Addendum to Lecture 3: why is min max = max min in our L1-L0 example?

The paper


shows how this problem can be cast as a special case of a duality theorem in the book by Ekeland + Temam, Convexity and Variational Problems, North Holland, 1976 (which is in reserve).

The following self-contained treatment is a little different – essentially a specialization of the argument given for a class of problems from plasticity in E. Christiansen, "Limit analysis in plasticity as a mathematical programming problem," Calcolo 17 (1980) 41-65

Given a closed domain $D \subset \mathbb{R}^n$ (with smooth enough boundary) and $f : D \rightarrow \mathbb{R}$ (sufficiently regular, see below), choose function spaces $X$ and $Y$ (natural to the problem) such that
\[
\sup_{\|\sigma\|_1 \leq 1} \inf_{f} \int \langle \sigma, \eta u \rangle = \inf_{f} \sup_{\|\sigma\|_1 \leq 1} \int \langle \sigma, \eta u \rangle
\]
\[
\text{for } \sigma \in \mathcal{X} \quad \text{and} \quad u \in \mathcal{Y}
\]

(Our \(L^1 - L^\infty\) example in Lecture 3 used \(f = 1\).)

**Choices:**

a) \(X = W^{1,p}(D)\), with \(p > n\) (so that \(\sigma \in X \Rightarrow \sigma\text{ is continuous}\))

b) \(Y = BV(\mathbb{R}^n) \cap \overline{\{u = 0 \text{ outside } D\}}\). While a naive formulation would ask that \(u = 0\) at \(\partial D\), we have to allow \(u\) to jump at the boundary (indeed, we saw using the co-area formula that the actual solution has this character). By definition, \(u \in BV \Rightarrow \nu_u\) is a vector-valued measure with finite total variation \(\int |\nabla u|\). Fact: \(\|u\|_{BV(D)} \leq C \int |\nabla u|\).

c) Since \(u\) may jump to 0 at \(\partial D\), \(\nu_u\) can "charge" \(\partial D\), so we must define
\[
\int \langle \sigma, \eta u \rangle = \int_{\mathbb{R}^n} \langle \sigma, \eta u \rangle = \int_{D} \langle \sigma, \eta u \rangle
\]

which is not the same in general as \(\int \langle \sigma, \eta u \rangle_{D}\).
With these choices it is easy to see that
\[
\sup u \leq \sup \left\{ \lambda : \exists \sigma \in W^{1, p} \cap \text{div} \sigma = \lambda f \right\}
\]
where \(10^{-1} \leq 1 \text{ p/twice} \).

The \( \inf \) is \( \inf \int_{D} 17u_1 \)
\[\int_{D} u \leq 1 \text{ \( u = 0 \) outside } D \]

(the arguments are exactly as we did in class for \( f = 1 \)).

To be sure the \( \sup \) is positive it is natural to assume that
\[d) \text{ there exists } \tau \in W^{1, p}(D) \text{ with } \text{div} \tau = f.\]

This is effectively a regularity hypothesis on \( f \), by elliptic regularity it suffices that \( f \in L^p \). We can then solve
\[\Delta \varphi = f \text{ in } D, \varphi = 0 \text{ at } \partial D, \text{ and take } \tau = \nabla \varphi.\]
Since elliptic theory \( \Rightarrow \frac{\| \varphi \|_{W^{1, p}(D)}}{\| \varphi \|_{L^p(D)}} \leq C \frac{\| f \|_{L^p(D)}}{\| \tau \|_{L^p(D)}} \)

this \( \tau \) is in \( W^{1, p}(D) \).

Since \( \sup u \leq \inf u \leq \sup u \) trivially, our task is to show that \( \inf \sup u \leq \sup \inf u \), i.e. that
\[\inf \int_{D} 17u_1 \leq \sup \left\{ \lambda : \exists \sigma \in W^{1, p} \cap \text{div} \sigma = \lambda f \right\} \]
where \( \int_{D} u \leq 1 \text{ \( u = 0 \) outside } D \).
It's convenient to define

\[ F(\sigma) = \inf_{u \in Y} \int \langle \sigma, u \rangle = \begin{cases} \lambda f & -\lambda \sigma^2 = \lambda f \\ -\infty & \text{otherwise} \end{cases} \]

and to observe that since \( F \) is an unfit of linear functionals, it is a concave function of \( \sigma \).

