Calculating Variations, Lecture 6, 3/6/2017

Today:

1) of what we did in 1D world plans, what extends to multidimensional plans of form
\[ \int W(Du) \quad \text{for } \mathbb{R}^2 + \mathbb{R} : \mathbb{R}^n \rightarrow \mathbb{R}^m ? \]

2) start discus of optimal control + Hamilton -
Jacobi's pas (this will surely spill over to
Lecture 7)

Well (a), in discussing 1D plan \( \int F(t, u, u') \, dx \)
we emphasized importance of \( F \) being convex
with \( u \). In fact we used this in showing
that

1) if \( u \) solves the EL eqn then \( u \) is
smooth (assuming \( F \) itself is smooth)

2) if \( u \) solves EL eqn + \( \frac{\partial F}{\partial u'} \) has a
negative eigenvalue when \( u \) and \( u'' \)
(\( t_0, u(t_0), u'(t_0) \)) are close to then
2nd var test is sure to fail

In higher dims (1) can fail: minimizers of
world plans are not always smooth, for
plans of form \( \int W(Du) \), even when \( W \) is
1 c \( \infty \)

- When $u$ is scalar valued and $W$ is strictly convex, minimizers of $\int W(\text{Du})$ are smooth as consequence of the DeGiorgi + Nash or De Giorgi's regularity for the plan.

- But when $u$ is vector valued there are counterexamples, e.g. Necas showed that $u_i = \frac{x_i x_i}{1x1}$ minimizes a convex valued plan involving maps $\mathbb{R}^n \to \mathbb{R}^m$ with a smooth, strictly convex integrand $W(\text{Du})$.

In higher dim there is always an analogue of (2). For maps $u : \mathbb{R}^n \to \mathbb{R}$ it shows that $W(\text{Du})$ should be convex. For maps $u : \mathbb{R}^n \to \mathbb{R}^m$ is shown instead that $W(\text{Du})$ should be rank one convex, i.e.

$$W(\theta F_1 + (1-\theta) F_2) \leq \theta W(F_1) + (1-\theta) W(F_2)$$

when $F_2 - F_1$ has rank one and $0 \leq \theta \leq 1$. 
or equivalently \[ \sum_{F \in \mathcal{F}} \frac{\partial W}{\partial f(x)} \xi \cdot \eta \, d\eta > 0 \quad \text{for all } \xi \in \mathbb{R}^n, \eta \in \mathbb{R}^n. \] (Students of the age will recognize this as being very similar to conclude that the Euler-Lagrange system is an elliptic system.)

Let's explain: recall that, in 1D, if \( L \) is not convex, the functional \( W(\rho) \) is not convex \( \rho \), if \( \xi \cdot \xi < 0 \) for some \( \xi \). We used \( a \) and \( \rho \) such that \( \Phi = \{ \xi : \xi \cdot \xi < \epsilon, \epsilon \leq \xi \cdot \xi \leq \delta \} \) and \( \rho(t) \) and saw that the second order term \( \rho \) was negative.

For \( \mathbb{R}^n \rightarrow \mathbb{R} \) we can do something similar, being more careful with the geometry.

Given \( \xi \in \mathbb{R}^n \), we can choose \( \eta \) so \( \eta = \xi \) or \( -\xi \) in two narrow strips \( \eta \in \mathbb{R} \) outside these strips. Proof: choose \( \rho \) close to \( \rho(1,0,...,0) \). Then let \( \eta \) be a suitable \( \xi \) of \( x_1 \).

Picture in \( \mathbb{R}^n \): \[ \begin{array}{c|c|c}
\eta_1 &=& 0 \\
\eta_2 &=& 0 \\
\eta_3 &=& -\xi \\
\eta_4 &=& 0 \\
\end{array} \]

must be \( \| \xi \| \).
This doesn't quite suffice - so the applied term and term we also want not to be well. Achieve that by locally the triple of the strips and introductory cuttoff at each end.

\[ \text{Width = 25} \]

\[ \text{can be piecewise linear (linear on each triangle)} \]

Note: 2nd term is \( \int \frac{\partial W}{\partial \text{w} \text{y}} \text{d}x \)

Part arising to strips is negative and less; part arising cutoff domains long no sign but it's of order \( \varepsilon^2 \). If \( \varepsilon << \delta << 1 \) then cutoff domains don't matter.

