to assume (for simplicity) $\mathbf{F}(\mathbf{x})$ stays in the $x_1$-$x_2$ plane, so $|\mathbf{F}_1|=1 \Rightarrow$
\[ \vec{t}_2 = (\cos \Theta(s), \sin \Theta(s)) \]

The rod's curvature is \( \Theta(s) \). (This will play a role similar to that of the "strain" \( \varepsilon_1 \) in our analysis of strings.)

Statics: slicing the rod at \( s = a \), each side acts on the other by

(i) a net force \( \vec{f}(s) \) (as with strings)

(ii) an additional bending moment \( \vec{m}(s) \)

(Think of a curved piece of paper, ignoring gravity: viewed as a 1D rod! Here \( \vec{f} = 0 \), and \( \vec{m}(s) \) is what maintains the bent state.)

Define \( \vec{m} \): part of beam at \( s > a \) exerts torque \( \vec{m}(s) \times \vec{r}(s) + \vec{m}(s) \) on the rest. Note that for deformations in the \( x_1 \)-\( x_2 \) plane, \( \vec{m} = (0, 0, M(s)) \) is essentially scalar-valued.

Balance of forces: \( \vec{f}(s) \cdot \vec{n}(s) + \int_a^d \vec{F}(s) \, ds = 0 \)

Balance of torques: \( \int_s^d \vec{F}(s) \times \vec{r}(s) \, ds - \int_s^t \vec{m}(s) \times \vec{r}(s) \, ds + \int_a^s \vec{F}(s) \times \vec{r}(s) \, ds = 0 \)
whence \[ \bar{\tau}_s + \vec{f} = 0 \]

\[ \bar{m}_s + (\vec{r} \times \vec{\tau}) + \vec{r} \times \vec{f} = 0 \]

or equivalently:

\[ \bar{\tau}_s + \vec{f} = 0 \]

\[ \bar{m}_s + \bar{c}_s \times \vec{n} = 0 \]

(Where does balance of torque come from? Well, dynamic version of balance of forces is cause of linear momentum, so it shouldn't be surprising that dynamic version of balance of torque is conservation of angular momentum. This will be clearer when we discuss Classical Mechanics; for now please take the balance laws as a starting pt.)

Notes: (1) if \( f = 0 \) then \( \bar{\tau} \) is constant (and clearly evident from the body center)

(2) a 1D rod can be held in a bent pose either by applying forces (horizontal) in the plane or below) or by applying bending
Constitutive law: since the rod is inextensible there is no constitutive law for $\bar{\sigma}$ (instead we get it by integrating $\bar{F}$ with constants of integration coming from boundary conditions). Note that $\bar{\sigma}$ need not be in the direction of $\bar{F}$.

The simplest law for $\bar{m}$ is the "physically linear" law

$$\bar{m} = (0, 0, M), \quad M = A \theta' \quad A = \text{constant}$$

One can of course consider more nonlinear laws (taking $\bar{M}$ to be a nonlinear function of $\theta'$). But this linear law is rich enough that we'll stick with it. (The linear law is in fact appropriate for thin bodies, since it is substantially conservative still means no length stretching or shrinking of surfaces parallel to the midline.)

When $\bar{F} = 0$ this leads us to the "elastic" (considered by Euler in 1727, but also by..."
J. Bernoulli in 1694; Antwerp has lost on the
question:

\[ \vec{r} = (\cos \theta(s), \sin \theta(s)) \]

\[ \vec{n} = 0 \Rightarrow \vec{n} = \text{const} = -\lambda (\cos \alpha, \sin \alpha) \]

for some \( \lambda, \alpha \)

\[ \vec{m}_s + \vec{e}_x \times \vec{n} = 0, \quad \vec{m} = (0,0,M) \Rightarrow \]

\[ (0,0,M_s) = \lambda (\cos \theta, \sin \theta, 0) \times (\cos \alpha, \sin \alpha, 0) \cdot \]

\[ \Rightarrow \quad M_s = \lambda \cos \theta \sin \alpha - \cos \alpha \sin \theta \cdot \]

\[ \Rightarrow \quad M_s = \lambda \sin (\alpha - \theta) \cdot \]

