Calculus of Variations, Lecture 3, 2/6/2017

[Start with example at end of Lecture 2 notes concerning least balls on 1st Dirichlet eigenvalue of a domain in \( \mathbb{R}^n \).]

Today:

(a) Another example of duality, this time involving \( L^1 \) and \( L^\infty \) type plans

(b) An example of "the calibration method" in the calculus of variations (not exactly convex duality, but closely related in concept)

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Starting pt for our \( L^1 - L^\infty \) example: what can we say about

\[
\begin{align*}
\min \ & \| \xi \|_{L^\infty} \\
\text{s.t.} & \quad \xi \in \mathbb{S}^n
\end{align*}
\]

where (to fix ideas) \( S^n \) is a bounded domain in \( \mathbb{R}^n \)?

Interpretation: it's raining uniformly on \( S^n \). How can rain flow to drain with least possible local accumulation? (Note:}
continuum plan has a natural analogue on any graph. Arguments presented below also have analogues in graph setting.

Observation: it is equivalent to solve

\[ \max \lambda \]
\[ 10^3 \leq 1 \]
\[ \text{dist} = \lambda \text{ (constant)} \]

Since \( \lambda_{\text{max}} \) is optimal for (**) then

\[ \text{dist} \lambda^2 = \lambda \text{ (const)} \Rightarrow \lambda \leq \lambda_{\text{max}} \]
\[ 10^3 \leq 1 \]

\[ \text{dist} \left( \frac{1}{\lambda^2} \right) = 1 \Rightarrow \frac{1}{\lambda} \leq \frac{1}{\lambda_{\text{max}}} \]
\[ \frac{1}{10^3} \leq \frac{1}{\lambda} \]

Thus \( \frac{1}{\lambda_{\text{max}}} \) is the optimal value for (**).

We identify a primal plan by the min/max procedure discussed in Lecture 2: (***) is equivalent to
\[
\max \lambda = \max \min \frac{\mathcal{N}}{10} \leq 1 \quad \text{if} \quad |u| \leq 1 \quad \text{at} \quad \partial \Omega
\]
\[
\text{div} \varepsilon = \lambda
\]
\[
\int_{\Omega} \varepsilon \, dx = 1
\]

(since \( \min \) over \( u \) is \(-\infty\) unless \( \text{div} \varepsilon = \lambda \) is constant, in which case it equals \( \lambda \).

Assuming \( \max \min = \min \max \), duality is

\[
\min \max \frac{\mathcal{N}}{10} \leq 1 \quad \text{if} \quad |u| \leq 1 \quad \text{at} \quad \partial \Omega
\]
\[
\int_{\Omega} u \, dx = 1
\]

Best \( \varepsilon \) has \(-\varepsilon, \mathcal{N} = 1 \mathcal{N} \), no dual is

\[
\min \int_{\Omega} 1 \mathcal{N} \, dx
\]
\[
\text{u = 0 at} \partial \Omega \quad \text{52}
\]
\[
\int_{\Omega} u \, dx = 1
\]

Is \( \max \min = \min \max \)? An inequality is elementary, as explained in Lecture 2.6. In this case

\[
|u| \leq 1, \quad \text{div} \varepsilon = \lambda \ (\text{constant}) \Rightarrow -\int_{\partial \Omega} \mathcal{N} \leq \int_{\Omega} \mathcal{N} \leq \int_{\Omega} \mathcal{N}
\]
\[
\text{u = 0 at} \partial \Omega, \quad \int_{\Omega} u \, dx = 1
\]
\[
\Rightarrow \lambda \leq \frac{\int_{\Omega} \mathcal{N}}{52}
\]
so \( \max P \leq \min D \). (In this example the "primal" is a concave maximization and the "dual" is a convex minimization.)

As for equality \( (\max \min = \min \max) \) it's not simple to prove in this case. I'll distribute a separate set of notes with a self-contained proof and some notes to relevant literature.

Another (actually related) feature of this problem: the optimal \( \mathbf{w} \) is rather singular - in fact, it's the characteristic function of a set (to be explained below).

Comment: This example captures an interesting feature of \( L^1 - L^\infty \) duality pairs: one problem is typically much easier to solve (perhaps explicitly) than the other.

In the present setting we can solve \( \mathbf{w} \) more or less explicitly, using the characterization

\[
\begin{align*}
\min_{\mathbf{w} \geq 0 \text{ of } \mathcal{S}} & \int 17x_1 = \min_{\text{length } (2D)} \\
\int_{\mathbb{S}} 2x_1 = & \text{min } \frac{\text{length } (2D)}{\text{Area } (2D)}
\end{align*}
\]
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= solution of a sort of "isoperimetric" problem!

For example, if Ω = square,

\[ \text{arc of a well-chosen circle} \]


Let me sketch the proof of (***). A key ingredient is the

**Coarea Formula**

\[ \int f(x) \mathcal{H}^1 \, dx = \int \left( \int f \, ds \right) \, dt \]

(this is easy to justify if u is nice enough - more or less it is just the "method of shells" from Calc III - but with slightly more careful notation it extends to any \( u \in BV \)).