For flows where \( \Omega : \mathbb{R}^n \rightarrow \mathbb{R}^m \), the argument similar, however it is only possible when \( \xi \) has real one i.e. \( \xi_x = \frac{\xi_a}{b_x} \). Explain why.

\[ \begin{align*}
\psi &= 0 \\
\varepsilon_1^x &= \frac{\xi_1}{b_x} \\
\varepsilon_2^x &= \frac{\xi_2}{b_x} \\
\varepsilon_3^x &= \frac{\xi_3}{b_x} \\
\psi &= 0
\end{align*} \]

\( \psi \) cuts at lower left.

\( \rightarrow \delta \xi_x \text{ at } x = 0 \) when it is tangent to lower...
For each $i$, $\varepsilon_i = x^i (x^i, \varepsilon_i)$
where $\varepsilon_i = \text{larger residual}$.

Thus, we need to rank one (say, $\varepsilon_i = x_i / 2$) + then we can use a layer perpendicular to $\varepsilon_i$.

New type: optimal control + HJ equ. Here the focus is still on $1$ real plane, but viewpoint is different from Lecture 5, leading to different types of objects (e.g., economics + finance) and different notions to play through the value function, Hamilton-Jacobi equations + theory of viscosity solutions of HJ equations).

My discussion will follow, more or less, my Spring 2014 PDE in Finance notes (Section 4). For related material see:

- LC Evans' PDE book (Chap 10) for explanation why the value fun of an optimal control problem is a viscosity solution of the adjoint H-J eq (Also the basic theory of viscosity solutions.)

- H. Hechinger, Optimal Control: An Intro to the Theory w/Apps (lots of examples best mostly via Pontryagin Max Fun not HJ eqs).
Typical examples of optimal control problems

A) \[ \min \int_0^T \left[ \rho(y(s), x(s)) \, ds + g(y(T)) \right] \]

where the maximization is over "controls" \( x(s) \), which determine evolution of the "state" via an ODE

\[ \dot{y}(s) = f(y(s), x(s)), \quad y(0) = y_0. \]

Typical engineering applications: send a spacecraft to the moon. Then \( y = (\text{position}, \text{velocity}) \), \( x(s) \) controls firing of rockets, \( f(y, x) \) is eqn of Newtonian mechanics, \( T \) = desired arrival time (treated here as fixed), \( g(y(T)) \) favors desired arrival location with velocity near 0, and \( h \) = fuel consumption.

Typical economics applications: max rather than min; \( x(s) \) controls investment policy and/or consumption of resources; \( h + g \) are utilities assoc to consumption + final - the wealth level to problems considered recently & var'd.
\[
\min \int_0^T \left( \left( x^2 - 1 \right)^2 + u^2 \right) \, dx
\]

can easily be put in this form:

\[
\min \int_0^T \left( x^2 - 1 \right)^2 + y^2 \, dt
\]

where \( x(t) \in \mathbb{R} \) is the control \( \frac{dy}{dt} = x(t) \), \( y(0) = 0 \) is the "state eqn" \( f(y, x) = (x^2 - 1)^2 + y^2 \) is the "running cost" \( g = 0 \) (no final true cost)

As we've noticed before, the nonconvexity in \( x \) leads to nonexistence of a minimizer, though the min value is perfectly well-defined (it is 0 in this case).

B) min arrival time

\[
\min \int t \text{ time at which } y(t) \text{ reaches some target net } 1 - \frac{x}{2}
\]

where \( y(s) \) solves an ode

\[
\dot{y} = f(y(s), x(s)), \quad y(0) = y_0
\]

and the optimization is over the "control" \( x(t) \).
This is more a special case of (A) (setting $T = \infty$ and $\rho = \sqrt{2}$) if target has not been reached.

but it's still worth enough to deserve special treatment.

**Special case that's easy to visualize:**

given $D \subset \mathbb{R}^n$ and $x \in D$, consider

$$\min \& \text{ try to exit } D \text{ starting from } x \text{ and travelling at velocity } \leq 1$$

Evidently state eqn in $j = x$, where $d(x) \in \mathbb{R}^n$ must satisfy $||x|| \leq 1$. Optimal path is of course straight line to nearest pt on $\partial D$, so

$$\min \text{ value } = d(x, \partial D)$$

(well-defined, though optimal path may not be unique).

