Derivative Securities - Fall 2012 - Section 12

The relevant parts of Hull are:

- chapter 8, on structured credit products + the recent financial crisis
- sections 23.9 - 23.10, on Gaussian copula approach to default correlations
- section 24.3 about CDS indexes
- sections 24.8 - 24.10 on CDO's

In §11 we focused on bonds issued by a specific corporation, or CDS on a single entity. Therefore only the prob of default by *this* entity was relevant.

In multiname setting, *correlation* btwn defaults is also importnt.

Example: Suppose you hold bonds issued by 10 companies, + each has default prob of 1% in 1st yr. If *independent*, then

prob of exactly k defaults in 1st yr

$$= \binom{10}{k} (.01)^k (.99)^{10-k}$$

so, for example, prob that *all* default is $(.01)^{10} = 10^{-20}$. (Exceedingly small!)

If fully correlated, then either there's no default or they all default; in this case

prob that all default in 1st year $= .01$ (not so small!)

Why do we care?

(1) A bank with many credit exposures must estimate its "value at risk." This amounts to estimating the likelihood of a very large loss (due to defaults) within a specific time period.

(2) An investor may seek to protect itself in a general, diversified way from deterioration of credit environment; or to profit by selling such protection. The standard tool is a basket CDS or index CDS contract.

Index CDS works like this:

- there's a specific list of companies (eg CDX NA IG corresp to list of 125 investment grade companies in North America)

- contract behaves like an (equally-weighted) CDO on each company. For example:

1^{st} default $\Rightarrow$ seller of protection pays $L(1-R)\cdot\frac{1}{125}$ to purchaser, and "spread payments" are reduced to $\frac{124}{125}$ of initial amount

2^{nd} default $\Rightarrow$ seller pays another $L(1-R)\cdot\frac{1}{125}$ to purchaser, + "spread payments" are reset to $\frac{123}{125}$ of initial amount.

To value this, we need expected # defaults at each payment date.

Note: recent huge trading loss by JPMorgan Chase involved contracts of this type. (Why the big loss? Huge trades $\Rightarrow$ they moved the market $\Rightarrow$ price paid was artificially high $\Rightarrow$ after a while they had a huge loss.)

(3) Before 2007, CDO's (collateralized debt obligations) backed by mortgages were widely

created + traded. Their <u>purpose</u>: to create high-quality bonds out of lower-quality loans (eg subprime). Also used for credit card debt, etc. Much less-used since 2008

<u>How it works</u>:

[collection of lower-quality loans]
total prin = 100M

senior tranche,
principal 80M
pays Libor + 60 basis pts

mezzanine tranche
principal 15M
pays Libor + 250 basis pts

Equity tranche
principal 5M
pays Libor + 2000 basis pts

(Hull, fig 8.1); here

equity tranche bears 1st 5% of defaults (after which it's dead);

mezzanine tranche bears next 15% of defaults (then it too is dead)

senior tranche starts seeing losses if defaults exceed 20%.

in practice there could be many more tranches

Typically (based on hypoth of not-too-large correlation) senior tranche would be high quality even if underlying instruments aren't. So: it's a way to draw capital that otherwise wouldn't have been available.

Worked badly for many reasons:

- correlations in 2007-2008 were much greater than mkt assumed
- creator of underlying loans has no incentive to be honest or maintain quality

(4) Also pre-2007: people also created synthetic CDO's, which were essentially tranched versions of CDS's on a portfolio.

eg a) equity tranche earns higher spread but bears full force of initial defaults (up to 5% say)

b) mezzanine tranche earns intermed spread + bears force of defaults beyond 5% up to 15% (say)