Now proceeding in steps:

**Step 1:** LHS of (***) = \( \min \int_{u=0}^{a+2\beta} \int f \, dx \)
(easy: if \[ \int_{\Omega} u \, dx = c \] to RHS is unchanged when we replace \( u \) by \( u/c \).)

**Step 2**

May suppose \( u \geq 0 \), since replacing \( u \) by \( -u \) leaves \( \int_{\Omega} u \) unchanged and it increases \( \int_{\Omega} u \, dx \).

**Step 3**

For \( u \geq 0 \),

\[
\int_{\Omega} u \, dx = \int_0^\infty \text{Area } \{ u \geq t \} \, dt
\]

Since

\[
\int_{\Omega} u \, dx = \int_{\Omega} 1 \, dt \, dx
\]

(new use Fubini's Thm). On the other hand

\[
\int_{\Omega} u \, dx = \int_0^\infty \text{length } \{ u = t \} \, dt
\]

(by Co-area formula with \( f = 1 \)). So: if RHS of (***) has min \( \leq \) Then

\[
\text{length } \{ u = t \} \geq \text{Area } \{ u \geq t \}
\]

for all \( t \) (by taking \( D = \{ u \geq t \} \)).
So \( \int_{\Omega} |u| \leq \int_{\Omega} \|
abla u\| \) for any

function \( u \) s.t. \( u \geq 0 \) and \( u = 0 \) at \( \partial \Omega \).

This shows

\[
\min_{u=0 \text{ at } \partial \Omega} \int_{\Omega} |u| = \min_{u \geq 0 \text{ in } \Omega} \frac{\int_{\Omega} \|
abla u\|}{\mathsf{area}(\Omega)}
\]

But the opposite inequality is easy (just take \( u = \text{char } \Omega \) of \( \Omega \)).

Digression: For a class of closely related problems

see G. Strang, "Maximum flows and minimum cuts in the plane," J. Global Opt 47 (2010) 527–535. One of the many topics there is a very efficient proof (due to Greenblatt) of

Cheeger's inequality: if \( \lambda_1 = \text{1st Dirichlet eigenvalue of } \Delta \) in \( \Omega \), and

\[
\lambda_1 = \min_{\mathsf{area}(\Omega)} \frac{\int_{\Omega} \|
abla u\|^2}{\mathsf{area}(\Omega)}
\]

Then

\[
\lambda_1 \leq \frac{\mu^2}{4}
\]
Pf. From prior discussion it is seen that $\|u\|_1 \leq 1$ and $\div \sigma = \mathbf{b}$. Let $u_0$ be the 1st Dirichlet eigenfunction. Then

$$
\frac{1}{2} \int u_0^2 = \int (\nabla u_0)^2 u_0^2 = -2 \int u_0 \langle \sigma, 7u_0 \rangle
$$

$$
\leq 2 \frac{\|u_0\|_1 \|7u_0\|_1}{2} \frac{\|u_0\|_1 \|7u_0\|_1}{2} \leq 2 \left( \frac{\int u_0^2}{2} \right)^{1/2} \left( \frac{\int 7u_0^2}{2} \right)^{1/2}
$$

So

$$
\|u_0\|_1 \leq 2 \left( \frac{\int 7u_0^2}{\left( \frac{\int u_0^2}{2} \right)^{1/2}} \right)^{1/2} = 2 \chi_0
$$


First, some orientation: essence of duality is a scheme for proving lower-bids on
minimization problems (or upper bounds on maximization problems) using just integration by parts + elementary inequalities (applied to a well-chosen test function).

Essence of "calibration method": sometimes this worked even for problems that are not (in any obvious way) convex.

The example I'll discuss comes from the theory of minimal surfaces: the 7-dimensional surface in $\mathbb{R}^8$

$$S = \frac{1}{2} x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2$$

(known as the Simmons cone) is area-minimizing in the sense that no compactly-supported perturbation lowers the (7-dimensional) surface area.

Rough sketch:
Importance of this: codim-one minimal surfaces are smooth in $\mathbb{R}^n$, $n \leq 7$. This example shows that $n=8$ is different. Original pt (much more complicated!) was due to Bombieri, De Giorgi, + Giausi in 1969.

Key idea : it's suffice to find a vector field $\xi$ in $\mathbb{R}^8$. 

\[ 10^2 \leq 1 \text{ ptwise}\]
\[ \omega = \text{unit normal at all pts on the Swyers cone} \]
\[ dx \xi \leq 0 \quad \text{"below" the Swyers cone} \]
\[ dx \xi \geq 0 \quad \text{"above" the Swyers cone} \]

("above" means $x_1^2 + x_2^2 + x_3^2 + x_4^2 > x_5^2 + x_6^2 + x_7^2 + x_8^2$, and "below" means the opposite sign equality.)

