**Derivative Securities – Fall 2007 – Section 9**
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**Interest-based instruments: bonds, forward rate agreements, and swaps.** This section provides a fast introduction to the basic language of interest-based instruments, then introduces some specific, practically-important examples, including forward rate agreements and swaps. Most of this material can be found in Hull (chapters 4, 6, and 7).

Steve Allen's version of these notes starts with several pages of background about bonds and bond markets, including how the LIBOR (London Interbank Offering Rate) is set, why we can basically ignore credit risk when working with LIBOR rates, why it's better to work with LIBOR rather than US Treasury rates, etc. I recommend reading that material, but I won't repeat it here.

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**Bond prices and term structure.** The time-value of money is expressed by the *discount factor*

$$B(t, T) = \text{value at time } t \text{ of a dollar received at time } T.$$

This is, by its very definition, the price at time $t$ of a zero-coupon bond which pays one dollar at time $T$. If interest rates are stochastic then $B(t, T)$ will not be known until time $t$. Prior to time $t$ it is random – just as in our discussion of equities, a stock price $s(t)$ or a forward price $\mathcal{F}(t)$ was random. Note however that $B(t, T)$ is a function of *two* variables, the initiation time $t$ and the maturity time $T$. Its dependence on $T$ reflects the *term structure* of interest rates. (The forward prices in our prior discussions also had a term structure – the forward price depends on the settlement date – but the settlement date was usually held fixed. With interest rates, by contrast, we'll be considering many maturities simultaneously.) We usually take the convention that the present time is $t = 0$; thus what is observable now is $B(0, T)$ for all $T > 0$.

A central principle for dealing with the term structure of interest rates is to use the same discount factor $B(t, T)$ for any cash flow that occurs at time $T$, regardless of what other cash flows it might be packaged with. For example, a cash flow occurring on June 15, 2012 will the discounted by the same factor whether it is a principal payment on a zero-coupon bond, a coupon payment on a 10 year bond, or a coupon payment on a 20 year bond. This principle follows from the law of one price; if it didn't hold we would be able to design arbitrage strategies, making money without taking any risk.

There are several equivalent ways to represent the time-value of money. The *yield* $y(t, T)$ is defined by

$$B(t, T) = e^{-y(t,T)(T-t)};$$

it is the unique constant interest rate that would have the same effect as $B(t, T)$ under continuous compounding. The *term rate* $R(t, T)$ is defined by

$$B(t, T) = \frac{1}{1 + R(t, T)(T - t)};$$

it is the unique interest rate that would have the same effect as $B(t,T)$ with no compounding. One can also define a term rate based on a specified compounding. For example, with annual compounding $R(t,T)$ will be the rate such that

$$B(t,T) = \frac{1}{[1 + R(t,T)]^{(T-t)}}.$$

The *instantaneous forward rate* $f(t,T)$ is defined by

$$B(t,T) = e^{-\int_t^T f(t,\tau)\,d\tau};$$

it is the unique deterministic time-varying interest rate that describes all the discount factors with initial time $t$ and various maturities.[1] We can easily solve for $y(t,T)$, $R(t,T)$, or $f(t,T)$ in terms of $B(t,T)$. Therefore each contains the same information as $B(t,T)$ as $t$ and $T$ vary. (Let us also mention one more: the discount rate $I(t,T)$, defined by $B(t,T) = 1 - I(t,T)(T-t)$. It has little conceptual importance; however interest rates for US Treasury bills are usually presented by tabulating these discount rates.)