Constitutive law say \( \quad M_s = (A \theta 's) \)' So one becomes (writing \( \theta = \theta(s) - \alpha \)

\[ [\dot{A} \theta 's] + \lambda \sin \theta(s) = 0 \cdot \]

Notice in general \( \lambda \) and \( \alpha \) are unknown, just like \( \theta(s) \); they must be determined from body data.
Example: deflection of a clamped beam (ignoring gravity). Take $x = 0$ to be clamped horizontally at R.H. edge $s = L$, which is otherwise free ($\theta'(L) = 0$). Then $\mathbf{t} = (0, -F)$ and
\[
(0, 0, H_4) = (\cos \theta, \sin \theta, 0) \times (0, F, 0) = F \cos \theta
\]
so
\[
\theta_4 = F \cos \theta \quad 0 < s < L\]
with $\theta(0) = 0$ and $\theta'(L) = 0$.

A more or less exact solution is possible:
\[
A \theta_4 = F \theta_4 \cos \theta
\]
so
\[
\theta_4 = \frac{2F}{A} \sin \theta + \text{const}
\]
\[
= \frac{2F}{A} [\sin \theta + \sin \theta] = -\theta(L) > 0
\]

Since $\theta(L)$ should be negative, a bit of arithmetic gives
\[
\int_{-\theta(L)}^{0} \frac{2F}{\sqrt{\sin \theta - \sin \theta}} \, d\theta = -A \sqrt{\frac{2F}{A}}.
\]
The value of $x$ is determined by the eqn
\[
\int_0^x \frac{d\phi}{(\sin \phi - \sin \phi_0)^{1/2}} = L \sqrt{E/\rho}
\]

(which has no explicit role but can easily be solved numerically).

Another example: the "xerox paper problem". Describe the profile of a standard 8.5\(\times\)11 sheet of paper, held at one edge so the tangent there is vertical.

Differences from the elasticity:

- gravity matters
- must specify \(\gamma\) and use a different \(\text{BC}\)

Now \(\dot{\gamma} + f = 0\), \(f = f_0 (0, -1, 0)\) - with conventions

\[s=L\text{ is held}\]

\[s=0\text{ is free}\]

The BC's are

- no force or moment at \(s=0\)
- specified angle \(\Theta = -\pi/2\) at \(s=L\)

Evidently, \(\tau = (a, b + f_0, 0)\) for constants \(a, b\)

\(\text{BC at } s = 0\text{ give } a = b = 0\)

\[\Rightarrow \tau = (0, f_0, 0)\]
Now eqn for $\bar{m}$ & linear constant law gives

$$\bar{m}'' + f.s \cos \theta(s) = 0 \quad 0 < s < L$$

$$\theta(0) = 0, \quad \theta(L) = -\pi/2$$

(On HW2 you'll be asked to estimate the value of $\bar{m}$ for a piece of xerox paper.)

What can we do with this that's interesting?

My choice: use it as an intuitive, physically natural example of bifurcation.

(Where to read more? Antman's Chapter 5 - is pretty good - you'll find versions of all that I do there - and much more. Howell-Kozyreff-Ockendon has a concise treatment in 7.4.9.3. Another good source: Ivar Stakgold, "Branching of solutions of nonlinear equations" SIAM Review 13 (1971) 289-332.)

Goal: consider the elastic column with compressive load $N$. If $N$ is large enough it will buckle. What is the critical load $N_c$?

How can we understand the buckled configurations?
Various choices of $b$ are possible; let's choose

$g(0) = 0$ and LHS is \underline{clamped}.

(\text{so } \theta(0) \text{ is fixed, and } \theta'(0) = 1,0)

$\theta'(1) = 0$ RHS is "pinned" to loading device (applied load is \(\xi,0\)) but applied bending moment is 0.

Eqn. (derived earlier) is

\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial \theta}{\partial x} \right) + x \sin \theta(\xi) = 0 \quad 0 < x < 1
\]

\[
\theta(0) = 0, \quad \theta'(1) = 0
\]

Clearly $\theta = 0$ is a solution for any $\lambda$. When $\lambda$ is small we expect it to be stable, when $\lambda$ is large enough we expect it to be unstable.

