Putnam Exam: Number Theory problems

These are from 1985 through 2002.

2002B5. A palindrome in base b is a positive integer whose base-b digits read the same backwards and forwards; for example, 2002 is a 4-digit palindrome in base 10. Note that 200 is not a palindrome in base 10, but it is a 3-digit palindrome 242 in base 9, and 404 in base 7. Prove that there is an integer which is a 3 digit palindrome in base b for at least 2002 different values of b.

2001A5. Prove that there are unique positive integers $a,\ n$ such that $a^{n+1} - (a+1)^n = 2001$.

2000A2. Prove that there exists infinitely many integers n such that $n,\ n+1,\ n+2$ are each the sum of two squares. [Example: $0 = 0^1 + 0^2,\ 1 = 0^2 + 1^2$, and $2 = 1^2 + 1^2$.

2000B1. Let a_j, b_j, c_j be integers for $1 \le j \le N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers $r,\ s,\ t$ such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of $j,\ 1 \le j \le N$.

2000B2. Prove that the expression

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \ge m \ge 1$.

1997B5. Prove that for $n \ge 2$,

$$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n \equiv \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}n-1 \pmod n$$

1996A5. If p is a prime number greater than 3 and $k = [2p/3]$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

1994B6. For each integer a, set

$$n_a = 101a - 100 \cdot 2^a.$$

Show that for $0 \le a, b, c, d \le 99,\ n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a, b\} = \{c, d\}$.

1993B1. Find the smallest positive integer n such that for every integer m with $0 < m < 1983$, there exists and integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}$$

1992A3. For a given positive integer m, find all triples (n, x, y) of positive integers, with n relatively prime to m, which satisfy $(x^2 + y^2)^m = (xy)^n$.

1988B1. A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \ldots)\}$. Show that every composite integer is expressible as $xy+xz+yz+1$, with x, y, and z positive integers.

1988B6. Prove that there exist an infinite number of ordered pairs (a, b) of integers such that for every positive integer t the number $at+b$ is triangular if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0, 1, 2, dots\}$.)

1986A2. What is the units (i.ie. rightmost) digit of $\left[\frac{10^{2000}}{10^{100}+3}\right]$? Here $[x]$ is the greatest integer $\leq x$.

1985A4. Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?