Section 11 – picking up where the typed notes leave off – Derivative Securities, Fall 2012

Pricing corporate bonds + credit default swaps

Must use defaultable discount rate

$\tilde{B}(0,T)$ = value today of a note worth \$1 at time T provided issuer hasn't defaulted by then

Modeling hypothesis:

$$\tilde{B}(0,T) = S_T \, B(0,T)$$

S_T: probability of survival to time T; $B(0,T)$: risk-free discount rate

Note: since

$$\tilde{B}(0,T) = \mathbb{E}_{RN}[\text{discounted payoff}]$$

we see

a) that S_T = risk neutral prob of survival to time T

b) essence of the "modeling hypothesis" is that default is independent of changes in interest rates

Evidently, modeling hypothesis leaves us with task of extracting S_T from market data.

A little more language:

d_i = prob of default between t_{i-1} + t_i, given survival up to t_i

$$t_0 = 0 \quad t_1 \quad t_2 \quad \cdots \quad t_N.$$

so $S_0 = 1$, $S_{i+1} = S_i(1 - d_{i+1})$

(Thus eg d_1 = prob of default in (t_0, t_1), $S_1 = (1-d_1)$ is prob of survival to t_1, $S_1 d_2$ = prob of default in (t_1, t_2), $S_2 = S_1 - S_1 d_2$ = prob of survival to t_2, etc)

New twist rel to risk-free bonds: a bond's coupon payments cease to happen after default (creditors cannot claim these in bankruptcy court) but some part of principal will be recovered (after liquidation or other bankruptcy settlement).

Value of a corporate bond: assume

principal L, fixed rate c,
payment frequency f (times/year)
recovery rate R
payment dates t_i (maturity t_N)

$$\Rightarrow \text{value} = \left[\sum_{i=1}^{N} \frac{c}{f} S_i B(0,t_i) + S_{t_N} B(0,t_N)\right] L + \sum_{i=1}^{N} S_{i-1} d_i B(0,t_i^*) R L$$

using notation $t_i^* = \frac{t_i + t_{i-1}}{2}$.

Term involving R is the "best estimate" for present value of the (partial) recovery of principal, if default occurs. (Why t_i^*? Well, default is presumably equally likely at any time in (t_{i-1}, t_i).)

Terminology: par coupon rate = the value of c that makes the bond's value $= L$

<u>Value a CDS</u>: only slightly different. Assume L, f, R, t_i, t_i^* are as above. Let fixed payments be at rate s (the "spread"). Then

Value of fixed payments (paid by buyer of protection) is

$$V_{fix} = L \sum_{i=1}^{N} \frac{s}{f} S_i B(0,t_i)$$

If there's a default, seller of protection pays buyer $L(1-R)$ at time of default (assuming, say, cash settlement) & receives accrued interest owed up to moment of default. So

$$V_{float} = L \sum_{i=1}^{N} S_{i-1} d_i B(0,t_i^*) \left[(1-R) - \frac{1}{2}\frac{s}{f}\right]$$

estimate of accrued interest

Present value of swap is $V_{fix} - V_{float}$. (This is the value to the seller, i.e. provider, of protection.)

Par CDS spread is the value of s that makes $V_{fix} - V_{float} = 0$. (Note that $V_{fix} - V_{float}$ has the form $as - b$, so $s = \frac{b}{a}$.)

See Hull §24.2 for a numerical example

using precisely this framework.

Variations are possible depending on details of CDS contract (eg it might provide a lump-sum payment instead of delivery of a bond)

Note that all the "default probabilities" above (typically extracted from mkt prices of bonds or swaps) are risk-neutral probs.

They are not the same as real-world probs. Usually

real-world default prob < RN default prob

(see §23.5 of Hull); in other words, defaultable bonds are cheaper than the historical default rates would suggest.

Possible reasons (somewhat speculative):

- "flight to quality" clearly had an effect during 2007-2008 crisis
- risk assoc with bonds is hard to diversify away; also, bond defaults

are correlated to rest of economy
=> can't be diversified away => by
CAPM, investors should be compensated
for assuming such risk.

So far, we left default probs to be
extracted from bond or CDS prices.

But why not try to get them from stock
prices?

Reasons to try:

a) stock prices are clearly related
(stock becomes worthless upon default)

b) stock prices are liquid + very
visible

c) Some instruments even combine elements
of stock + bond (eg convertible bonds)

First attempt to do this was by Black + Scholes
(1973) + Merton (1974) - now known as "Merton's

model." A related idea is in widespread current use: the CreditGrades model, developed about 2002 by Deutsche Bank, Goldman Sachs, JPMorgan, + RiskMetrics. Let's discuss both.

The Merton Model (Merton, J Finance 29, 1974, 449-470) views company's equity as an option on its assets. Let

V_0 = value/share of company's assets today
V_T = " " " " at time T

KNOWN → E_0 = value/share of company's stock today
E_T = " " " " at time T

KNOWN → D = debt/share the company owes (assumed const)

σ_V = volatility of V (assumed constant).
KNOWN → σ_E = volatility of stock price

<u>Basic hypothesis</u>: $E_T = (V_T - D)_+$

since if $V_T - D < 0$ the company is bankrupt at time T;
else value of equity $= V_T - D$.

Consequences, by Black-Scholes:

a) $E_0 = V_0 N(d_1) - De^{-rT} N(d_2)$

where $d_1 = \dfrac{\ln(V_0/D) + (r + \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}$ $\quad d_2 = d_1 - \sigma_V \sqrt{T}$; also

b) RN prob of default $= 1 - N(d_2)$.