Most long-term bonds have *coupon payments* as well as a *final payment*. The value of the bond at time 0 is the sum of the present values of all future payments. For a *fixed-rate* bond the coupon payments (amount $c_j$ at time $t_j$) are fixed in advance, as is the final payment (amount $F$ at time $T$). The value of the bond at time $t$ is thus

$$\text{cash price} = \sum c_j B(t,t_j) + FB(t,T).$$

This is known as the *cash price*; it is a consequence of the principle of no arbitrage. Notice that the cash price is a discontinuous function of time: it rises gradually between coupon payments, then falls abruptly at each coupon date $t_j$ because the holder of the bond collects the coupon payment. The cash price is not the value you'll see quoted in the newspaper. What you find there is the difference between the cash price and the interest accrued since the last coupon date:

$$\text{quoted price} = \sum c_j B(t,t_j) + FB(t,T) - \text{accrued interest}.$$

Notice that the quoted price is a continuous function of time, since the accrued interest is discontinuous (it resets to zero at each coupon payment) and the two discontinuities cancel. Another name for the cash price is the *dirty price*; another name for the quoted price is the *clean price*.

A *floating-rate* bond is one whose interest rate (coupon rate) is reset at each coupon date. By definition, after each coupon payment its value returns to its *face value*. A typical example is a one-year floating-rate note with semiannual payments and face value one dollar, pegged to the LIBOR (London Interbank Offer) rate. Suppose at date 0 the LIBOR term interest rate for six-month-maturity is 5.25 percent per annum, but at the six-month reset the LIBOR six-month-maturity rate has changed to 5.6 percent per annum. Then the coupon payment due at six months is $.0525/2 = .02625$, and the coupon payment due at one year

[1] Do not confuse this with the forward term rate, introduced below and called $f_0(t,T)$.

is $.056/2 = .028$; in addition the face value (one dollar) is repaid since the bond matures. Note the convention: interest is paid at the end of each period, using the interest rate set at the beginning of the period.

The value of the fixed-rate bond was the discounted value of its future income stream. The same is true of the floating-rate bond, provided that we *discount using the LIBOR rate*. In other words for this purpose $B(t,T)$ should be the value at time $t$ of a LIBOR contract worth one dollar at time $T$. In fact, the value of the floating-rate bond at six months (just after the first coupon payment) is the value at that time of the payments to be made at one year. If $t$ is six months and $T$ is 1 year then this is

$$B(t,T)(.028+1) = \frac{1}{1+.028}(.028+1) = 1.$$

The bond could be sold for this value – so holding it at six months is exactly the same as having one dollar of income at six months. The value of the bond at time 0 is similarly

$$B(0,t)(\text{first coupon} + \text{value at six months}) = \frac{1}{1+.02625}(.02625+1) = 1.$$

Our calculation is clearly not special to the example; it resides in the fact that $B(t,T) = 1/(1+R(t,T)(T-t))$.

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**Duration**. A useful tool for capturing the dependence of a bond price on interest rates is the *duration* of the bond.

Consider the derivative of the value of a single coupon payment with respect to the yield implicit in the associated discount rate. The present value of the coupon payment is $c_jB(0,t_j) = c_je^{-y(0,t_j)t_j}$, so its derivative with respect to a change in the rate $y(0,t_j)$ is $-t_jc_je^{-y(0,t_j)t_j} = -t_jc_jB(0,t_j)$. The derivative of the bond price with respect to a change in *each* yield will be just the sum of these individual derivatives:

$$\sum -t_jc_jB(0,t_j) \; + \; (-T)FB(0,T).$$

The *duration* of a set of cash flows is defined as the weighted average of the time to maturity the cash flows, using weights proportional to the present values. For a bond, the duration is:

$$\frac{\sum t_jc_jB(0,t_j) \; + \; TFB(0,T)}{\sum c_jB(0,t_j) \; + \; FB(0,T)}.$$

Evidently, the duration of a bond is equal to minus the derivative of the bond price (with respect to yield) divided by the bond price.

The duration of a bond increases with longer time to maturity and with lower coupon payments (the lower the coupon payment, the greater is the proportion of the present value of the cash flows that is attributable to the final return of principal).

