Derivative Securities, Fall 2012 – Homework 2. *Revised 9/27/2012, changing problem 6.* Originally distributed 9/20/2012. Due by classtime on 10/3/2012.

(1) Which functions $f(S_T)$ can be the value-at-maturity of a portfolio of calls? By this I mean a portfolio consisting of a_i call options with strike price price K_i, $1 \leq i \leq N$, all having the same maturity date T. (We permit short as well as long positions, i.e. a_i can be positive or negative. We may suppose $0 < K_1 < \ldots < K_N$. The value of this portfolio at maturity is $f(S_T) = \sum_{i=1}^{N} a_i (S_T - K_i)_+$.)

   (a) Show that f is a continuous, piecewise linear function of S_T, with $f(S_T) = 0$ for S_T near 0, and $f(S_T) = a_\infty S_T + b_\infty$ when S_T is sufficiently large.

   (b) Show that any such f can be realized by a suitable portfolio, and the portfolio is uniquely determined by f. (Hint: think about the graph of f. How does it determine K_i and a_i?)

   (c) Show that $a_\infty = \sum_{i=1}^{N} a_i$ and $b_\infty = -\sum_{i=1}^{N} a_i K_i$.

(2) Context: you believe there's a 40% chance that Google will attempt to take over Facebook a month from now. Facebook is presently trading at \$25/share, and the forward price for delivery a month from now is also \$25/share. (This is consistent with the fact that Facebook pays no dividend, and the risk-free rate is essentially 0.) Assume that if, a month from now, Google attempts the takeover, Facebook stock will jump to \$35/share; if not, the stock will (in line with its recent slide) be worth just \$20/share.

   Now the problem: suppose you hold 1000 European calls on Facebook with strike \$25 and maturity a month from now, and you wish to fully hedge the calls using forward contracts. What, exactly, should you do? (Please use forwards with delivery price = the forward price, and delivery time = a month from now.)

(3) Consider a one-period market in which the stock price is described by a trinomial tree: the price today is $s_0 = 100$; at the final time T the three possible states are $s_u = 120$, $s_m = 90$, and $s_d = 80$. Assume the risk-free rate is $r = 0$. Consider an option whose payoff is $f_u = 30$ if the final-time state is s_u, $f_m = 0$ if the final-time state is s_m, and $f_d = 0$ if the final-time state is s_d. We're interested in the value f_0 of the option today.

   (a) Show that this option cannot be replicated by a combination of stock and risk-free bond. (So f_0 cannot be determined by considerations based only on the absence of arbitrage.)

   (b) Consider changing the problem by altering the value of f_m (leaving all other numbers unchanged). For which choice(s) of f_m would the payoff be replicatable? (Equivalently: for which f_m could the option be valued using only the absence of arbitrage?)

   (c) The Section 2 notes discuss arbitrage-based bounds on f_0, using portfolios that either super-replicate or sub-replicate the option. Evaluate those bounds, for the given option (with $f_m = 0$, as originally stated). (Hint: the Section 2 notes formulate the upper and lower bounds as linear programming problems, and also discuss the associated dual problems. Part (c) can be done *either* by solving the linear programming problems *or* by solving the dual problems.)

(4) This problem concerns the pricing of a European option in a two-period multiplicative binomial tree market. Suppose the present stock price is $s_0 = 50$, the time interval is $\delta t = 6$ months, and the multiplicative factors are $u = 1.2$, $d = 0.8$ (so the stock price 6 months from now is either $s_0 u$ or $s_0 d$, and the stock price 12 months from now is either $s_0 u^2$, $s_0 ud$, or $s_0 d^2$). Assume the risk-free rate is 4 percent per annum.

(a) Consider a European put with strike 45 and maturity 1 year. Find its value now, by working backward through the tree.

(b) The Section 3 notes discuss a "formula" for value of an option, as the discounted expected payoff using a suitable binomial distribution. What does that formula specialize to in the present case? Check that it gives the same value you got in part (a).

(c) Describe the trading strategy that replicates this option, using stock and the risk-free bond. (In particular: specify how many shares of stock you should hold at each node after rebalancing.)

(5) Here is another problem concerning the pricing of a European option in a two-period multiplicative binomial tree market. Suppose the present forward price (for delivery 1 year from now) is $\mathcal{F}_0 = 100$, the time interval is $\delta t = 6$ months, and the multiplicative factors are $u = 1.3$ and $d = 0.7$ (so the forward price 6 months from now is either $\mathcal{F}_0 u$ or $\mathcal{F}_0 d$, and the price a year from now is either $\mathcal{F}_0 u^2$, $\mathcal{F}_0 ud$, or $\mathcal{F}_0 d^2$). Assume the risk-free rate is 4 percent per annum.

(a) Consider a European call with strike 100 and maturity 1 year. Find its value now, by working backward through the tree.

(b) Describe the trading strategy that replicates this option, using forwards (with delivery price = forward price, and delivery time = maturity date of the option) and the risk-free bond. (In particular: specify how many forwards you should hold at each node after rebalancing.)

(c) Suppose the market price of the option is 16. This is different from what you (should have) obtained in (a), so there is an arbitrage. Explain, by specifying how one can take advantage of this mispricing to obtain a risk-free profit.

(6) *Revised 9/27/2012* At the end of the Section 3 notes, we discussed the pricing and hedging of an American put in a particular two-period multiplicative binomial market. (The initial stock price was $s_0 = 100$; the multiplicative factors were $u = 2$ and $d = 1/2$; and the risk-free rate was such that $e^{r\delta t} = 3/2$.) Let's value and hedge another American put using the same market model.

(a) Consider an American put with strike 200, maturing at the end of the second time period. What is its value at the initial time? *(Comment: you'll find that an investor holding this option should exercise it at the initial time. Of course such an option would not usually be found in the marketplace.)*

(b) *Suppose you're the bank that issued this option, and the investor holding the option failed to exercise it at the initial time as he should have. Describe a trading strategy you can use, involving stock and the risk-free bond, that takes advantage of the investor's mistake to achieve a risk-free profit.*

(c) *Same question as part (b), but this time specify a trading strategy involving forwards (with delivery price = forward price and delivery time = maturity time of the option) and the risk-free bond.*