Derivative Securities – Fall 2012– Section 2
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Binomial and trinomial one-period models. This section explores the implications of arbitrage for the pricing of contingent claims in a one-period setting. It has three distinct parts:

(1) A fairly standard discussion of the one-period binomial setting (similar for example to Hull's Sections 12.1 and 12.2). The focus here is on hedging using a risk-free bond and the underlying.

(2) A less-standard discussion of the one-period trinomial setting. I include this because it provides the simplest example of a market that's not complete. Moreover it permits me to introduce (ever so briefly) the connection between risk-neutral pricing and the duality theory of linear programming. This material is not in Hull or Baxter/Rennie (though it is well-known to experts).

(3) Returning to the one-period binomial setting, we then discuss what changes if the hedging is done using forwards or futures instead of the underlying. Most books postpone this topic until later (Hull addresses it in Chapter 17). But there are both theoretical and practical advantages to hedging by forwards or futures, as we'll explain when we start this discussion.

Don't be misled by the order. A proper understanding of topics (1) and (3) will be crucial for what we'll do later on. Topic (2) is in some sense a digression. It too has value – for understanding the difference between complete and incomplete markets, and for gaining some perspective on the risk-neutral measure. But the HW and exam will go light on topic (2) (for example, they will not involve linear programming or duality).

We do not claim, of course, that any real-world market can realistically be modelled using a one-period binomial or trinomial framework. But we will argue soon that many markets can be modelled using *multiperiod* trees, in much the same way that diffusion can be modelled by random walk on a lattice. A good understanding of the single-period setting will lead, with just a little extra work, to an understanding of the multiperiod models.

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Topic (1): The binomial model. We consider a one-period market which has

- just two securities: a stock (paying no dividend, initial unit price per share s_1 dollars) and a bond (interest rate r, one bond pays one dollar at maturity).
- just one maturity date T
- just two possible states for the stock price at time T: s_2 and s_3, with $s_2 < s_3$

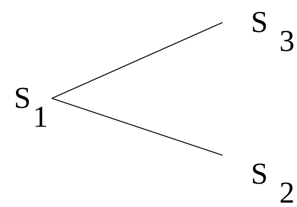

Figure 1: Prices in the one-period binomial market model.

(see the figure). We could suppose we know the probability p that the stock will be worth s_3 at time T. This would allow us to calculate the expected value of any contingent claim. However we will make no use of such knowledge. Pricing by arbitrage considerations makes no use of information about probabilities – it uses just the list of possible events.

The reasonable values of s_1, s_2, s_3 are not arbitrary: the economy should permit no arbitrage. This requires that

$$s_2 < s_1 e^{rT} < s_3.$$

It's easy to see that if this condition is violated then an arbitrage is possible. The converse is extremely plausible; a simple proof will be easy to give a little later.

In this simple setting a contingent claim can be specified by giving its payoff when $S_T = s_2$ and when $S_T = s_3$. For example, a long call with strike price K has payoff $f_2 = (s_2 - K)_+$ in the first case and $f_3 = (s_3 - K)_+$ in the second case. The most general contingent claim is specified by a vector $f = (f_2, f_3)$ giving its payoffs in the two cases.

Claim 1: In this model every contingent claim has a replicating portfolio. Thus arbitrage considerations determine the value of every contingent claim. (A market with this property is said to be "complete".)

In fact, consider the portfolio consisting of ϕ shares of stock and ψ bonds. Its initial value is

$$\phi s_1 + \psi e^{-rT}.$$

Its value at maturity replicates the contingent claim $f = (f_2, f_3)$ if

$$\begin{aligned} \phi s_2 + \psi &= f_2 \\ \phi s_3 + \psi &= f_3. \end{aligned}$$

This is a system of two linear equations for the two unknowns ϕ, ψ. The solution is

$$\phi = \frac{f_3 - f_2}{s_3 - s_2}, \quad \psi = \frac{s_3 f_2 - s_2 f_3}{s_3 - s_2}.$$

The initial value of the contingent claim f is necessarily the initial value of the replicating portfolio:

$$V(f) = \phi s_1 + \psi e^{-rT} = \frac{f_3 - f_2}{s_3 - s_2} s_1 + \frac{s_3 f_2 - s_2 f_3}{s_3 - s_2} e^{-rT}.$$

Claim 2: The value can conveniently be expressed as

$$V(f) = e^{-rT}[(1-q)f_2 + qf_3] \quad \text{where} \quad q = \frac{s_1 e^{rT} - s_2}{s_3 - s_2}.$$

Moreover, the condition that the market admit no arbitrage is $0 < q < 1$, which is equivalent to $s_2 < s_1 e^{rT} < s_3$.