There are other measures of sensitivity of bond price to changes in yield. For example, the *value of a basis point* tells you how much the price of a bond (quoted as the price per

\$100 in principal) will change given a one basis point change in yield (e.g., a change from a 5.52% yield to a 5.53% yield). Since the derivative of the bond price is relative to a change in the yield of 1 unit (= 100%), the value of a basis point is equal to the derivative divided by 10,000. For a bond with price close to 100, the value of a basis point will be very close to duration divided by 100. For example, a 5% coupon bond with 10 years to maturity selling at par will have a duration of about 8 years and a value of a basis point of about .08, meaning that a one basis point decrease in yield will raise the price from 100.00 to about 100.08. But, a zero coupon bond with 10 years to maturity with a yield of 5.00% will have a duration of 10 years but a value of a basis point of about .06; the difference is that the duration, the weighted average time to maturity, gives the derivative of the price divided by the bond price and the bond price in this case is about \$60 per \$100 of principal.

More detail on duration can be found in Hull, section 4.8.

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**Forward rates and forward rate agreements.** When interest rates are deterministic $B(0,t)B(t,T) = B(0,T)$ (this was on Homework 1). When they are random this is clearly not the case, since $B(0,t)$ and $B(0,T)$ are known at time 0 while $B(t,T)$ is not. However the ratio

$$F_0(t,T) = B(0,T)/B(0,t)$$

still has an important interpretation: it is the discount factor (for time-$t$ borrowing, with maturity $T$) that can be locked in now, at no cost, by a combination of market positions. In fact, consider the following portfolio:

(a) long a zero-coupon bond worth one dollar at time $T$ (present value $B(0,T)$), and

(b) short a zero-coupon bond worth $B(0,T)/B(0,t)$ at time $t$ (present value $-B(0,T)$).

Its present value is 0, and its holder pays $B(0,T)/B(0,t)$ at time $t$ and receives one dollar at time $T$. Thus the holder of this portfolio has "locked in" $F_0(t,T)$ as his discount rate for borrowing from time $t$ to time $T$.

This discussion makes reference to just three times: 0, $t$ and $T$. So it is natural and conventional to work with term rates rather discount rates. Defining $f_0(t,T)$ by

$$F_0(t,T) = \frac{1}{1 + f_0(t,T)(T-t)}$$

we have shown that $f_0(t,T)$ is the *forward term rate* for borrowing from time $t$ to time $T$.[2] In other words, an agreement now to borrow or lend later (at time $t$, with maturity $T$) has present value zero, if it stipulates that the term rate is $f_0(t,T)$.

What about a contract to borrow or lend at a rate $R_K$ other than $f_0(t,T)$? This is known as a *forward rate agreement.* We can value it by an easy modification of the argument used

[2] Do not confuse this with the instantaneous forward rate discussed earlier.

above. Suppose the principal (the amount to be borrowed at time $t$) is $L$. Then the contract provides a payment at time $T$ of

$$(1 + R_K \Delta T)L = (1 + f_0 \Delta T)L + (R_K - f_0)\Delta T L$$

where $f_0 = f_0(t,T)$ and $\Delta T = T - t$. So the contract is equivalent to a forward rate agreement at rate $f_0(t,T)$ on principal $L$ *plus* an additional payment of $(R_K - f_0)\Delta T \cdot L$ at time $T$. The forward agreement at rate $f_0$ has present value 0, so the contract's present value (to the lender) is

$$B(0,T)(R_K - f_0)\Delta T \cdot L.$$

The following observations are useful in connection with swaps (which we'll discuss shortly):

(1) *A forward rate agreement is equivalent to an agreement that the lending party may pay interest at the market rate $R(t,T)$ but receive interest at the contract rate $R_K$.* Indeed, the lender pays $L$ at time $t$ and receives $(1 + R_K \Delta T)L$ at time $T$. We may suppose that the payment at time $t$ is borrowed at the market rate. Then the lender is (a) borrowing $L$ at the market rate $R$ at time $t$, repaying $(1 + R\Delta T)L$ at time $T$, and (b) lending $L$ to the counterparty at time $t$, receiving repayment $(1 + R_K \Delta T)L$ at time $T$. Briefly: the lender is *exchanging* the market rate $R$ for the contract rate $R_K$.

