Putnam Exam: Series problems

These are from 1985 through 2002.

2002A6. Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

2001B3. For any positive integer n, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n}.$$

2000A1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive real numbers for which $\sum_{j=0}^{\infty} x_j = A$?

1999A3. Consider the power series expansion

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is and integer m such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

1999A4. Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n 3^m + m 3^n)}.$$

1999B3. Let $A = \{(x, y) : 0 \leq x, y < 1\}$. For $(x, y) \in A$, let

$$S(x, y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs (m, n) of positive integers satisfying the indicated inequalities. Evaluate

$$S(x, y) = \sum_{\substack{(x,y) \to (1,1) \\ (x,y) \in A}} (1 - xy^2)(1 - x^2 y) S(x, y)$$

1996B4. For any square matrix A, we can define $\sin A$ by the usual power series

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}$$

Prove or disprove: there exists a 2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

1990B2. Prove that for $|x| < 1, |z| > 1$,

$$1 + \sum_{j=1}^{\infty} (1 + x^j) \frac{(1-z)(1-zx)(1-zx^2)\cdots(1-zx^{j-1})}{(z-x)(z-x^2)(z-x^3)\cdots(z-x^j)} = 0$$

1989A6. Let $\alpha = 1 + a_1 x + a_2 x^2 + \ldots$ be a formal power series with coefficients in the field of two elements. Let

$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } \alpha \\ & \text{has an even number of zeros in the block,} \\ & \\ 0 & \text{otherwise.} \end{cases}$$

(For example, $a_{36} = 1$ because $36 = 100100_2$, and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + x\alpha + 1 = 0$.

1988A3. Determine, with proof, the set of real numbers x for which

$$\sum_{n=0}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1 \right)^x$$

converges.

1988B4. Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

1987A6. For each positive integer n, let $a(n)$ be the number of zeros in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?