The formula for $V(f)$ in terms of q is a matter of algebraic rearrangement. This simplification seems mysterious right now, but we'll see a natural reason for it a bit later.

The condition that the market supports no arbitrage has two parts:

(i) a portfolio with nonnegative payoff must have a nonnegative value; and

(ii) a portfolio with nonnegative and sometimes positive payoff must have positive value.

In the binomial setting every payoff (f_2, f_3) is replicated by a portfolio, so we may replace "portfolio" by "contingent claim" in the preceding statement without changing its impact. Part (i) says $f_2, f_3 \geq 0 \Rightarrow (1-q)f_2 + qf_3 \geq 0$. This is true precisely if $0 \leq q \leq 1$. Part (ii) forces the sharper inequalities $q > 0$ and $q < 1$.

Notice the form of Claim 2. It says the present value of a contingent claim is obtained by taking its "expected final value" $(1-q)f_2 + qf_3$ then discounting (multiplying by e^{-rT}). However the "expected final value" has nothing to do with the probability of the stock going up or down. Instead it must be taken with respect to a special probability measure, assigning weight $1-q$ to state s_2 and q to state s_3, where q is determined by s_1, s_2, s_3 and r as above. This special probability measure is known as the "risk-neutral probability" associated with the market. When (as in the present setting) the risk-neutral probability is unique, the pricing formula is commonly written

$$\text{option value} = V(f) = e^{-rT} E_{\text{RN}}[f] = \text{discounted risk-neutral expected payoff.}$$

How to remember the formula for q? It's easy. The pricing formula must hold for *all* contingent claims. In particular, it must price the stock correctly. Therefore we must have

$$s_1 = e^{-rT}[(1-q)s_2 + qs_3].$$

Rearranging this equation gives once again the formula for q. (Note that the formula automatically prices a bond correctly, since $e^{-rT} = e^{-rT}[q + (1-q)]$ regardless of the value of q.)

I like to view q as nothing more than a convenient way of representing $V(f)$. However the term "risk-neutral probability" can be understood as follows. In the literature on financial economics (see e.g. John Cochrane's book *Asset Pricing*), a common viewpoint is that the present value of an uncertain future income stream is determined by its *discounted expected utility*. The utility should be concave; however, one can consider as an extreme case the linear utility $U(f) = f$. This is called the "risk neutral" utility, because in this case the *expected utility* of future income is identical to the *expected value* of future income (whereas for a strictly convex U, the expected utility $E[U(f)]$ would depend on the probability distribution of f as well as on its mean). Our formula $V(f) = e^{-rT}[(1-q)f_2 + qf_3]$ gives the value of the option as the discounted expected utility of the payoff, using the linear (i.e. risk-neutral) utility, and the ("risk-neutral") probability distribution determined by q.

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Topic (2): The trinomial model. This is the simplest example of an *incomplete* market. It resembles the binomial model in having

- just two securities: a stock (paying no dividend, initial unit price per share s_1 dollars) and a bond (interest rate r, one bond pays one dollar at maturity).
- just one maturity date T.

However it differs by having three final states rather than two:

- the stock price at time T can take values s_2, s_3, or s_4, with $s_2 < s_3 < s_4$

(see the figure). The reasonable values of $s_1, \ldots, s_4$ are not arbitrary: the economy should

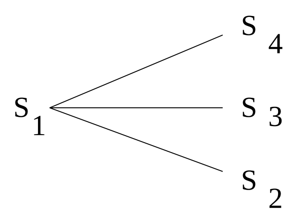

Figure 2: Prices in the one-period trinomial market model.

permit no arbitrage. This requires that

$$s_2 < s_1 e^{rT} \quad \text{and} \quad s_1 e^{rT} < s_4.$$

In other words, the stock must be able to do better than or worse than the risk-free return on an initial investment of s_1 dollars. It's easy to see that if this condition is violated then an arbitrage is possible.