(2) *A forward rate agreement can be priced by assuming that the market rate $R(t,T)$ will be the forward rate $f_0(t,T)$.* Indeed, the pair of loans just considered have net cash flow 0 at time $t$, and the lender receives $(R_K - R)\Delta T \cdot L$ at time $T$. The value of $R$ is not known at time 0. But substitution of $f_0$ in place of $R$ gives the correct value of the contract at time 0.

Almost all forward rate agreements are cash settled, meaning that there is no actual exchange of principal, just a payment of $(R_K - R)\Delta T \cdot L$ at time $T$. Note that $R$ is determined at time $t$ but the payment is not made until time $T$.

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**Swaps.** A *swap* is an exchange of one income or payment stream for another. The most basic example is a (plain vanilla) interest rate swap, which exchanges the cash flow of a floating-rate debt for that of a fixed-rate debt with the same principal. We shall restrict our attention to this case.

A swap is, in a sense, the floating-rate bond analogue of a forward contract. It permits the holder of a floating-rate bond to eliminate his interest-rate risk. This risk arises because the future interest payments on a floating-rate bond are unknown. It can be eliminated by entering into a swap contract, exchanging the income stream of the floating-rate bond for that of a fixed-rate bond. What fixed rate to use? Any rate is possible – but in general the associated swap contract will have some (positive or negative) value. However at any given time there is a fixed rate that sets the present value of the swap to 0. This is rate that would normally be used. It is called the *par swap rate*. We'll use the notation $R_{\text{swap}}$ for the par swap rate.

It is clear from the definition that a swap is equivalent to a portfolio of two bonds, one short and the other long, one a fixed-rate bond and the other a floating-rate bond. Real bonds would have coupon payments then would return the principal at maturity. In a swap the coupon payments don't match, so there is a cash flow at each coupon date; however the principals do match, so there is no net cash flow at maturity. But the principal of the associated bonds isn't irrelevant – we need it to calculate the interest payments. It is called the *notional principal* of the swap.

A swap can also be viewed as a collection of forward rate agreements. Indeed, we showed above that the value of a floating-rate bond is equal to its principal just after each reset. So being short the floating-rate bond and long the fixed-rate bond is equivalent to paying the market interest rate and receiving the fixed interest rate. This amounts to a collection of forward rate agreements – one for each coupon payment – all with the same principal (the notional principal of the swap) and the same interest rate (the fixed rate of the swap).

Valuing a swap is easy: it suffices to value each associated bond then take the difference. (An alternative, equivalent procedure is to value each associated forward rate agreement and add them up; we'll discuss this below.) Suppose an institution receives fixed payments at 7.15% per annum and floating payments determined by LIBOR. We assume there are two payments per year, the maturity is two years, and the notional principal is $L$. To value the fixed side of the swap we must find the present value of the future coupon payments. It is natural to use the LIBOR discount rate for $B(0,T)$. Let us assume

$$B(0,t_1) = .9679, \quad B(0,t_2) = .9362, \quad B(0,t_3) = .9052, \quad B(0,t_4) = .8749$$

where $t_1 = 182$ days, $t_2 = 365$ days, $t_3 = 548$ days, and $t_4 = 730$ days are the precise payment dates. The value of the fixed side of the swap is then

$$\begin{aligned} V_{\text{fix}} &= L\{.9679 \times .0715 \times (182/365) + .9362 \times .0715 \times (183/365) \\ &\quad + .9052 \times .0715 \times (183/365) + .8749 \times .0715 \times (182/365)\} = (0.1317)L \end{aligned}$$

Notice that we counted only the coupon payments, with no final payment of principal.

Now let's value the floating side of the swap. Of course we cannot know its cash flows at each time – this would require knowledge of $B(t_i, t_{i+1})$ for each $i$, which cannot be known at time 0. However to value the swap all we really need to know is $B(0,t_4)$. Indeed, the value of the floating bond at time 0 is just its notional principal $L$. But we did not count the return of principal $V_{\text{fix}}$, so we must not count it here either. Thus the value of the floating side of the swap is

$$V_{\text{float}} = L - B(0,t_4)L = (0.1251)L.$$

The value of the swap is the difference, namely

$$V_{\text{swap}} = V_{\text{fix}} - V_{\text{float}} = (0.0066)L.$$

This is, of course, the value of the swap to the party receiving the fixed rate and paying the floating rate. The value to the other party is $-(0.0066)L$.