Our goal is:

(i) a scheme for guessing the form of the solution and proving that the guess
is correct

(2) a far-from-obvious link between optimal control (which intrinsically involves problems in one variable, typically "time") and PDE (namely Hamilton-Jacobi equations).

The two goals will be achieved together (they are interdependent).

For problem class (A) the trick is to study the dependence of the optimal value on the initial position and time; we define

\[ u(x,t) = \min_{x(t) \in A} \int_{t}^{T} f(y(s), x(s)) \, ds + g(y(T)) \]

where

\[ \dot{y}(s) = f(s, y(s), x(s)) \quad \text{for} \quad t < s < T \]

with

\[ y(t) = x \]

We'll derive a role for \( u \). The main tool is the dynamic programming principle

\[ u(x,t) = \min_{x(t) \in A} \left\{ \int_{t}^{t'} f(y(s), x(s)) \, ds + u(y(t'), t') \right\} \]

\[ t < t' \]
Interpretation: the optimal strategy must do something between \( t \) and \( t' \) starting from \( t' \). It should solve the same problem with a new starting time position.

We can derive a pde for \( u \) (formally, i.e. assuming more differentiability, than might really be true) by applying this with \( t' = t + \Delta t \) in the limit \( \Delta t \to 0 \). Here is the formal argument:

- Let's guess that over \( t < s < t + \Delta t \) the optimal \( c(s) \) is (more or less) constant. Then

\[
 u(x,t) = \min_{a \in A} \left\{ p(x,a) \Delta t + u(x+f(t,a) \Delta t, t + \Delta t) \right\}
\]

dropping corrections of order \((\Delta t)^2\)

- Let's assume \( u \) is differentiable, and expand via Taylor series

\[
 u(x,t) = \min_{a \in A} \left\{ p(x,a) \Delta t + u_x(t,a) \Delta t + \frac{1}{2} u_{xx}(t,a) \Delta t^2 + \cdots + \cdots \right\}
\]

- Cancellation the \( \Delta t \) terms, we get
\[ u_t + \min_{a \in A} \{ P(x, a) + \gamma u \cdot f(t, a) \} = 0 \]

is a pde of the form \( u_t + H(t, x, \gamma u) = 0 \). It is to be solved for \( t < T \), \( x \in \mathbb{R}^d \), with initial-time data:

\[ u(x, T) = g(x) \]

(since if starting \( u(x, t) = 0 \) then \( u(x, t) = g(x) \) from the initial data \( u \)). Note that the \( H \) we get this way is concave in \( \gamma u \) (being the sum of linear terms of \( \gamma u \)).

If we had started with a max problem instead of a min problem, same calculus would have given:

\[ u_t + \max_{a \in A} \{ P(x, a) + \gamma u \cdot f(t, a) \} = 0 \]

is a pde of form \( u_t + H(t, x, \gamma u) = 0 \) with \( H \) concave in \( \gamma u \).

Example: The Hopf-Lax formula for \( u_t + H(\gamma u) = 0 \) with convex \( H \). We have only to write:

\[ H(\gamma) = \max_{\bar{a}} \langle \bar{a}, \gamma \rangle + P(\bar{a}) \]

(evidently, \( P = -H^* \) where \( H^* \) is the Fenchel transform; we defined a couple of lectures ago).
to guess that the relevant value of the pde with 
\[ u = q \] at \( t = T \) is 
\[ u(x, t) = \max \left\{ \int_0^t h(x(s)) \, ds + g(y(T)) \right\} \]

using eqn of state
\[ \dot{y} = x, \quad y(T) = x \]

Fact: given any choice of \( y(t) \), the best path is the one with constant \( x \). This follows from concavity of \( h \), and Jensen's inequality

\[ h \left[ \text{average velocity} \right] \geq \text{average of } h \left[ \text{velocity} \right] \]