In this case a contingent claim is specified by a 3-vector $f = (f_2, f_3, f_4)$; here f_j is the payoff at maturity if the stock price is s_j. Question: which contingent claims are replicatable? Answer: those for which the system

$$\begin{aligned} \phi s_2 + \psi &= f_2 \\ \phi s_3 + \psi &= f_3 \\ \phi s_4 + \psi &= f_4 \end{aligned}$$

has a solution. This specifies a two-dimensional space of f's. So the market is not complete, and "most" contingent claims are not replicatable.

If a contingent claim f is not replicatable then arbitrage does not specify its price $V(f)$. However arbitrage considerations still *restrict* its price:

$$V(f) \leq \text{the value of any portfolio whose payoff dominates } f;$$
$$V(f) \geq \text{the value of any portfolio whose payoff is dominated by } f.$$

In other words,

$$\begin{array}{rcl} \phi s_2 + \psi & \geq & f_2 \\ \phi s_3 + \psi & \geq & f_3 \\ \phi s_4 + \psi & \geq & f_4 \end{array} \Longrightarrow V(f) \leq \phi s_1 + e^{-rT}\psi$$

$$\begin{array}{rcl} \phi s_2 + \psi & \leq & f_2 \\ \phi s_3 + \psi & \leq & f_3 \\ \phi s_4 + \psi & \leq & f_4 \end{array} \Longrightarrow V(f) \geq \phi s_1 + e^{-rT}\psi.$$

We obtain the strongest possible consequences for $V(f)$ by solving a pair of linear programming problems:

$$\max_{\substack{\phi s_j + \psi \leq f_j \\ j=2,3,4}} \phi s_1 + e^{-rT}\psi \leq V(f) \leq \min_{\substack{\phi s_j + \psi \geq f_j \\ j=2,3,4}} \phi s_1 + e^{-rT}\psi.$$

These bounds capture *all* the information available from arbitrage concerning the price of the contingent claim f. (The actual price of an option in an incomplete market must be determined by additional considerations besides arbitrage. As mentioned earlier, a standard approach involves discounted expected utility, see e.g. John Cochrane's book *Asset Pricing*.)

Every linear program has a dual linear program. If you don't already know something about duality, I suggest skipping the following, proceeding directly to the discussion of hedging by forwards. (If you'd like to read about duality for linear programming, Peter Lax's book *Linear Algebra* has a concise treatment. My favorite textbook on linear programing is V. Chvatal's *Linear Programming*; a more modern, more advanced choice is R. Vanderbei's *Linear Programming: Foundations and Extensions*.)

Recall that our upper bound for $V(f)$ was the optimal value of a linear programming problem. Let's find the form of the dual linear programming problem, by expressing the original problem as a min-max then interchanging the min and the max. The upper bound is:

$$\begin{aligned} \min_{\substack{\phi s_j + \psi \geq f_j \\ j=2,3,4}} \phi s_1 + e^{-rT}\psi &= \min_{\phi,\psi} \max_{\pi_j \geq 0} \phi s_1 + e^{-rT}\psi + \sum_{j=2}^{4} \pi_j (f_j - \phi s_j - \psi) \\ &= \max_{\pi_j \geq 0} \min_{\phi,\psi} \phi s_1 + e^{-rT}\psi + \sum \pi_j (f_j - \phi s_j - \psi) \end{aligned}$$

$$
\begin{aligned}
&= \max_{\pi_j \geq 0} \min_{\phi,\psi} \ \phi(s_1 - \sum \pi_j s_j) + \psi(e^{-rT} - \sum \pi_j) + \sum \pi_j f_j \\
&= \max_{\substack{\sum \pi_j s_j = s_1 \\ \sum \pi_j = e^{-rT} \\ \pi_j \geq 0}} \sum \pi_j f_j.
\end{aligned}
$$

The first line holds because

$$\max_{\pi_j \geq 0} \pi_j (f_j - \phi s_j - \psi) = \begin{cases} 0 & \text{if } \phi s_j + \psi \geq f_j \\ +\infty & \text{otherwise.} \end{cases}$$

The second line holds by the duality theorem of linear programming, which says in this setting that $\min\max = \max\min$. The third line is obtained by rearrangement, and the fourth line by an argument similar to the first.