OK, that was easy. But the answer didn't come out zero. What fixed rate could have been used to make the answer come out zero – in other words, what is the par swap rate? That's easy: we must replace .0715 in the above by a variable $x$, set the value of the swap to 0, and solve for $x$. This gives

$$x \cdot \{.9679 \times (182/365) + .9362 \times (183/365) + .9052 \times (183/365) + .8749 \times (182/365)\} = 0.1251,$$

which simplifies to $1.8421x = 0.1251$ whence $x = .0679$. Thus the par swap rate is 6.79% per annum.

While exact day counts are used (as above) in industry calculations, for the rest of these lectures, we will use a simpler approximation, which is to assume that we just divide by the frequency $f$ of payments per year. For example, if payments are made quarterly, we divide a 6% annual coupon payment into four 1.5% payments. With this convention, the preceding discussion about valuing a swap can be summarized as follows:

$$V_{\text{swap}} = V_{\text{fix}} - V_{\text{float}}$$

where

$$V_{\text{fix}} = L \sum \frac{c}{f} B(0, t_j), \quad V_{\text{float}} = L[1 - B(0, T)],$$

whence

$$V_{\text{swap}} = L \left[ \sum \frac{c}{f} B(0, t_j) + B(0, T) - 1 \right].$$

Now consider two swaps with different coupon rates $c$ and $c'$. We get

$$V_{\text{swap, c}} = L \left[ \sum \frac{c}{f} B(0, t_j) \right] - V_{\text{float}}, \quad \text{and} \quad V_{\text{swap, c}'} = L \left[ \sum \frac{c'}{f} B(0, t_j) \right] - V_{\text{float}}$$

whence

$$V_{\text{swap, c}} - V_{\text{swap, c}'} = L \sum \frac{c - c'}{f} B(0, t_j).$$

If we take $c' = R_{\text{swap}}$ then $V_{\text{swap, c}'} = 0$, by the definition of $R_{\text{swap}}$. This gives the following convenient alternative formula for the value of a swap with coupon rate $c$:

$$V_{\text{swap}} = L \sum \frac{c - R_{\text{swap}}}{f} B(0, t_j).$$

(This is the alternative formula mentioned earlier, generalizing the formula for the value of a forward rate agreement.)

We have discussed only the simplest kind of swap – a "plain vanilla interest rate swap". But the general principle should be clear. For example a forward-starting swap that begins at time $t_k$ can be valued by the following modification of the formulas:

$$V_{\text{fix}} = L \sum \frac{c}{f} B(0, t_j) \quad \text{where the sum starts at period } k+1 \text{ and ends at } T;$$

$$V_{\text{float}} = L[B(0, t_k) - B(0, T)];$$

$$\begin{aligned} V_{\text{swap}} &= L\left[\sum \frac{c}{f}B(0,t_j) + B(0,T) - B(0,t_k)\right] \quad \text{with the same convention on the sum} \\ &= L\sum \frac{c - R_{\text{swap}}}{f}B(0,t_j) \quad \text{with the same convention on the sum.} \end{aligned}$$

Another widely used instrument is the"plain vanilla foreign currency swap," which exchanges a fixed-rate income stream (or floating-rate income stream) in a foreign currency for a fixed-rate income stream (or floating-rate income steam) in dollars. Such an instrument can be used to eliminate foreign currency risk. It can be valued using the formulas we just developed, but each side of the swap must be valued using the discount factors appropriate to the currency in which payments are being made. At the end the valuation of the foreign-currency side is translated into dollars by applying the current spot FX rate. See Hull sections 7.8 & 7.9 for a more detailed discussion.