Note: this form (with $\lambda$ fixed) has a \underline{wave} formulation: $\theta(\xi)$ is a \underline{critical} pt.
\[ E = \int_0^1 \frac{1}{2} \theta'^2 + \lambda \cos \theta \, ds \]

subject to \( \theta(0) = 0 \)

(The cond. \( \theta'(1) = 0 \) arises as a "natural" bc. at \( s = 1 \).) Interpret this as

\[ E = (\text{elastic energy}) + (\text{work done by load}) \]

since

\[ \int_0^1 \frac{1}{2} \theta'^2 \, ds = \text{"energy" due to curvature} \]

and

\[ \int_0^1 \lambda \cos \theta \, ds = \lambda \int_0^1 \theta' \cdot (1,0) \, ds = F(1) \cdot (1,0) \]

is force \( (\text{displacement of loaded pt.}) \).

Natural physical ext. is to increase \( \lambda \) gradually, starting from 0. Amounts mathematically to a "continuation method" for obtaining solns \( \theta = \theta(s, \lambda) \). Diff of eqn wrt \( \lambda \) gives eqn for \( \theta = \partial \theta / \partial \lambda \), which we can expect to integrate (as ode in \( \lambda \)) to get \( \theta \). This procedure is especially simple in the given example: diffn of eqn wrt \( \lambda \) gives
\[ (A \dot{\Theta}) + \lambda (\cos \Theta) \dot{\Theta} + \sin \Theta = 0 \]
\[ \dot{\Theta}(0) = 0, \quad \dot{\Theta}(1) = 0 \]

If \( \Theta(4) = 0 \) then \( (*) \) implies \( \dot{\Theta}(5) = 0 \) so long as
\( \lambda \) is 1st eigenvalue of linearized plane

\[ A \dot{\Theta} + \lambda \dot{\Theta} = 0, \quad \dot{\Theta}(0) = \dot{\Theta}(1) = 0 \]

From new eqn let's take \( \Theta = 1 \) for simplicity. Then
1st eigenvalue is
\[ \lambda = \frac{\pi^2}{4}, \text{ associated to eigenfunction} \]
\[ \Theta(5) = \sin \left( \frac{\pi}{2} \right) \]

Conclusion so far: when \( \Theta = 1 \), crit load is \( \frac{\pi^2}{4} \). (For general \( a \geq 0 \), crit load would be \( \frac{\pi^2a^2}{4} \) by same argument.)

But: exp can't stop at \( a = \infty \), and
neither should we. However: we need a
viewpt that permits \( \theta = \Theta(4, a) \) to be
nonunique. In fact, we'll show that
the bifurcation diagram is locally, near \( a = a_0 \), like this:

\[ \text{stable} \rightarrow \text{unstable} \rightarrow \text{stable} \]
Variational perspective: For $\lambda \gg 1$, the word problem has a saddle point $z$ (nearby) local min

![Diagram showing two curves: one for $\lambda \ll \lambda_0$ and the other for $\lambda \gg \lambda_0$. $\theta(z) = 0$ is a saddle point.]

Big ozone procedure for analysis uses "Lyapunov-Schmidt reduction" (will sketch it later). But situation can be captured very concretely by the following more elementary calculation (see Atman's $\varepsilon \approx 35.6$ or Howell et al. $\varepsilon \approx 4.9, 3$): try ansatz

$$\lambda(\varepsilon) = \frac{\pi^2}{4} + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \ldots$$

$$\theta(\varepsilon) = 0 + \varepsilon \theta_1(\varepsilon) + \varepsilon^2 \theta_2(\varepsilon) + \ldots$$

and expand in powers of $\varepsilon$. Full eqn is

$$0 = (\varepsilon \theta_1 + \varepsilon^2 \theta_2 + \ldots) + \left(\frac{\pi^2}{4} + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + \ldots\right) \sin \left(\varepsilon \theta_1 + \varepsilon^2 \theta_2 + \ldots\right)$$

and we have (using $\sin x \approx x - \frac{1}{6}x^3 + \ldots$)
\[ \sin(\Theta_1 + \varepsilon^2 \Theta_2 + \cdots) = \Theta_1 + \varepsilon^2 \Theta_2 + \varepsilon^3 \left( \Theta_3 - \frac{1}{6} \Theta_1^3 \right) + \cdots \]