The preceding argument is correct, but if you're not well-versed in duality theory then the assertion "min max = max min" may seem rather mysterious. To demystify it, let's explain by an entirely elementary argument why

$$\min_{\phi s_j + \psi \geq f_j} \phi s_1 + e^{-rT}\psi \geq \max_{\substack{\sum \pi_j s_j = s_1 \\ \sum \pi_j = e^{-rT} \\ \pi_j \geq 0}} \sum \pi_j f_j.$$

(The opposite inequality is more subtle; the main point of linear programming duality is to prove it.) Indeed, consider any ϕ and ψ such that $\phi s_j + \psi \geq f_j$ for each $j = 2, 3, 4$; and consider any $\{\pi_j\}_{j=2}^4$ such that $\pi_j \geq 0$, $\sum_{j=2}^4 \pi_j s_j = s_1$, and $\sum_{j=2}^4 \pi_j = e^{-rT}$. Multiply each inequality $\phi s_j + \psi \geq f_j$ by π_j, then add and use the hypotheses on π_j to see that $\phi s_1 + e^{-rT}\psi \geq \sum \pi_j f_j$. Minimizing the left hand side (over all admissible ϕ, ψ) and maximizing the right hand side (over all admissible π_j) gives the desired inequality.

Making the minor change of variables $\hat{\pi}_j = e^{rT}\pi_j$, our duality argument has shown that

$$
\begin{aligned}
V(f) \leq \max\{e^{-rT}[\hat{\pi}_2 f_2 + \hat{\pi}_3 f_3 + \hat{\pi}_4 f_4] \ &: \ \hat{\pi}_2 s_2 + \hat{\pi}_3 s_3 + \hat{\pi}_4 s_4 = e^{rT} s_1 \\
&\quad\ \hat{\pi}_2 + \hat{\pi}_3 + \hat{\pi}_4 = 1, \quad \hat{\pi}_j \geq 0\}.
\end{aligned}
$$

The lower bound is handled similarly. The only difference is that we are maximizing in ϕ, ψ and minimizing in π_j. An argument parallel to the one given above shows

$$
\begin{aligned}
V(f) \geq \min\{e^{-rT}[\hat{\pi}_2 f_2 + \hat{\pi}_3 f_3 + \hat{\pi}_4 f_4] \ &: \ \hat{\pi}_2 s_2 + \hat{\pi}_3 s_3 + \hat{\pi}_4 s_4 = e^{rT} s_1 \\
&\quad\ \hat{\pi}_2 + \hat{\pi}_3 + \hat{\pi}_4 = 1, \quad \hat{\pi}_j \geq 0\}.
\end{aligned}
$$

Thus the upper and lower bounds on $V(f)$ are obtained by maximizing and minimizing the "discounted expected return" $e^{-rT}[\hat{\pi}_2 f_2 + \hat{\pi}_3 f_3 + \hat{\pi}_4 f_4]$ over an appropriate class of "risk-neutral probabilities" $(\hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4)$. The incompleteness of the market is reflected in the fact that there is more than one risk-neutral probability: in the present trinomial setting the 3-vector $(\hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4)$ is constrained by two inequalities, so the class of risk-neutral probabilities is one-dimensional (a line segment in the three-dimensional space of triplets $(\hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4)$). The conditions determining this line segment are nonnegativity, plus the conditions that the risk-neutral probabilities price both the bond and the stock correctly.

We noted earlier the condition $s_2 < s_1 e^{rT} < s_4$, which is required for the economy to be "reasonable" – i.e. not to admit an arbitrage. This is precisely the condition that there be at least one risk-neutral probability, i.e. a triplet $(\hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4)$ such that $\sum \hat{\pi}_j = 1$, $\sum \hat{\pi}_j s_j = e^{rT} s_1$, and $\hat{\pi}_j > 0$ for each j.

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Topic (3): Hedging by forwards or futures. We return now to the one-period binomial setting. We showed at the beginning of this section that in the one-period binomial context, any contingent claim is equivalent to (is "replicated by") a suitable combination of a bond and the underlying. This led us to the "risk-neutral pricing formula" for the value of a contingent claim.