Claim : Any such $\xi$ provides an elementary integration-by-parts-based proof that (in the notation of our previous sketch)

\[ |\xi| \leq |\xi| \]

(Apology to convex duality: while $\xi$ does not solve a "dual plan", it provides a bound by arguments similar to those used in proving that $\xi$ from a dual plan to bound the primal.)
Proof of the claim: given $S$, let

$$A = \{ \text{pts below } \tilde{S} + \text{ above } S \}$$

$$B = \{ \text{pts below } \tilde{S} + \text{ above } S \}$$

and let $\tilde{n}_A, \tilde{n}_B$ be unit normals pointing "upward".

Observe that

$$\text{div } S = \int_S \sigma \cdot \tilde{n}_A \quad \text{using the outward normal } \tilde{n}_A$$

$$\quad = \int_{\partial B} \sigma \cdot \tilde{n}_A$$

$$\quad = \int_{\partial A} \sigma \cdot \tilde{n}_A$$

$$\text{div } S = \int_{\partial B} \sigma \cdot \tilde{n}_B \quad \text{using outward normal } \tilde{n}_B$$

$$\quad = \int_{\partial B} \sigma \cdot \tilde{n}_B$$

$$\quad = \int_{\partial A} \sigma \cdot \tilde{n}_B$$
Therefore

\[ \iiint_A \text{div} \sigma \, dV - \iiint_B \text{div} \sigma \, dV = \int_{S_2 \cap B_R} \sigma \cdot n \, dS - \int_{S_1 \cap B_R} \sigma \cdot n \, dS \]

(\text{pts where } S = \overline{S} \text{ aren't in either } \partial A \text{ or } \partial B, \text{ but such points enter into both terms on RHS and the contributions cancel}).

Note: if \( \sigma \) has properties listed above, then LHS terms are positive, so

\[ 0 \leq \iiint_A \text{div} \sigma \, dV - \iiint_B \text{div} \sigma \, dV \]

\[ = \int_{S_2 \cap B_R} \sigma \cdot n \, dS - \int_{S_1 \cap B_R} \sigma \cdot n \, dS \]

This is exactly area

\[ \leq \text{area of } S \text{ in } B_R, \text{ since } \sigma \cdot n \leq 1 \text{ on } S \]

So \( \text{area of } S \text{ in } B_R \leq \text{area of } S \text{ in } B_R \)

as asserted.

Where to find \( \sigma \)? It's easy: writing
$z = (x_1, x_2, x_3, x_4) \rightarrow \mathbf{w} = (x_5, x_6, x_7, x_8)$

Let

$$f(z, \mathbf{w}) = \frac{\left| z^4 \mathbf{w} \right|^4}{4}$$

and consider

$$\sigma^2 = \frac{\sqrt{f}}{17f_{13}}.$$

(Note: "above $\leq \Rightarrow f > 0\)."

An elementary (dimension-dependent!) calculation gives

$$17f_{13} \text{ det } \frac{\sqrt{f}}{17f_{13}} = \left( \frac{1}{17f_{13}} \right)^{10} \left[ 3121 - 3121 \mathbf{w}^2 + 3121 \right]$$

Clearly $\geq 0$

so $\sigma^2$ has same sign as $f$ (as desired). And Simon's case is $\xi f = 0^2$, so $\sigma = \xi$

on $\xi$ as required.

Some exercises:

1. Show that the problems

2. Minimize $f(w, z) = \frac{1}{2} \sum_{i=1}^{10} (z_i - w_i)^2$

   $\sigma, \mu = f \text{ at } \Sigma$
and

$$\max \int_{\Omega} u \cdot f \, dx - \int_{\Omega} u \cdot F \, dx$$

are a dual pair, if $\int_{\Omega} F \, dx = \int_{\Omega} f \, dx$.

How should $\sigma + \nu \mu$ be related if equality is to hold?

(Hint: $\max_{\sigma \in \mathbb{R}^2} \langle \xi, \sigma \rangle = -\infty$ if $|\xi| \leq 1$.

$\max_{\sigma \in \mathbb{R}^2} \langle \xi, \sigma \rangle = \infty$ if $|\xi| > 1$.)

Rule: if $\Omega \subset \mathbb{R}^2$ and $F = 0$ then $\sigma$ can be solved explicitly in simple cases using the co-area formula. Why?

(2) The $L^1 - L^\infty$ example discussed in the last half of these notes concerned

$$\min_{\sigma} \|\sigma\|_{L^\infty} \text{ s.t. } dw \sigma = 1 \text{ on } \Omega \subset \mathbb{R}^2.$$

Can you do something similar for

$$\min_{\sigma} \max_{x \in \Omega} (10^1 (x) + 10^2 (x)) \text{ s.t. } dw \sigma = 1 \text{ on } \Omega \subset \mathbb{R}^2.$$

where $\Omega \subset \mathbb{R}^2$?