Since any velocity depends only on ends \( T \), ie
\[ \frac{1}{T-t} \int_t^T \frac{\partial h}{\partial x} \, ds = \frac{1}{T-t} [y(T) - y(t)] \]

we arrive at the "Hopf-Lax formula"
\[ u(x, t) = \max \left\{ (T-t) \frac{x}{T-t} + \frac{3-x}{3} \right\} \]

which reduces solving the pde \( u_t + H(x_u) = 0 \) \( (t<T) \) with \( u = q \) at \( t = T \) to a 1-D optimization at each time \( t \) and spatial \( x \).
For problem class $B$, i.e. with arrival time problems, the situation is similar: if

$$u(x) = \min \left\{ \text{tie when } y(x) \text{ reaches target } \Gamma \right\}$$

using epr of state

$$y = f(y(x), a(x)), \quad y(0) = x$$

[note that starting time is now fixed!] Then dyn program says

$$u(x) = \min_{\substack{a(x), x \in \mathbb{R}^n \ \forall \Delta t < T}} \left\{ u(y = f(y(x), a(x)), x) + \Delta t \right\}$$

Arguing as before (taking $\Delta t = 0$ and using Taylor expansion) we get

$$u(x) = \min_{a \in A} \left\{ u(x + f(x, a) \Delta t) + \Delta t \right\}$$

$$= \min_{a \in A} \left\{ u(x) + f(x, a) \cdot \Delta t + \Delta t \right\}$$

$$= \min_{a \in A} \left\{ 7u \cdot f(x, a) + 1 \right\} = 0$$

an eqn of form $H(7u) + 1 = 0$ (with $H$ concave), to be solved for $x \notin \Gamma$, with bc $u = 0$ at $\Gamma$. 
Example: in the special case where state eqn is
\[ \dot{y} = a(s) \]
and speed is \(|x(s)| \leq 1\) we know optimal path is straight + goes toward pt of \(P\) closest to \(x\), so that
\[ u(x) = \text{dist}(x, P) \]

Assoc HJ eqn is
\[ \min_{u} \left\{ J_u \cdot a \right\} + 1 = 0 \]
\[ \text{i.e.} \]
\[ -17u + 1 = 0 \]
\[ \text{i.e. the internal eqn} \]
\[ 17u = 1 \]
\[ u = 1 \text{ at } P \]
\[ u = 0 \text{ at } \]

Our discussion thus far has ignored some very important issues:

1. The soln \(u(x,t)\) we want may not be
differentiable (calling into question the observability of the pole). For example, the

\[ |u_x| = 1 \text{ on } \mathbb{R}^n \]
\[ u = 0 \text{ at } \partial \Omega \]

has no \( C^1 \) solve.

(2) The pole may have many \( C^1 \) solves; for example

\[ |u_x| = 1 \text{ on } [-1, 1]^n, \quad u = 0 \text{ at } \partial \\
\]

has lots of solves.

How to know which one we want?

(3) Our real goal was to solve problems; can the pole be used for this (either by hand or numerically)?

Sketch of answers:

To (1) and (2): there's a notion of a \textit{viscosity}
solution of the pole. Viscosity solutions are unique, and they are the special \( u_c \) solution of the pole that gives the value \( u_0 \).

(This is explained in Evans' chap 10)

To (3): The observation of the pole gives us a pretty good idea how the control should be related to \( u_t \) (it should achieve the option that determined \( H(x, u_t) \)).

Once we have a conjectured solution, we can often harness the argument that showed the pole to prove that it is optimal (using what is sometimes called a "verification argument").

Before starting (3), let's work an example:

\[
U(x, t) = \max_{a(t)} \int_{\mathbb{T}} e^{-\beta \alpha - t} a(s) \, ds
\]

where \( 0 < \beta < 1 \), \( \beta > 0 \), and where the state equation is

\[
\frac{dy}{ds} = \gamma - a, \quad y(+) = x
\]

and the control + state must satisfy \( a(s) \geq 0, y(s) \geq 0 \).

(Here \( \gamma > 0 \) is a constant interest rate.)
Interpretation: an investor has initial wealth $x$ at time $t$, and plans his consumption $c(t)$ to maximize his discounted "utility," up to a fixed final time $T$. (We use the power law utility $u(x) = x^g$, $0 < g < 1$, because it makes the HTB solvable by separation of variables.)