We now return to that calculation to see what happens if we hedge using a forward or futures contract rather than the underlying. There are both practical and theoretical reasons for doing this. *Practical:* Hedging might require taking a negative ("short") position. Some investors (e.g. mutual funds) are not permitted to take short positions; it might therefore be more convenient to sell forwards or futures instead. Forwards have a further advantage: if we only buy or sell forwards with delivery price equal to the forward price then no money changes hands at the time of sale; rather, money changes hands only at the settlement time. (However futures are more liquid than forwards; therefore real-world hedging would usually use futures not forwards.) *Theoretical:* in the multiperiod setting, option pricing formulas are simplest when expressed in terms of the forward price. The fundamental reason is again that if hedging is done using forwards then no money changes hands until the settlement time. (This is important when interest rates are random, because at time 0 we know the discount rate $B(0,T)$ but we don't know $B(t,T)$ for $0 < t < T$.)

It is convenient to change notation slightly. We write s_{now} (rather than s_1) for the initial stock price, and $s_{\text{down}} < s_{\text{up}}$ (rather than $s_2 < s_3$) for the two possible stock prices at time T. Similarly, we write $\mathcal{F}_{\text{now}}$, $\mathcal{F}_{\text{down}}$, and $\mathcal{F}_{\text{up}}$ for the forward prices in the initial, down, and up states.

Our one period market model has initial time 0 ("now") and final time T. Our forwards, however, need not have settlement time T. Rather, their settlement time can be any time $T' \geq T$. When (in Section 3) we price an option using a multiperiod model, it will be natural to let T' be the final time under consideration (the maturity date of the option). We shall consider only forwards with *delivery price equal to the forward price*, so the value is zero at the time of "purchase."

Recall that the forward price is fully determined by the price of the underlying: if the settlement time is T' and the interest rate is (constant) r, then $\mathcal{F}_{\text{now}} = e^{rT'} s_{\text{now}}$ while

$\mathcal{F}_{\text{down}} = e^{r(T'-T)} s_{\text{down}}$ and $\mathcal{F}_{\text{up}} = e^{r(T'-T)} s_{\text{up}}$. Also recall that (when the interest rate is constant) the futures price is the same as the forward price.

We want to explain the following points:

(a) The forward price satisfies $\mathcal{F}_{\text{now}} = q\mathcal{F}_{\text{up}} + (1-q)\mathcal{F}_{\text{down}}$, where q is the risk-neutral probability of the up state. (Remember: q is unique in the binomial setting.) Put differently: $\mathcal{F}_{\text{now}} = E_{\text{RN}}[\mathcal{F}_T]$.

(b) If a contingent claim has payoff f_{up} in the up state and payoff f_{down} in the down state, then its value initially is $V(f) = e^{-rT}[qf_{\text{up}} + (1-q)f_{\text{down}}]$.

(c) This contingent claim is replicated by a portfolio consisting of α units of the forward and a bond worth β at time T, where

$$\alpha = e^{r(T'-T)} \frac{f_{\text{up}} - f_{\text{down}}}{\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}}, \quad \beta = qf_{\text{up}} + (1-q)f_{\text{down}}.$$

(d) This contingent claim is alternatively replicated by a portfolio consisting of α futures contracts and a bond worth β at time T, where

$$\alpha = \frac{f_{\text{up}} - f_{\text{down}}}{\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}}, \quad \beta = qf_{\text{up}} + (1-q)f_{\text{down}}.$$

Concerning (a): Notice that there is no factor of e^{-rT} on the right hand side! That's not a typographical error – the forward price is *not* the price of a contingent claim. One can demonstrate (a) by simply writing all the forward prices in terms of $s_{\text{now}}, s_{\text{up}}, s_{\text{down}}$ then doing some algebraic manipulation. However we can get more insight by applying our pricing formula to a forward contract (with, by convention, delivery price $K = \mathcal{F}_{\text{now}}$ and settlement time T'). Its value at time 0 is 0. Let's find its "payoff" at time T. In the up state a forward with delivery price $\mathcal{F}_{\text{up}}$ would have been valueless, so our instrument is worth the same as a sure payment of $\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}}$ at time T'. Thus: its value in the up state is $e^{-r(T'-T)}(\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}})$. Similarly its value in the down state is $e^{-r(T'-T)}(\mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}})$. Now apply the pricing formula to conclude that

$$0 = e^{-rT'}\left[q(\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}}) + (1-q)(\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}})\right].$$

Algebraic rearrangement gives $\mathcal{F}_{\text{now}} = q\mathcal{F}_{\text{up}} + (1-q)\mathcal{F}_{\text{down}}$, as asserted. Notice that this formula determines the value of q:

$$q = \frac{\mathcal{F}_{\text{now}} - \mathcal{F}_{\text{down}}}{\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}}$$

which is of course equivalent to our previous formula $q = \frac{e^{rT}s_{\text{now}} - s_{\text{down}}}{s_{\text{up}} - s_{\text{down}}}$. (Comment to those who know enough that the following makes sense: in the multiperiod setting this argument shows that the forward price is a martingale. But we have used the hypothesis that the interest rate is constant, or at least deterministic. When the interest rate is random, it is the futures price not the forward price that's a martingale under the risk-neutral measure.)