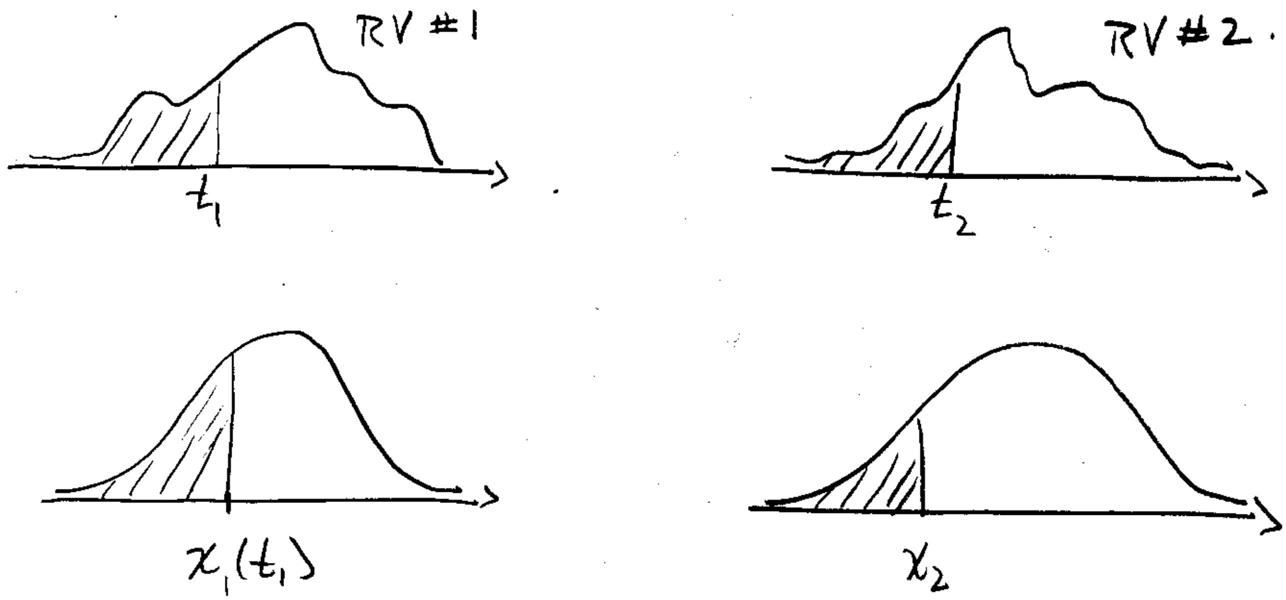
c) senior tranche gets smallest spread + bears force of defaults only after 20%

why? Basically same reasons as regular CDS's in a portfolio. But much more complex to price or hedge.

Why were they popular? Mostly as a speculative tool, I think!

Stepping back: tranched instruments are no longer so popular, but correlations btwn defaults are still imptnt for basket or index CDS, or estimating VAR on a debt portfolio.

Basic tool: <u>Gaussian</u> <u>copula</u> permits discn of correlation btwn two unrelated, non-Gaussian RV's.



define $x_1(t_1)$ by

$$N(x_1(t_1)) = \text{prob that RV\#1} \leq t_1$$

+ similarly $X_2(t_2)$ (using RV#2). Now assume x_1 + X_2 are <u>bivariate</u> <u>normal</u> with corr. ρ.

Let's apply this viewpoint to estimate default risk involving a pool of entities, assuming for simplicity

- <u>homogeneity</u> (all entities are equivalent)
- <u>a one-factor model</u> (to be explained below)

(Think of the common factor as the "state of the economy.")

- <u>The pool is large</u> (permitting us to use law of large nos)

Goal is an expression for fraction of the pool that has defaulted by time t (or just the expected number of defaults by time t), given inputs

- prob that a single entity defaults by time t is given, say $D = D(t)$
- correlation ρ (and Gaussian copula).

Let x_j be the standard Gaussian assoc to the j^{th} entity. Then

- j^{th} entity defaults by $t \iff x_j < N^{-1}(D)$
- $x_j = \sqrt{\rho}\, F + \sqrt{1-\rho}\, Z_j$ (the one-factor hypothesis)

where F + Z_j are indep standard Gaussians
(Note that $\mathbb{E}\, x_j = 0$, $\mathbb{E}(x_j^2) = 1$, $\mathbb{E}(x_j x_\ell) = \rho$.)