Step 0: Let's show the value function must have the form

$$u(x,t) = g(t) x^g$$

in some $g(t)$. Suffice to show

$$u(\lambda x, t) = \lambda^g u(x,t)$$

for all $\lambda > 0$ (with $g(t) = u(1,t)$). To see (x), consider control $\lambda a(s)$ for problem starting from $\lambda x$, whereas $a(s)$ is optimal choice starting from $x$. Associate future state expectation $y_x(s) = y_{\lambda x}(s) = \lambda y_x(s)$. Using form of utility, we conclude that

$$u(\lambda x, t) = \lambda^g u(x,t).$$

Same result with $\lambda$ replaced by $\lambda^{-1}$ gives

$$u(x,t) = \lambda^{-g} u(\lambda x, t).$$
Together, these give (1).

**Step 1:** Find HTB eqn. Almost a special case of calm down before — except now we have a discount term $e^{-\delta (1-t)}$. Arguing as before, finally

$$u(x, t) = \max \left\{ \frac{\delta}{\alpha} \Delta t \pm e^{-\delta \Delta t} u(x + (rX - \alpha) \Delta t, \tau + \Delta t) \right\}$$

$$= \max \left\{ \frac{\delta}{\alpha} \Delta t + (1 - e^{-\delta \Delta t}) (u(x, \tau) + \frac{\Delta t}{\alpha} \frac{\partial u}{\partial t}) + (rX - \alpha) u_x \Delta t \right\}$$

$$= u(x, \tau) + \Delta t \max \left\{ \frac{\delta}{\alpha} - \rho u + \frac{\partial u}{\partial t} + (rX - \alpha) u_x \right\}$$

No as $\Delta t \to 0$ we get

\[ \text{(1.5)} \]  

$$u_t + \max \left\{ \frac{\delta}{\alpha} + (rX - \alpha) u_x \right\} - \rho u = 0. \]

**Step 2:** Optimal consumption policy is easy to find. Clearly $u_x > 0$ (clear from Step 1), so up till $x$ is positive, namely

$$x = \left( \frac{1}{\delta} u_x \right)^{\frac{1}{\alpha - 1}}.$$
Recalling that \( u = g(t) x^2 \), we get

\[ x(t) = g(t)^{\frac{1}{3-t}}, \]

To find \( g(t) \) we substitute this into the pole + do some arithmetic:

\[ q_t x^2 - pq x^3 + (q^{\frac{3}{3-t}}(1-q) + rq g) x^2 = 0 \]

i.e.

\[ \frac{q_t}{q} + (rq - pq) g(t) + (1-q) g(t)^{\frac{3}{3-t}} = 0. \]

Multiplying by \( (1-q)^{-\frac{3}{3-t}} \), we find that \( H(t) = g(t)^{\frac{1}{3-t}} \) satisfies the linear eqn

\[ H_t - \mu H + 1 = 0 \quad \text{with} \quad \mu = \frac{p - rq}{1-q}. \]

**Step 3.** We must to note the final-tie condition, which in this form is \( u(x, T) = 0 \) (since there is no final-tie term in the optimization).

Due to its simple form, we can easily solve \((**\)) with \( H = 0 \) at \( t = T \). Solution is

\[ H(t) = \mu^{-1} (1 - e^{-\mu(T-t)}). \]
This determines $H$, whence $g(t) = H^{-b}$ and $u = g(t) x^b$.

But our derivation of the HJB eqn was final. So, is the value just found really the optimal value, or is it really

$$u(x,t) = \max \int_{a(t)}^{g(t)-t} e^{-g(s-t)} g(s) ds .$$

Answer is yes, by the following verification argument. Let $\bar{u}(x,t)$ be the optimal value, and $\tilde{u}(x,t)$ = conjectured optimal value assoc our explicit soln. Then

(A) $u(x,t) \geq \tilde{u}(x,t)$ because $\tilde{u}(x,t)$ is the value assoc with a particular consumption plan, namely the one we found in Step 2. (This can be checked directly, but we'll also see why it's true in part B.)

(B) to show $u(x,t) \leq \tilde{u}(x,t)$ let's calculate

$$\frac{d}{dt} \tilde{u}(y_{x,a}(t), t)$$ where $y_{x,a}$ is given state egal for any (fixed) policy $a(t)$. We get
\[ \frac{\partial}{\partial t} \tilde{u}(y(t), t) = \tilde{u}_t + \tilde{u} \cdot (\gamma y - a) \leq \rho \tilde{u} = \frac{\partial}{\partial t} \tilde{u} \]

using that \( \tilde{u} \) solves the HJB eqn (**) in the last step. (Note that this color has \( a(\cdot) \) in the last step when \( a(\cdot) \) is the optimal policy found in step 2.)