So we get:

at order \( \varepsilon \)
\[ \Theta_1'' + \frac{\pi^2}{4} \Theta_1 = 0, \quad \Theta_1(0) = \Theta_1'(1) = 0 \]

\[ \Rightarrow \Theta_1(x) = \varphi \sin \left( \frac{\pi}{2} x \right) \quad \varphi = \text{any constant} \]

at order \( \varepsilon^2 \)
\[ \Theta_2'' + \frac{\pi^2}{4} \Theta_2 = -x \Theta_1 = -x \varphi \sin \left( \frac{\pi}{2} x \right) \]

\[ \Theta_2(0) = 0, \quad \Theta_2'(1) = 0 \]

Solution exists iff \( \text{RHS is null vector of LHS} \),
i.e. if \( \int_0^1 \varphi \sin \left( \frac{\pi}{2} x \right) \; dx = 0 \). So (assuming \( \varphi \neq 0 \)) \( x_1 = 0 \) and \( \Theta_2 \) is again a multiple of \( \varphi \sin \left( \frac{\pi}{2} x \right) \).

at order \( \varepsilon^3 \)
\[ \Theta_3''' + \frac{\pi^2}{4} \Theta_3 = -x_2 \Theta_1 - x_3 \varphi^2 + \frac{\pi^2}{24} \varphi^3 \]

Solution exists iff
\[ \int_0^1 \left( -x_2 \varphi^2 + \frac{\pi^2}{24} \varphi^3 \right) \; dx = 0 \]

This simplifies to
\[ x_2 \varphi = \frac{\pi^2}{32} \varphi^3 \]

since \( \int_0^1 \sin^2 \left( \frac{\pi}{2} x \right) = \frac{1}{2} \), \( \int_0^1 \sin^4 \left( \frac{\pi}{2} x \right) = \frac{3}{8} \).
We could continue, but there's no need: we've shown that bifurcation is locally a parabola, opening to the right (since $\frac{d^2}{dx^2} > 0$).

\[ \lambda = \lambda_c \quad \text{for} \quad \Theta \text{ component} \]

\[ \lambda = \lambda_c = \frac{1}{2} \left( \frac{2}{5} \right) \]

\[ \lambda = 0 \]

(Bifurcation is called "supercritical" because the parabola opens to the right.)

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Here's a sketch of how Laxner-Schmidt reduction works in this case (see eg. Stahberg for more detail): 

Let's look for \( \Theta = \Theta_c + \tilde{\Theta} \quad \tilde{\Theta} \neq 0 \)

where \( \Theta_c = \text{expansion} = \sin \left( \frac{\pi}{2} \right) \). Write eqn \( \Theta'' + \lambda \sin \Theta = 0 \) as

\[ \Theta'' + \lambda \Theta + (\lambda - \lambda_c) \Theta + \lambda (\sin \Theta - \Theta) = 0 \]

i.e.

\[ \Theta'' + \lambda \Theta = - (\lambda - \lambda_c) \Theta - \lambda (\sin \Theta - \Theta) \]
Consistency condi is

$$ \lambda (\lambda - \Delta) q + \lambda (\sin \theta - \theta) = 0 $$

If this holds, there's a unique $\Theta = q$ solving (1). So we can view $\Theta = \Theta(\lambda, q)$ as being defined (for $\lambda$ near $\lambda_0$ and $q$ near $q_0$) by

$$ \Theta'' + \frac{\lambda}{\Theta} q = \Theta_1 [- (\lambda - \Delta) q - \lambda (\sin \theta - \theta)] $$

$$ \Theta(q, \Theta(0) = a, \Theta(1) = 0. $$

The eqn (2) gives the rcln between $\lambda$ and $q$ that describes the bifurcation diagram. One can show (using $\sin \theta - \theta = -\frac{1}{2} \theta^3$) that $\lambda(\lambda_0) < \lambda_0^3$, so the leading order character of bifurcation relation is

$$ \int (\lambda - \Delta) (q q + \frac{\Theta}{q}) q + \lambda \left( -\frac{1}{6} \frac{q^4}{q^2} \right) q = 0 $$

i.e.,

$$ \lambda - \Delta q \int q^2 dq - \frac{1}{6}\lambda q^3 \int q^4 dq = 0. $$

as we obtained earlier by expansion. (Essence of this approach: $\Theta = q q + \Theta$ represents the nontrivial solution as a graph over the $1D$ axis $q q$.)