Concerning (b): this is of course just a restatement of our familiar pricing formula. We have restated it because the present discussion gives an independent proof. In fact, (b) follows from (c) or (d) since the initial value of the forward or futures contract is 0 and the discount factor for the bond is e^{-rT}.

Concerning (c): Consider the portfolio consisting of α units of the forward and a bond worth β at time T. It has the same value as the contingent claim at time T if

$$\alpha e^{-r(T'-T)}(\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}}) + \beta = f_{\text{up}} \quad \text{and} \quad \alpha e^{-r(T'-T)}(\mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}}) + \beta = f_{\text{down}}.$$

These are two equations in two unknowns (α and β), so we should be able to solve for α and β. But rather than proceed blindly, let's be clever. Multiplying the first equation by q and the second by $1 - q$ then adding gives

$$\beta = q f_{\text{up}} + (1-q) f_{\text{down}},$$

while subtracting the two equations gives

$$\alpha e^{-r(T'-T)}(\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{down}}) = f_{\text{up}} - f_{\text{down}}.$$

Simplification yields (c).

Concerning (d): The futures contract has initial value 0, and its value at time T is $\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}}$ in the up state, and $\mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}}$ in the down state. So α futures and a bond worth β at time T replicates the contingent claim if

$$\alpha(\mathcal{F}_{\text{up}} - \mathcal{F}_{\text{now}}) + \beta = f_{\text{up}} \quad \text{and} \quad \alpha(\mathcal{F}_{\text{down}} - \mathcal{F}_{\text{now}}) + \beta = f_{\text{down}}.$$

This is identical to the system considered in (c) with $T' = T$. Therefore the proper values of α and β are as given by (d).

You might be puzzled. We promised a discussion of hedging, whereas the preceding discussion appears to be about replication and pricing. But it also tells us how to hedge. Indeed, suppose you buy a contingent claim with payoff f. Suppose further that at the same time you "sell" α units of the forward contract (receiving nothing, since the delivery price is the forward price), with α as given by (c). Then you have entirely eliminated all risk: your position is equivalent to holding a bond worth $\beta = q f_{\text{up}} + (1-q) f_{\text{down}}$ at time T. The same applies if instead you "sell" α futures contracts, with α given by (d).

The discussion can also be viewed differently, as a prescription for finding an arbitrage if the option is mispriced. Indeed, if the market price P of the option is different from $e^{-rT}\beta = V(f)$, our calculation identifies a risk-free portfolio (combining the option with a position in forwards or futures) whose return is different from the risk-free rate. If the return is larger than the risk-free rate, the associated arbitrage opportunity involves borrowing money and buying this portfolio. If it is smaller, then it involves selling this portfolio and lending the resulting income at the risk-free rate.

Food for thought:

1. How does this specialize to a contingent claim whose payoff is the same in both states, i.e. for which $f_{\text{up}} = f_{\text{down}}$?

2. Does it matter whether the value of the contingent claim increases or decreases when the stock price increases ($f_{\text{up}} > f_{\text{down}}$ versus $f_{\text{up}} < f_{\text{down}}$)?

3. Does it matter whether the payoffs of the contingent claim (f_{up} and f_{down}) are positive or negative?

4. Does the size of changes in the stock price or claim value matter (e.g. does anything change if $f_{\text{up}} - f_{\text{down}}$ or $s_{\text{up}} - s_{\text{down}}$ is very large or very small)?

5. What would things be different if we knew at the initial time whether the stock was going up or down?

6. Have we made any use of the actual probability that stock goes up or down?

7. A one-period trinomial model reduces to a one-period binomial model if the probability of one of the final states is set to zero. Yet the trinomial one-period market was incomplete, while the binomial one-period market was complete. Why is this not a contradiction?