So

\[ \frac{\partial}{\partial t} e^{-\rho t} \tilde{u}(y(t), t) \leq -e^{-\rho t} a^*(t) \]

Intergrate on \( u \) so that \( \tilde{u}(y(t), T) = 0 \) to get

\[ -e^{-\rho t} \tilde{u}(x, t) \leq -\int_t^T e^{-\rho s} a^*(s) \, ds \]

\[ \Rightarrow \tilde{u}(x, t) \geq \int_t^T e^{-\rho (s-t)} a^*(s) \, ds \]

Maximizign RHS over all choices of \( a(\cdot) \) we see that

\[ \tilde{u}(x, t) \geq u(x, t) \]

as desired.

Similar work shows more generally that if order of HJB eqn is \( C^1 \) then it is indeed the optimal control.
A classic example: recall from pg. 10 that "min time plan" case: "travel starting from \( x \), with max speed 1, until you arrive in a target set \( T \), has optimal eqn \( 1^T u = 1 \). \( u = 0 \) at \( 1^T \) as its HT3 eqn. This has many \( c \) rules, but none that are smooth, if \( 1^T 0 \). \( D = |0^T 1| \).

\[ D = |0^T 1| \]

\[ D = |0^T 1| \]

Two \( c \) rules if \( |u| = 1 \) in \( D \),

\[ u = 0 \text{ at } \partial D \]

Can verification any still be used? Yes as follows. The goal would be to give a verita-style pf that:

\[ u(x) = \text{dist} (x, \partial D) \]

\[ u(x) = \min \{ \text{reach time to } \partial D \} \]

where the state eqn is \( \dot{y}(t) = x(t) \), \( y(0) = x \).
Obviously, $\tilde{u}(x) \geq u(x)$ since we know how to achieve $\tilde{u}$ (namely: travel at constant speed 1 toward nearest body $p$).

To see $\tilde{u}(x) \leq u(x)$ we first argue formally as before (pretending $\tilde{u}$ is differentiable): for any $\alpha(t)$ s.t. $1(\alpha(t)) \leq 1$,

$$\frac{d}{dt} \tilde{u}(y(t)) = \nabla \tilde{u} : \frac{dy}{dt} = \nabla \tilde{u} \cdot \alpha(t).$$

$$\geq \min_{1(\alpha(t)) \leq 1} \nabla \tilde{u} \cdot \alpha(t) = -1.$$

where at arrival occurs at $t = T$ then

$$\tilde{u}(y(T)) - \tilde{u}(y(0)) \geq \int_0^T -1$$

$$\tilde{u}(x) \leq T.$$

Optimizing over all $\alpha(t)$ gives $\tilde{u}(x) \leq u(x)$.

To make this honest, observe that we pretended $\tilde{u}$ was smooth (which isn't true) but we only used the HTS ep as an inequality:

$$\min_{1(\alpha(t)) \leq 1} \nabla \tilde{u} \cdot \alpha = -1 \nabla \tilde{u} \cdot 1 \geq -1$$

for all we used.
So our a.g.t. shows (quite honestly) that

if $w \in C^{1}$, $N = 0$ at $DD$, and $|7w| \leq 1$

then $N(x) \leq u(x)$.

Apply this next to $\tilde{u} = \dot{u} + (x, DD)$ but rather to a "smoothed out" approx.

As $N \rightarrow u$ we conclude (honestly) that $\tilde{u}(x) \leq u(x)$, as desired.

Generalization of this: assertion that a singular

solutions of HJB epn is the opt'l value can often be

achieved by regular a.g.t. applied to a smooth approx.

I have barely mentioned viscosity solns of HJB epn.

It would take us too far afield, and Evans' treatment is excellent. But briefly: situation is

a bit like study of shock waves (e.g. Burgers' epn).

a) Though HJB epn may have no smooth solns

many ae solns, there's a special one
(called the "viscosity soln", though artificial viscosity is not the most convenient analytical tool here).

b) value in of an optimal control plan is always the viscosity soln. (Thus: no need for any verisimilitude if we can manage to find the viscosity soln.)

