Putnam Integration Problems.

These are all from 1985–2000.

A4-2000. Show that the improper integral

$$\lim_{B\to\infty}\int_0^B \sin(x)\sin(x^2)dx$$

converges.

A5-1999. Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$p(0) \le C\int_{-1}^{1} |p(x)|dx.$$

A2-1995. For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a}-\sqrt{x}} - \sqrt{\sqrt{x}-\sqrt{x-b}}\right) dx$$

converge?

A5–1993. Show that

$$\int_{-100}^{-10}\left(\frac{x^2-x}{x^2-3x+1}\right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}}\left(\frac{x^2-x}{x^2-3x+1}\right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}}\left(\frac{x^2-x}{x^2-3x+1}\right)^2 dx$$

is a rational number.

B4–1993. The function $K(x,y)$ is positive and continuous for $0 \le x \le 1,\ 0 \le y \le 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \le x \le 1$. Suppose that for all $x,\ 0 \le x \le 1$,

$$\int_0^1 f(y)K(x,y)dy = g(x) \text{ and } \int_0^1 g(y)K(x,y)dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \le x \le 1$.

A2–1992. Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series expansion of $(1+x)^\alpha$. Evaluate

$$\int_0^1 C(-y-1)\left(\frac{1}{y+1}+\frac{1}{y+2}+\frac{1}{y+3}+\cdots+\frac{1}{y+1992}\right) dy.$$

Continued on other side.

A5–1991. Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 \leq y \leq 1$.

B1–1990. Find all real-valued differentiable functions f on the real line such that for all x

$$(f(x))^2 = \int_0^x \left((f(t))^2 + (f'(t))^2\right) dt + 1990.$$

A2–1989. Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dy\, dx$ where a and b are positive.

B3–1989. Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 and x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x) dx$$

(sometimes called the nth moment of f).

a. Express μ_n in terms of μ_0.

b. Prove that the sequence $\left\{\mu_n \frac{3^n}{n!}\right\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

B1–1987. Evaluate $\displaystyle\int_2^4 \frac{\sqrt{\ln(9-x)}dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$

A5–1985. Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\ldots\cos(mx)dx$. For which integers m, $1 \leq m \leq 10$, is $I_m \neq 0$?

B5–1985. Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})} dt$. You may assume that $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$.