Now we start the real work. Let \( \mu \) be the value of the sup inf (i.e., \( \mu = \sup_{\sigma \in X} \inf_{\|u\|_L^2 \leq 1} F(\sigma) \)), and consider

\[ S_1 = \left\{ (\sigma, r) \in X \times \mathbb{R} : F(\sigma) - \mu \geq r \right\} \]
\[ S_2 = \left\{ (\sigma, r) \in X \times \mathbb{R} : \|\sigma\|_{L^2} \leq 1, r \geq 0 \right\} \]

Then

- \( S_1 + S_2 \) are both convex (we use here the concavity of \( F \))
- \( S_2 \) has nonempty interior (in fact, \( \sigma = 0, r = 1 \) is an interior point; we use here that \( p > 0 \) so that \( \|\sigma\|_{L^p} \leq C \|\sigma\|_{W^{1,p}} \)).

Therefore, there is a linear functional on \( X \times \mathbb{R} \) that
"separates" $S_1 + S_2$ (This is a corollary of the Hahn-Banach Theorem, see e.g. Royden [in my copy = 3rd edn., it is Thm 20 in Chap 10]). This means $\exists \theta \in (W^p)^*$ and constants $\bar{r}, r \in \mathbb{R}$ st

(i) $\theta(\sigma) + r \bar{r} \geq r$ for all $(\sigma, r) \in S_1$

(ii) $\theta(\sigma) + r \bar{r} \leq r$ for all $(\sigma, r) \in S_2$

Claim: $r \bar{r} < 0$. In fact, $(0, r) \in S_2$ for all $r > 0$; substitution into (ii) shows (as $r \to 0^+$) that $\bar{r} > 0$.

Can $\bar{r} = 0$? If so then since $(0, -\mu) \in S_1$, (i) forces $\mu \leq 0$. But since $(0, 0) \in S_2$, when $\mu$ is sufficiently small, (ii) forces $\mu > 0$. This is a contradiction.

So $\bar{r} < 0$, the claim is proved.

Rescaling $l + c$, we may suppose $\bar{r} = -1$; then

(i) $\theta(\sigma) + c \geq c$ for $(\sigma, r) \in S_1$

(ii) $\theta(\sigma) + c \leq c$ for $(\sigma, r) \in S_2$.
We're almost done. Recall (from bottom of p. 3) that our task is to show

$$\inf \int |f| \leq \mu, \quad \sup |f| = 1, \quad x \in Y$$

We do this by showing

**Claim 1:** \( \sup_{\|\sigma\|_L^1 \leq 1} \|\sigma\|_X \leq 1 \) \( \sigma \in X \)

**Claim 2:** \( \|f\| = \int \langle \sigma, 7u_0 \rangle \) for some \( u_0 \in Y \)

**Claim 3:** \( \int uf = 1 \)

These suffice, since

$$\sup_{\|\sigma\|_L^1 \leq 1} \int \langle \sigma, \sigma u_0 \rangle = \int |\sigma u_0|$$

\( \sigma \in X \)

**Proof of Claim 1:** Observe first that since \( (0, -\mu) \in \Sigma_1 \), we have \( c \leq \mu \) from (i).
Now observe that $F(\sigma_j) \uparrow \mu$. Setting $\tilde{\sigma} = F(\sigma_j) - \mu$ we have

$$l(\sigma_j) - \tilde{\sigma}_j \geq c \quad \text{by (i)}$$

$$l(\sigma_j) \leq c \quad \text{by (ii)}$$

whence $l(\sigma_j) \to c$. This shows

$$\sup_{\|\sigma\|_{\infty} \leq 1} l(\sigma) \geq c$$

$$\sigma \in X$$

but the opposite inequality is obvious from (ii).

So

$$\sup_{\|\sigma\|_{\infty} \leq 1} l(\sigma) = c$$

$$\sigma \in X$$

and Claim 1 is proved.

Proof of Claim 2: since $\sup_{\|\sigma\|_{\infty} \leq 1} l(\sigma)$ is bounded, there is a vector-valued measure $\tilde{\mu} = \sum \mu_i$ with finite total variation such that

$$l(\sigma) = \int_D \sum \sigma_i \, d\mu_i.$$

Our task is to show that $\tilde{\mu} = \nu$ for some $(\tilde{\sigma}, \tilde{\mu})$. 

