Problems on symmetric functions of roots of a polynomial equation.

In what follows, $a_1, a_2, \ldots a_n$ are the roots of the equation

$$(z - a_1)\cdots(z - a_n) = z^n - \alpha_1 z^{n-1} + \cdots + (-1)^p \alpha_p z^{n-p} + \cdots + (-1)^n \alpha_n = 0$$

The α_i are the elementary symmetric functions of the roots:
(1) $\alpha_1 = \sum_{i=1}^{n} a_i = \sum a_1$. (This unorthodox notation uses a typical term a_1 and is useful.)
(2) $\alpha_2 = \sum_{i<j}^{n} a_i a_j = \sum a_1 a_2$.
...
(n) $\alpha_n = a_1 a_2 \cdots a_n$.

In addition, we define the functions

$$S^k = \sum_{i=1}^{n} a_i^k = \sum a_1^k.$$

S_k is defined for all non-negative integers. If all $a_i \neq 0$ for all i, or equivalently if $\alpha_n \neq 0$, S_k is defined for all integers, positive, negative and zero.

In lecture, we computed S_2 in terms of the elementary symmetric functions. Briefly, the method:

$$\alpha_1^2 = (a_1 + \ldots)(a_1 + \ldots) = \sum a_1^2 + 2\sum a_1 a_2 = S_2 + 2\alpha_2$$

so $S_2 = \alpha_1^2 - 2\alpha_2$. (It is critical that you understand where the factor 2 comes in!)

Here are some problems.
1. Express S_3 in terms of the elementary symmetric functions. (Done in lecture.)
2. Express $\sum a_1^2 a_2)$ in terms of the elementary symmetric functions. This is, of course, $\sum_{i,j=1}^{n} a_i^2 a_j$. Here we do not have $i < j$ because, for example $a_1^2 a^2 \neq a_2^2 a_1$.
3. If $\alpha_n \neq 0$, express $\sum 1/a_1$ in terms of the elementary symmetric functions.
4. If $\alpha_n \neq 0$, express $\sum a_1/a_2$ in terms of the elementary symmetric functions.
5. Express $\sum a_1^3 a_2$ in terms of the elementary symmetric functions.

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In 6-9, take $n = 3$, and let a, b, c be the roots of the equation $x^3 + px + q = 0$.
6. Find $(a - b)^2 + (b - c)^2 + c - a)^2$
7. Show that $S_{n+3} = -pS_{n+1} - qS_n$ for $n \geq 0$.
8. Using the above recursion and the values of S_0, S_1, S_2, find S_3, S_4, S_5 as functions of the coefficients p and q.
9. The recursion in 7 is valid for all integers n if $q \neq 0$. Why? Using this fact, compute

$$1/a + 1/b + 1/c,\ 1/a^2 + 1/b^2 + 1/c^2, \text{ and } 1/a^3 + 1/b^3 + 1/c^3.$$

(Assume $p \neq 0$.)

10. In the quadratic $x^2 + px + q = 0$, let the roots be a and b. Using the technique of Exercise 7, find a recursion relating S_{n+2}, S_{n+1} and S_n. In particular, find $a^5 + b^5$ by this method.
11. As in 10, but find $1/a^4 + 1/b^4$. Do 2 ways. (Assume $q \neq 0$.)

12. If $r + s = 1$ and $r^4 + s^4 = 4$, what is $r^3 + s^3$?

13. If $r + s + t = 1$, $r^2 + s^2 + t^2 = 2$, and $r^3 + s^3 + t^3 = 3$, what is the value of $r^4 + s^4 + t^4$?