Derivative Securities – Fall 2007 – Section 11 addendum
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Supplement to the discussion of single-name credit topics. Steve Allen's Section 11 notes discuss the modeling of defaultable bonds, credit default swaps, as well as other issues involving the possibility that a single firm might default. Rather than repeat that material, I simply provide a brief overview and some additional examples. (Thus, these are not stand-alone notes but rather a supplement to the Section 11 presently posted on Blackboard.)

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Overview. Here is a compact overview of the main discussion points:

(1) *The big picture.* Risk of default is a huge issue for banks and other insitutitions with large bond portfolios. So we need models (for regulators and risk managers) and methods for reducing risk (by hedging part of it, or by diversification). The main methods for reducing risk are now credit default swaps (a single-name product) and collateralized debt obligations (a multi-name product). These are rapidly growing markets. The fact that large bankruptcies like Enron and Worldcom had relatively little impact on any single institution was largely due to the use of instruments like these.

In discussing default probabilities, be careful to distinguish between the *conditional probability of default in time period i* d_i, and the *unconditional probability of default in time period i*. The latter is $S_{i-1}d_i$ where S_i is the probability of surviving until the end of year i. Notice that $S_{i+1} = S_i(1 - d_i)$ and $S_0 = 1$.

(2) *Instruments to be discussed.*

(a) Defaultable bonds. More expensive than risk-free bonds, due to risk of default. Valued by adding up discounted cash flow as usual, but each future payment gets multiplied by the probability that the payer has not yet defaulted. (Thus, we are calculating the *expected* discounted cash flow.)

(b) Options on defaultable bonds. Use Black's formula. No new ideas required.

(c) Credit default swaps. Essentially, a CDS provides insurance that the holder of a defaultable bond will get a suitable payment (a predetermined fraction of the principal, typically) if/when the issuer defaults.

(d) Collateralized debt obligations. Discussed in Section 12. Breaks a portfolio of defaultable bonds into tranches. Each tranch has a different exposure to default, hence a different price. Useful because unlike a CDS, its exposure (though maybe substantial) is highly diversified.

Also worth mentioning: total return swaps. Prior to CDS's, this was the primary mechanism for lending money while reducing the lender's exposure to default by the borrower.

(3) *How to model such instruments?* The key lies in indentifying and using the (risk-neutral) default probabilities. Three approaches:

(a) Extract default probabilities from bond prices. This is a lot like the "bootstrapping" process we discussed for getting the risk-free yield curve from prices of risk-free bonds.

(b) Extract default probabilities from the market prices of CDS's. This is a lot like getting the implied volatility of a stock from option prices.

(c) Estimate default probability from information about the stock price. Attractive because the stock price is visible and liquid. Main idea: Merton's model. Too crude in its raw form, but this lies at the foundation of both main commercial products (KMV and CreditGrades).

(d) Items (a)-(c) refer to instruments with a single issuer. In pricing CDO's, many additional issues arise. The correlations between defaults by different issuers are crucial. Simplest tool for capturing this: Gaussian copula (see Section 12).

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An example. Steve's Section 11 explains how to price a coupon bond or CDS (and how to find the par CDS spread or par coupon rate), given the default probabilities and risk-free discount factors. Those calculations can be reversed, to find the default rate given a bond or CDS price (or a par CDS spread, or a par coupon rate) – provided we make a simplifying assumption such as that the conditional default probabilities d_i are independent of i.

Here is another example, in which several bond prices are used to extract several default probabilities:

Question: Suppose the risk-free yield curve is flat at 6% with annual compounding. One-year, two-year, and three-year bonds yield 7.2%, 7.4%, and 7.6% with annual compounding. All pay 6% coupons. Assume that in case of default the recovery is 40% of principal, with no payment of accrued interest. Find the risk-neutral probability of default during each year.

Answer: We assume the coupons are paid annually. Moreover if a bond defaults during a certain year, we assume the coupon due at the end of the year is not paid, and the recovery (40% of principal) is paid at the end of the year.

To begin, let's find the risky bond prices from the yield information in the problem. We use the standard relationship between a bond's yield and its price (ignoring the possibility of default); for example the yield y of a two-year bond with principal P and coupon payment c is related to the price of the bond by

$$\text{price} = c/(1+y) + c/(1+y)^2 + P/(1+y)^2.$$

Using the data in the problem, for bonds with principal one dollar, we find:

$$\begin{aligned}
\text{price of 1-yr bond} &= .06/1.072 + 1/1.072 = .9888 \\
\text{price of 2-yr bond} &= .06/1.074 + .06/(1.074)^2 + 1/(1.074)^2 = .9748 \\
\text{price of 3-yr bond} &= .06/1.076 + .06/(1.076)^2 + .06/(1.076)^3 + 1/(1.076)^3 = .9585.
\end{aligned}$$

Now we find the default probabilities from the fact that these bond prices must be the RN-expected discounted values of the associated cash flows. Let p_i be the probability that a default occurs in year i.

Focusing on the one-year bond first, we have

$$.9888 = (1 - p_1)(.06/1.06 + 1/1.06) + p_1(.4/1.06) = 1 - p_1 + .3774 p_1$$

so $p_1 = (.9888 - 1)/(.3774 - 1) = .0180$.

Considering the two-year bond next, we have

$$.9748 = p_1(.4/1.06) + p_2[.06/1.06 + .4/(1.06)^2] + (1 - p_1 - p_2)[1]$$

using for the last term the fact that $.06/1.06 + .06/(1.06)^2 + 1/(1.06)^2 = 1$. Since we know p_1, this reduces to a linear equation for p_2, namely

$$.9748 = .0068 + .4126 p_2 + (.9820 - p_2).$$

It follows that $p_2 = (.9748 - .0068 - .9820)/(.4126 - 1) = .0238$.

Finally we use the three-year bond price to find p_3. We have

$$\begin{aligned} .9585 &= p_1(.4/1.06) + p_2[.06/1.06 + .4/(1.06)^2] \\ &\quad + p_3[.06/1.06 + .06/(1.06)^2 + .4/(1.06)^3] + (1 - p_1 - p_2 - p_3)[1]. \end{aligned}$$

Using the known values of p_1 and p_2 this becomes

$$.9585 = .0166 + .4459 p_3 + (.9582 - p_3)$$

so $p_3 = (.9585 - .0166 - .9582)/(.4459 - 1) = .0294$.

Notice that the probabilities p_i used above are actual probabilities, not conditional probabilities. In the notation used at the beginning of this document, $p_i = S_{i-1} d_i$, i.e.

$$\begin{aligned} p_1 = \text{prob of default in year 1} &= d_1 \\ p_2 = \text{prob of default in year 2} &= (1 - d_1) d_2 \\ p_3 = \text{prob of default in year 3} &= (1 - d_1)(1 - d_2) d_3. \end{aligned}$$

It was most natural for this problem to solve directly for the unconditional probabilities p_i; but the conditional probabilities d_i can easily be found, if desired, from the formulas just above.