Need inputs σ_V + V_0

• How to get σ_V? By Ito,

$E(t) = E(V(t), t)$ given by Black-Scholes formula for a call option

$\Rightarrow dE = \dfrac{\partial E}{\partial V} dV + (\text{stuff})\, dt$

$\Rightarrow \sigma_E E = \dfrac{\partial E}{\partial V} \sigma_V V$ by comparing coeffts of "dw" on both sides of preceding SDE

Also: $\dfrac{\partial E}{\partial V} = N(d_1)$ from our discn of the 'Greeks'; and σ_E is visible in market as implied volatility of options.

So

$$\sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0 = N(d_1) \sigma_V V_0$$

is a nonlinear for relating V_0 and σ_V to market observables.

Stepping back: we now have two nonlinear eqns in V_0 and σ_V

$$\boxed{\begin{aligned} E_0 &= V_0\, N(d_1) - D e^{-rT} N(d_2) \\ \sigma_E E_0 &= N(d_1)\, \sigma_V\, V_0 \end{aligned}} \qquad d_1 = \frac{\ln(V_0/D) + (r + \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}$$

which we can expect to solve (eg by Newton's method), given observed values of r, E_0, σ_E, and D.

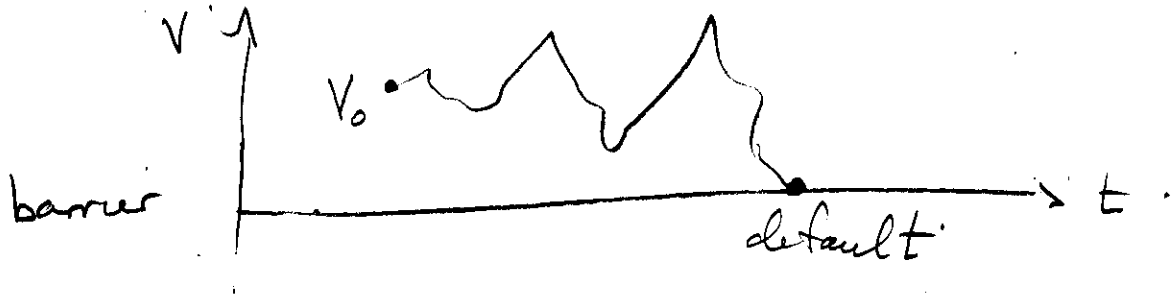
How well does this work? Well, it's not reasonable to take $1 - N(d_2)$ as the "exact" RN default prob. (For example: predicted probs of default in 1st year are much too low.) But the ordering of default probs obtained this way is about right (as the model is applied to different firms).

Easy to criticize: eg doesn't distinguish between default at T + default before T. Also, why should $V(t)$ be lognormal? (Defaults often occur due to surprises; why shouldn't $V(t)$ have jumps?).

The CreditGrades model strives to keep the simplicity of Merton's model but make it more predictive. Described by "CreditGrades Technical Document", C. Finger et al, Riskmetrics 2002 (easy to find via google).

Starts like Merton's model, however:

- $dV = \sigma_V V\,dw$ (ie assume drift $= 0$).
- view default as occurring when $V(t)$ hits a barrier



- Merton took barrier D; here we take barrier to be random (ie not entirely known).

$$\text{barrier} = \bar{L} D e^{\lambda Z - \frac{1}{2}\lambda^2} \qquad Z = \text{standard Gaussian}$$

where we view

$$L = \bar{L} e^{\lambda Z - \frac{1}{2}\lambda^2}$$

(lognormal, with mean $\bar{L}$ + vol λ).

as the <u>recovery rate</u> (so $\bar{L}$ and λ are obtainable from market data).

2nd bullet says

default time = 1st time that $V_0 e^{\sigma_V W(t) - \sigma_V^2 t/2}$ crosses barrier

$= 1^{st}$ time that X_t crosses $\log\left(\frac{\bar{L}D}{V_0}\right) - \lambda^2$

where $\quad X_t = \sigma_V W_t - \lambda Z - \frac{\sigma^2 t}{2} - \frac{\lambda^2}{2}$.

<u>There are formulas for such things</u>

We still have to relate V_0 and σ_V to market observables; $V_0 = E_0 + \bar{L}D$, but what abt σ_V ?

Recall from arg't for Merton that

$$E\,\sigma_E = \sigma_V V \frac{\partial E}{\partial V}$$

but we no longer have such simple dependence of E on V. Argue instead as follows:

define "distance to default" η by

$$\eta = \frac{1}{\sigma_V} \log\left(\frac{V}{\bar{L}D}\right).$$

("# of standard deviations away from default")

$$= \frac{V}{\sigma_E E} \frac{\partial E}{\partial V} \log\left(\frac{V}{\bar{L}D}\right)$$

and assume

$$\eta = \frac{E + \bar{L}D}{\sigma_E E} \log\left(\frac{E + \bar{L}D}{\bar{L}D}\right)$$

(why? well, it has plausible behavior in the limits $E \ll \bar{\mathcal{L}}D$ and $E \gg \bar{\mathcal{L}}D$; see §2.2 of the CreditGrades technical document)

So combining the 1st & last expr for σ_V we get

$$\frac{1}{\sigma_V} = \frac{E_0 + \bar{\mathcal{L}}D}{\sigma_E E} \quad \text{ie} \quad \boxed{\sigma_V = \frac{\sigma_E E_0}{E_0 + \bar{\mathcal{L}}D}}$$

(RHS is known!)

For some examples + descr., see e.g. "An empirical implementation of CreditGrades", Journal of Credit Risk 6 (2010) 89–98, by Andy Jia-Yuh Yeh [apparently done as a MS project for the Haas School's MFE program]