A key observation is that

\[ l(\sigma) = 0 \text{ whenever } \text{div}\sigma = 0 \]

Indeed, if \( \text{div}\sigma = 0 \) then \( F(\sigma) = 0 \) and moreover \( F(t\sigma) = 0 \) for any \( t \in \mathbb{R} \). So

\[ \text{div}\sigma = 0 \implies (t\sigma, -\mu) \in \mathcal{S} \text{ for all } t \]

\[ \implies t l(\sigma) + \mu = c \text{ for all } t \]

\[ \implies l(\sigma) = 0. \]

Thus \( l \) can be viewed as a linear functional on \( W^{1,p}(\Omega) / \{ \sigma : \text{div}\sigma = 0 \} \). But this quotient is isomorphic to \( L^p \), via the map

\[ T : \sigma \mapsto \text{div}\sigma. \]

(This follows from the Closed Graph Theorem once we recognize that \( T \) is onto. To see that, note that if \( g \in L^p(\Omega) \), the solution of \( \Delta \phi = g \) in \( \mathcal{D} \), \( \phi = 0 \) at \( \partial \mathcal{D} \) has

\[ \| \phi \|_{L^p} \leq C \| g \|_{L^p} \text{ so that } \sigma = \nabla \phi \in W^{1,p} \text{ has } T(\sigma) = g. \)

Since \( L^p(\Omega) = L^p \), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( \exists u_0 \in L^q(\mathcal{D}) \) s.t.

\[ l(\sigma) = -\int (\text{div}\sigma) u. \]

Extending \( u_0 \) by 0 outside \( \mathcal{D} \), we can write...
Thus as

\[ l(\sigma) = \int \langle \sigma, \nabla u_0 \rangle \]

Evidently the measure \( \mu \) introduced when we started this proof is really \( \mu = \nabla u_0 \).

We have \( u_0 \in Y \), since

\[ \left\| \nabla u_0 \right\|_1 = \sup_{\| \sigma \| \leq 1} \int \langle \sigma, \nabla u_0 \rangle \]

is known to be finite. Claim 2 is proved.

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Proof of Claim 3: Let \( T \in W^{1,p} \) satisfy \( -\text{div} T = f \). Then

\[ F(T) = 1 \]

\[ \text{so} \]

\[ (\lambda T, \lambda - \mu) \in S_1 \quad \text{for any} \quad \lambda \in \mathbb{R} \]

Therefore

\[ \lambda l(T) - \lambda + \mu \geq c \quad \text{for any} \quad \lambda \in \mathbb{R} \]

which can hold only if \( l(T) = 1 \). Since

\[ l(\sigma) = \int \langle \sigma, \nabla u_0 \rangle \]

we have

\[ 1 = \int \langle T, \nabla u_0 \rangle = \int -\text{div} T \cdot u_0 = \int f u_0 \].
Claim 3 is now complete.

The preceding argument is special in its choice of function spaces, but typical in the sense that the "separating hyperplane theorem" (or some other version of the Hahn-Banach theorem) underlies most proofs of duality, other than those done by explicit examination of the solutions (as we did for quadratic cases).

It is natural to ask whether support and sup can fail. The article by Christiansen gives an example where the failure is due to a poor choice of function spaces, namely restriction of both $\sigma + u$ to be $C^1$ functions. In fact

$$\inf_{u \in C^1} \sup_{\sigma \in C^1} \int_D \langle u, \sigma \rangle = \inf_{u \in C^1} \int_D 17u_1$$

is strictly positive (in fact $\geq \mu$, since the inf is being taken over a subset of $Y$). However
we have
\[
\inf_{u \in C^k} \int \langle \sigma, \nabla u \rangle = \begin{cases} 
\lambda f & \text{if } -\lambda \Delta u = \lambda f \\
-\infty & \text{otherwise}
\end{cases}
\]
\[
u = 0 \text{ at } \partial D
\]
\[
\int u f = 1
\]
from which it follows that if \( f \notin C^{k-1} \) then
\[
\sup_{\sigma \in C^k} \inf_{u \in C^k} \int \langle \sigma, \nabla u \rangle = 0
\]
\[
u = 0 \text{ at } \partial D
\]
\[
\int u f = 1
\]
For other examples where \( \sup u \neq \inf u \) in problems closely related to our \( L_1 - L_\infty \) example, see R. Nogawa, "Examples of max-flow and min-cut problems with duality gaps in continuous networks," Math Prog 63 (1994) 213-234.