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**Inferring discount factors from market prices.** An important operation in dealing with interest rate instruments is to infer a set of discount factors from observed market prices. These market prices can come in many forms, but the most common are quoted interest rates – typically, term rates for nearby dates, par swap rates for longer dates, and forward term rates for intermediate dates. Discount factors can be inferred from these quotes one at a time, starting with the shortest dates and working towards longer ones. In this process, known as "bootstrapping," discount factors already inferred for shorter dates are utilized as part of the calculation of longer-date discounts.

The full methodology for doing this in practice is rather complicated. Issues such as overlapping periods, exact day counts, accrued interest calculations, and inferring discount factors for several dates from a single price observation all need to be handled and a method needs to be chosen for interpolating discount factors for dates in between those for which prices are available. Here we just give a relatively simple example that demonstrates the basic technique.

Suppose the following data are available: term rates for 6 month and 1 year maturities, forward term rates for the period 1 year to 1.5 years and the period 1.5 years to 2 years, and par swap rates for semi-annual coupon swaps of 2.5 years and 3 years. Moreover suppose the numbers are

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|---:|---|
| 6 month term rate | 5.00% |
| 1 year term rate | 5.25% |
| 1 year – 1.5 year forward term rate | 5.50% |
| 1.5 year – 2 year forward term rate | 6.00% |
| 2.5 year par swap rate | 6.25% |
| 3.0 year par swap rate | 6.50%. |

Then the first few discount rates are determined as follows:

$$B(0,.5) = 1/(1 + 5\%/2) = .97561$$
$$B(0,1.0) = 1/(1 + 5.25\%) = .95012$$
$$B(0,1.5) = B(0,1.0)/(1 + 5.5\%/2) = .95012/(1 + 5.5\%/2) = .92469$$
$$B(0,2.0) = B(0,1.5)/(1 + 6.0\%/2) = .92469/(1 + 6.0\%/2) = .89776.$$

Now the PV of the first 4 coupon payments of the 2.5 year swap is

$$6.25\% \times .5 \times (.97561 + .95012 + .92469 + .89776) = .11713.$$

So

$$B(0,2.5) = (1 - .11713)/(1 + 6.25\%/2) = .85612.$$

Similarly, since

$$6.50\% * .5 * (.97561 + .95012 + .92469 + .89776 + .85612) = .14964$$

we have

$$B(0,3.0) = (1 - .14964)/(1 + 6.50\%/2) = .82359.$$

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**Caps, floors, and swaptions.** Easiest first: a swaption is just an option on a swap. When it matures, its holder has the right to enter into a specified swap contract. He'll do so of course only if this swap contract has positive value. Since a swap is equivalent to a pair of bonds, a swaption can be viewed as an option on a pair of bonds. Similarly, since a swap is equivalent to a collection of forward rate agreements, a swaption can be viewed as an option on a collection of forward rate agreements.

Now let's discuss caps. The borrower in a floating-rate loan does not know his future expenses, since they depend on the floating interest rate. He could eliminate this uncertainty entirely by entering into a swap agreement. But suppose all he wants is insurance against the worst-case scenario of a high interest rate. The cap was invented for him: it pays the difference between the market interest rate and a specified cap rate at each coupon date, if this difference is positive. By purchasing a cap, the borrower insures in effect that he'll never have to pay an interest rate aboved the cap rate. The cap can be viewed as a collection of caplets, one associated with each coupon payment. Each caplet amounts to an option on a bond. It is roughly speaking a call option on the market rate at the coupon time.

A floor is like a cap, but it insures a sufficiently high interest rate rather than a sufficiently low one. It can be viewed as a collection of floorlets, one associated with each coupon payment. Each floorlet is again an option on a bond – roughly speaking a put option on the market rate at the coupon time.

There is a version of put-call parity in this setting: cap-floor=swap, if the fixed rate specified by all three instruments is the same.

Thus caps and floors are collections of options on bonds; swaptions are options on collections of bonds. We'll discuss them in more detail in the next section, and we'll explain how they can be priced using a variant of the Black-Scholes formula.