<u>So</u>: if value of F is fixed, then

name j defaults $\iff x_j < N^{-1}(D)$

$$\iff Z_j < \frac{N^{-1}(D) - \sqrt{\rho}\, F}{\sqrt{1-\rho}}$$

$\iff$ prob of default is

$$N\left(\frac{N^{-1}(D) - \sqrt{\rho}\, F}{\sqrt{1-\rho}}\right).$$

To find prob that any indiv. credit defaults we're thus left to do a numerical integration

$$\text{prob} = \int N\left(\frac{N^{-1}(D) - \sqrt{\rho}\, f}{\sqrt{1-\rho}}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-f^2/2}\, df$$

If pool is large, it's a decent approxn to identify this with the <u>fraction</u> of the pool that has defaulted by time t.

Uses of this:

a) for estimating VAR:

what is the likelihood that 10% of the pool defaults by time t?

$\Updownarrow$ (under large-pool approxn)

what is the likelihood that

$$N\left(\frac{N^{-1}(D) - \sqrt{\rho}\,F}{\sqrt{1-\rho}}\right) > .1$$

$\Updownarrow$

what is the likelihood that a Gaussian F has

$$F < \frac{N^{-1}(D) - \sqrt{1-\rho}\;N^{-1}(.1)}{\sqrt{\rho}}$$

b) for pricing a CDS on an index:

protection purchaser pays

$$L\,\frac{s}{f}\;\mathbb{E}\left[\begin{matrix}\text{fraction of names}\\ \text{still alive at } t_j\end{matrix}\right] \quad \text{at } t_j$$

and receives

$$L(1-R)\left(\mathbb{E}\left[\begin{matrix}\text{fraction alive}\\ \text{at } t_{j-1}\end{matrix}\right] - \mathbb{E}\left[\begin{matrix}\text{fraction alive}\\ \text{at } t_j\end{matrix}\right]\right)$$

due to defaults, plus "accrued interest"

$$L \frac{1}{2f}\left(\mathbb{E}\left[\substack{\text{fraction alive}\\ \text{at } t_{j-1}}\right] - \mathbb{E}\left[\substack{\text{fraction alive}\\ \text{at } t_j}\right]\right)$$

(at time $(t_{j-1}+t_j)/2$, by the usual fudge).

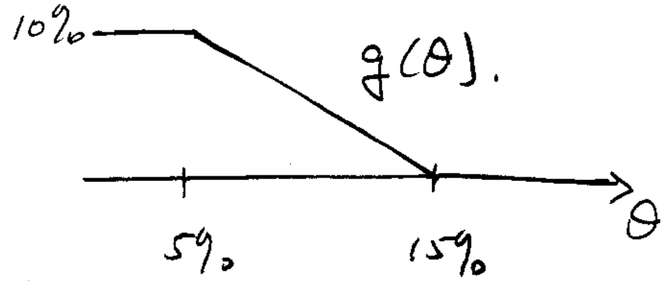
Note that calcs of $\mathbb{E}[\text{fraction alive at } t_j]$ are all "the same" (as explained above), except for use of different values of D = prob that any single name has defaulted by time t.

c) valuation of a tranched CDO is only a little different. Let

$$\theta_t(F) = N\left(\frac{N^{-1}(D_t) - \sqrt{\rho}\,F}{\sqrt{1-\rho}}\right)$$

= fraction of defaults by time t, given F (under large-pool approxn)

Then eg for mezzanine tranche in our example, protection purchaser's payment at time at t_j is $L \frac{1}{f}\, g(\theta_{t_j})$



which gets valued as

$$L \frac{1}{f} \mathbb{E}\left[g(\theta_{t_j}) \right] B(0, t_j)$$

Gaussian expectation, since θ_t is a fn of the Gaussian RV F

and protection purchaser receives (at $\frac{t_j + t_{j-1}}{2}$)
$L(1-R) + L\frac{1}{2f}$ times $g(\theta_{t_{j-1}}) - g(\theta_{t_j})$,
valued using

$$\mathbb{E}\left[g(\theta_{t_{j-1}}) \right] - \mathbb{E}\left[g(\theta_{t_j}) \right]$$

(were expectations wrto Gaussian F).

For all these calcns, choice of ρ is crucial. Lesson of JPM Chase blowups and also 2007-8 crisis: inferring ρ from mkt data is dangerous (the market could have wrong beliefs, or be skewed by supply/demand issues, for example.)