Suggested exercises

1. In our example involving optimal consumption (pp 5.16-5.18) we got an explicit soln of the HJB eqn, but the formula doesn't make sense if \( g - r g_b = 0 \). What is the solution in that case?

2. Show that \( u_0(x) = \lim_{T \to \infty} u(x,T) \) as the final-time horizon \( T \to \infty \) is:

\[
\begin{cases}
    G_0 x^b & \text{if } g - r g_b > 0 \\
    \infty & \text{if } g - r g_b < 0
\end{cases}
\]

3. What is the optimal consumption strategy, in the limit \( T \to \infty \)?
Consider the analogous example when the integral to maximize is
\[ \int_{\mathcal{D}} e^{f(a,t)} \ln(a) \, ds \]
and let \( u(x,t) \) be the associated value function.

(a) Show that for any \( \lambda > 0 \),
\[ u(\lambda x, t) = u(x, t) + \frac{1}{\lambda} \ln(\lambda) \left( 1 - e^{-\frac{1}{\lambda}(x-t)} \right) \]

(b) Using (a), conclude that
\[ u(x, t) = g_0(t) \ln(x) + g_1(t) \]
for some functions \( g_0 \) and \( g_1 \).

(c) What ODE's and final time conditions should \( g_0 + g_1 \) solve? (The ODE's can be solved explicitly; \( g_0 \) is pretty simple but \( g_1 \) is a little messier.)

We discussed a "minimum travel time" problem whose value function \( u \) solves \( \nabla u = 1 \) in \( D \) and \( u = 0 \) at \( \partial D \).

Find a related problem whose value function solves \( \nabla u = 1 \) in \( D \) and \( u = g \) at \( \partial D \), where \( g \) is a specified function.
2) Consider the 2D case, i.e., let D be a domain in $\mathbb{R}^2$ (assumed 2D is smooth). Describe the optimal controls and paths, if $g$ is smooth and its derivative w.r.t. arc length has $|g'| < 1$.

c) What changes if $|g'| > 1$ on some part of D?

4) This problem is a special case of the "linear-quadratic regulator" widely used in engineering applications. The state is $y(t) \in \mathbb{R}^n$ and the control is $u(t) \in \mathbb{R}^m$ (with no phase restriction). The state eqn is

$$\frac{dy}{dt} = Ay + u, \quad y(0) = x$$

where $A$ is a given (constant) matrix. The goal is to find

$$u(x, t) = \min_u \int_0^T \left( |y(s)|^2 + |u(s)|^2 \right) ds + |y(T)|^2$$

(Thus: we prefer $y = 0$ along the trajectory and at the $T$ but we also prefer not to use too much control.)

a) Find the HJB eqn. Explain why we should expect the relation $\partial V / \partial t = -\frac{1}{2} \text{trace} (L V)$ to hold along optimal trajectories.
b) Since the problem is quadratic, it is natural to guess that
\[ u(x, t) = \langle K(t)x, x \rangle \]
where \( K(t) \) is a symmetric-natrix-valued function. Show that it solves the HJB equation
\[
\frac{dk}{dt} = K^2 - I - (K^T A + A^T K) \quad \text{for } t < T
\]
with \( K(T) = I \) (the nxn identity matrix).
[Hint: two quadratic forms agree exactly if their symmetric matrices agree.]

2) Show by a suitable verification argument that this \( u \) is indeed the value function of the control problem.

3) We showed (pp 6.22-6.24) how a verification argument can be used to show that \( \tilde{u}(x) = \text{dist}(x, \partial D) \)
   is the value function of a simple "minimum travel time" optimal control problem.
   In 1D with \( D = [-1, 1] \) and \( \tilde{u}(x) \) as shown:
   \[
   \tilde{u}(x) \quad \text{(shaped as \( \pm 1 \))}
   \]
   we could try to use a similar argument to show that \( \tilde{u}(x) \) is the value function of this problem.
Of course we must fail (since \( \mathbb{u}(x) \neq \text{dist}(x, \partial D) \)) even though \( \mathbb{u} \cdot \mathbf{x} = 1 \) a.e. What goes wrong?